Sobolev Spaces and Partial Differential Equations
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Lecture 81
The Heat Equation

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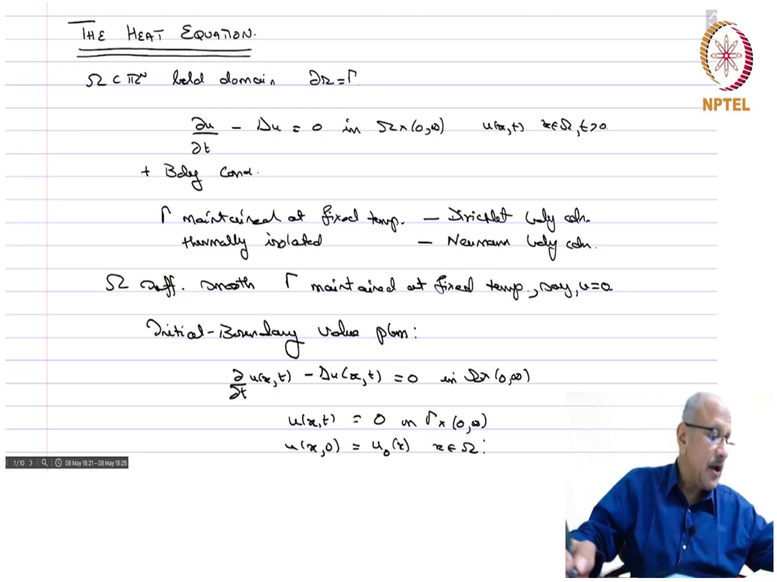

So, we will now look at applications of semigroups to the study of evolution equations. So, we start with the heat equation. So, let $\Omega \subset \mathbb{R}^N$ be a bounded open set bounded domain $\partial\Omega = \Gamma$. The heat equation is given by

$$\frac{\partial u}{\partial t} - \Delta u = 0 \; in \; \Omega \times (0, \infty).$$

So, this omega refers to the x so u is a function of x and t. So, $x \in \Omega$ and $t > 0$.

So, with appropriate boundary conditions, so this heat equations with variations there of occur in several physical phenomena involving diffusion. It is a simplest example of a parabolic differential equation in case of the heat equation u represents the temperature of a body which and the boundary and is a function of x is in omega and time t greater than 0. The boundary conditions depend on the physical situation which we are considering.

If we maintain gamma maintained at fixed temperature then you have a Dirichlet boundary condition. And if gamma is thermally isolated with no heat exchange with the external thing then we have Neuman boundary condition. So, in if case the system reserves heat from an external source then the 0 on the right-hand side of the equation will have a inhomogeneous term f of x t which will depend on the heat source which we are supporting.

So, we will assume now that omega is sufficiently smooth. And that gamma maintained at fixed temperature say $u = 0$ we maintain it at the 0 temperature. Then we have the following initial boundary value problem. So, you have

$$\frac{\partial u}{\partial t} - \Delta u = 0 \; in \; \Omega \times (0, \infty).$$

$$u(x, t) = 0 \; on \; \Gamma \times (0, \infty)$$

$$u(x, 0) = u_0(x) \; on \; for \, x \in \Omega$$

So, we will study this in the framework of semigroups.

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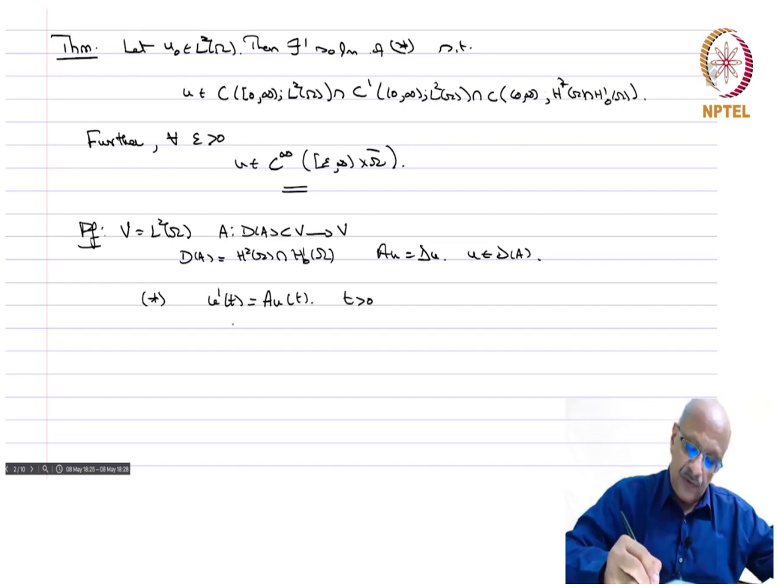

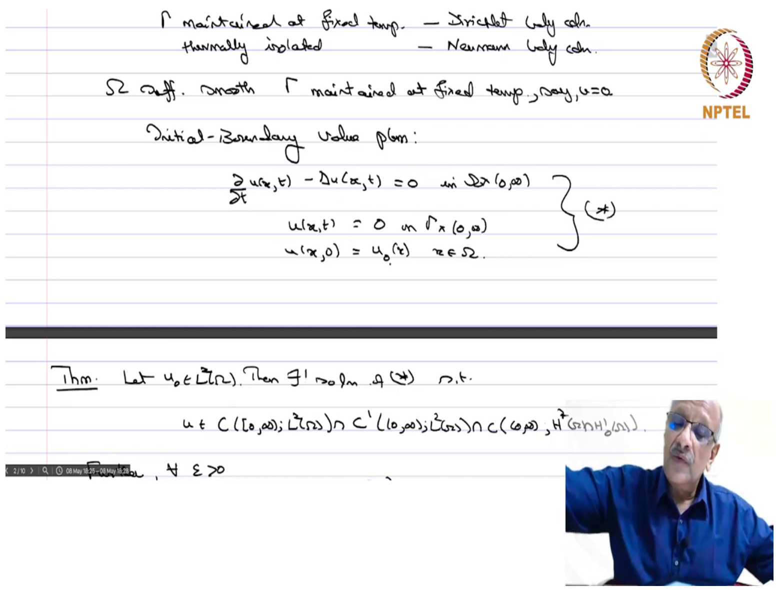

So, we have the following theorem.

Theorem: Let $u_0 \in L^2(\Omega)$. So, we are having very rough initial data. Then there exists a unique solution of star. So, this is the start which is the heat equation. Such that

$$u \in C([0,\infty); L^2(\Omega)) \cap C^1((0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H^1_0(\Omega)).$$

further for every $\varepsilon > 0$, $u \in C^\infty([\varepsilon,\infty) \times \overline{\Omega})$.

So, you see the heat equation you have u_0 is just a L^2 function but the solution for positive time is C^1 and it has takes values in H^2 intersection H^1_0. So, u of t will belong to $H^2 \cap H^1_0$ as soon as t is positive. And furthermore, if you take epsilon infinity cross omega bar if you do not go near 0 the function is infinitely differentiable. Instantaneously however, if the initial data is instantaneously this solution becomes infinitely smooth.

Proof. So, you take $V = L^2(\Omega)$. So, this is our Hilbert space $A: D(A) \subset V \to V$

by $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. And $Au = \Delta u$ in $D(A)$. Then the star equation can be written as you

$u'(t) = Au(t),\ t > 0,\ and\ u(0) = u_0.$

So, this is the equation here du by dt equals Δu. So, that is u dash t is equal to A of u t u belonging to h 1 0. So, it is 0 on the boundary and then its initial value is u 0. So, you have this 0.

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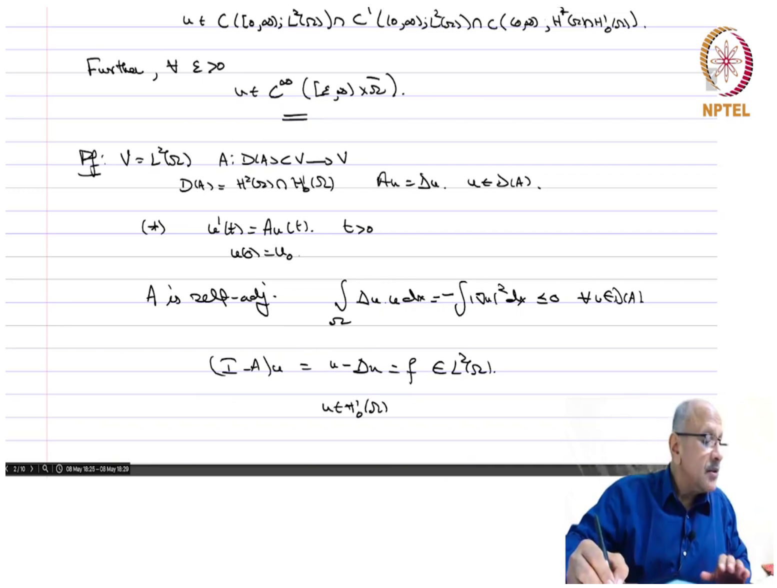

So, then you have then we know that A self-adjoint we have already seen this. And we also have that integral on omega Laplacian u times u dx is equal to minus integral grad u square dx which is less than or equal to 0 for all u in $D(A)$. Sorry since you have $H^1_0(\Omega)$ that is no boundary term. Also, if you have $(I - A)u$ is nothing but u minus Laplacian u equal to F in $L^2(\Omega)$. So, then u is in $H^1_0(\Omega)$. Then we know that this has a unique solution.

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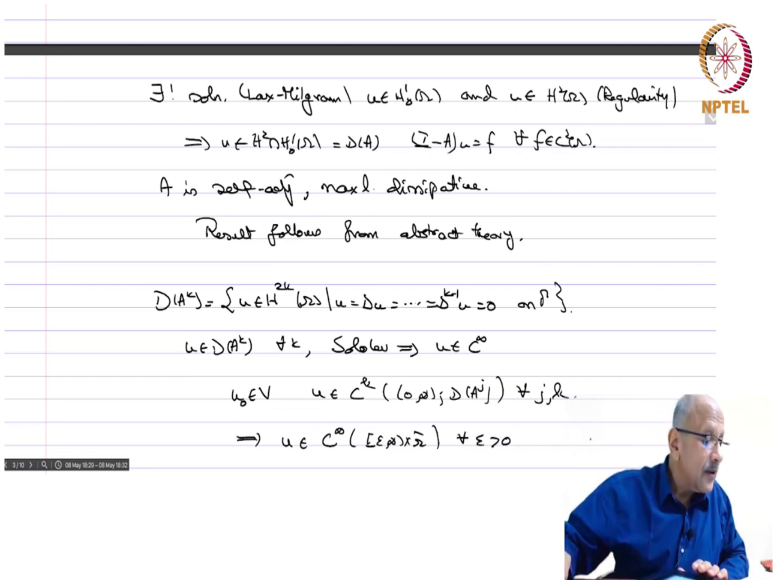

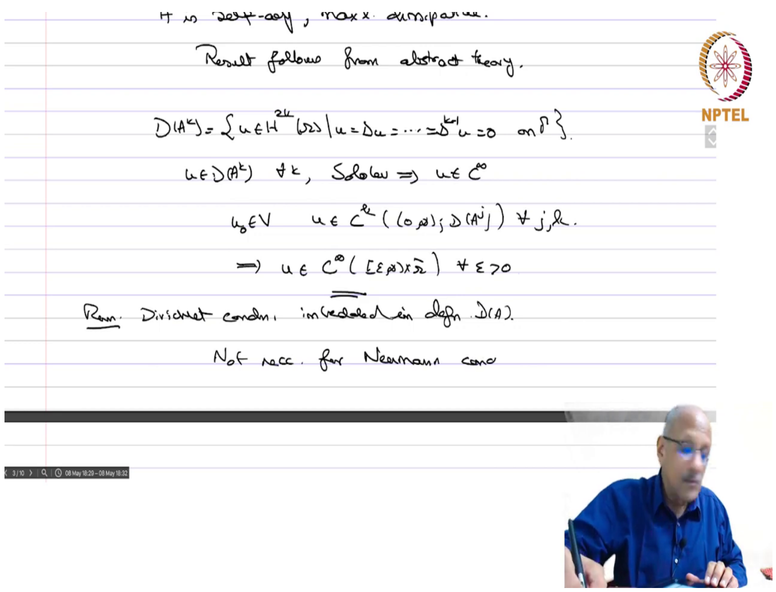

By Lax Rilgram lemma $u \in H^1_0(\Omega)$. And also, u belongs to H^2 (Ω) by regularity. Therefore, u belongs to $H^2(\Omega) \cap H^1_0(\Omega) = D(A)$ and you have I minus A u of A equal to f for every f in $L^2(\Omega)$. So, hence you have that this. So, A is self-adjoint and maximal dissipative. And therefore, result follows from abstract theory. So, for any u 0 in the space you can solve the heat equation with we have seen that complete.

$$D(A^k) = \{u \in H^{2k}(\Omega) \ | u = \Delta u = ... = \Delta^{k-1}u = 0 \ on \ \Gamma\}.$$

So, if $u \in D(A^k)$ for all k then by Sobolev implies that you belongs to C infinity. And this is true. So, we have seen that if u is in u naught is in v then u belongs to C k of epsilon is a open sorry 0 infinity with $D(A^j)$ for all j and k.

So, we have seen this regularity theory I did not prove it. And therefore, this implies that u belongs to C infinity of z epsilon epsilon infinity cross omega bar for every epsilon positive. So, by the sobolev embedding theorem. So,

Remark: so Dirichlet condition embedded in definition of D. So, you know that this essential boundary condition. So, we have to impose it. So, in the definition of $D(A)$ itself we have built it and if it whether the Neuman condition we this would not have been necessary not necessary for Neuman condition.

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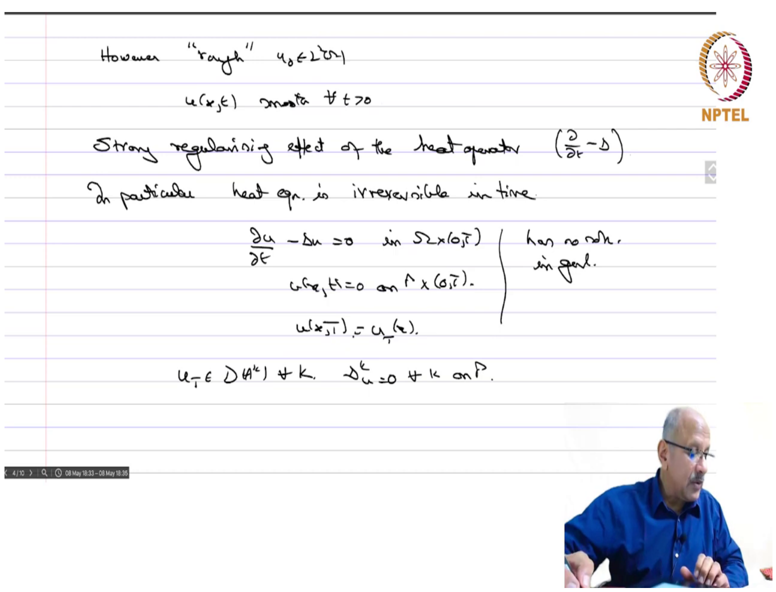

So, an important thing to notice. However, rough the initial data u_0 is the solution $u(x, t)$. So, however rough 0 in $L^2(\Omega)$ $u(x, t)$ smooth for every t positive. So, this is called the strong regularizing effect of the heat operator. Heat operator is d by dt minus delta. So, this strongly regularizing so if your moment you solve it instantaneously the solution becomes very very smooth in particular heat equation is irreversible in time.

So, that is

$$\frac{\partial u}{\partial t} - \Delta u = 0 \; in \; \Omega \times (0, T).$$

$$u(x, t) = 0 \; on \; \Gamma \times (0, T)$$

$$u(x, T) = u_T(x) \; on \; for \, x \in \Omega.$$

So, has no solution in general because we know that if u naught the time at times 0 it is however rough instantaneously it must be smooth. So, it is necessary in particular the $u_T \in D(A^k)$.

And even that and $\Delta^k u = 0$ for all k on gamma. And even if all this is there it is not necessary that the you can solve the heat equation backwards in time. So, you need something extremely smooth functions only. So, given an L^2 function there is no hope to solve this equation backwards.

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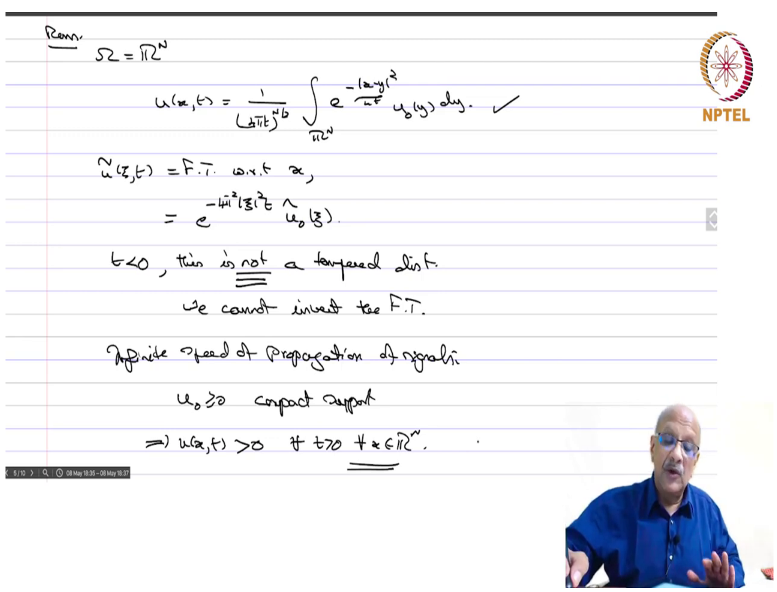

Now, this irreversibility in time can also be seen from the formula for the heat equation if $\Omega \subset \mathbb{R}^N$. For instance, if you have omega equals $\mathbb{R}^N$ so

Remark we have already seen this formula

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy.$$

So, the Fourier transform

$$\tilde{u}(\xi, t) = e^{-4\pi|\xi|^2 t} \hat{u}_0(\xi).$$

And if t is less than 0, this is not a tempered distribution it is not of slow growth. It is of it grows exponentially fast and therefore this is not a tempered distribution So, we cannot invert the Fourier transform. So, this is the reason why we cannot solve the heat equation backwards.

Now, if we have already seen infinite speed of propagation of signals. So, u naught greater or equals 0 compact support implies $u(x,t) > 0$ for all t positive for all $x \in \mathbb{R}^N$. So, this is a comes from this formula which we have here. And therefore, this is called the infinite speed of propagation of signals. So, this is we will this is about the heat equation how we solve for any data and then we have infinite smoothness which just comes from the theory of semigroups for maximal dissipative self-adjoint operators.