Sobolev Spaces and Partial Differential Equations
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Lecture 80
Self-Adjoint Case and the Case of Isometries

So, we were looking at contraction semigroups in Hilbert spaces. And we found that the infinitesimal generators were precisely the maximal dissipative operators. After this we said two important classes and the first one we were looking at is when the case when A the infinite decimal generator was also self adjoint.

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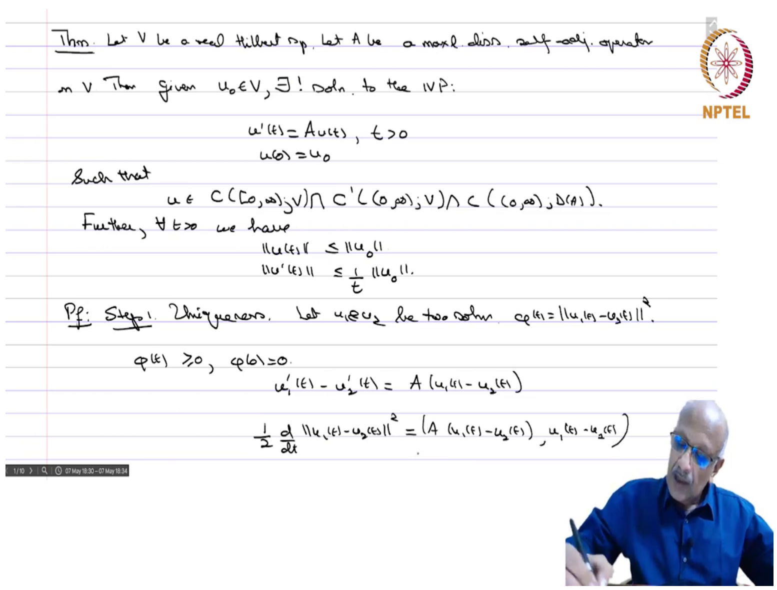

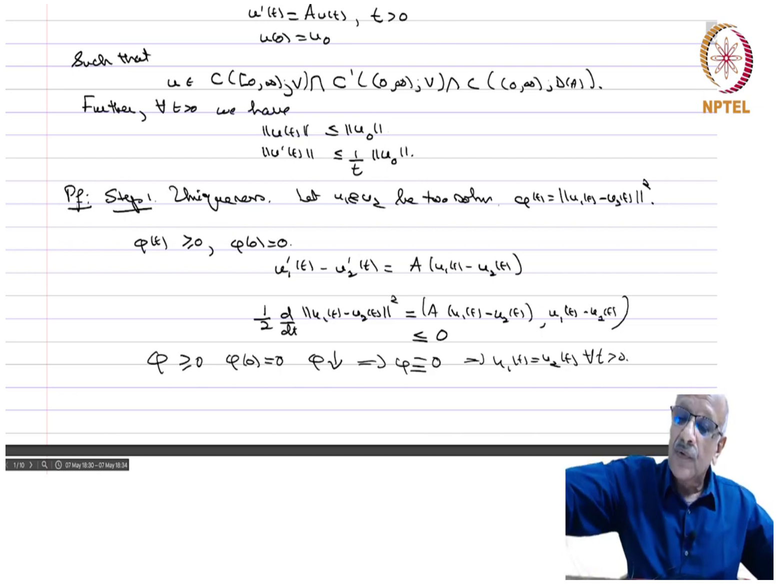

So, here is the very nice theorem which we prove. So,

Theorem: let V be a real Hilbert space. Let A be maximal dissipative self-adjoint operator on V. Then given $u_0 \in V$ there exists a unique solution to the initial value problem

$$u'(t) = Au(t)$$

$$u(o) = u_0.$$

such that $u \in C([0, \infty): V) \cap C^1((0, \infty); V) \cap C((0, \infty); D(A))$. Further for all $t > 0$, we have

$||u(t)|| \leq ||u_0||$ and

$$||u'(t)|| = ||Au(t)|| \leq \frac{1}{t}||u_0||.$$

So, this is the theorem. So, the important thing to note is that so far we have been saying we can solve this initial value problem provided the initial data is in the domain of A. Then otherwise we called it a generalized solution. But then here we in the self-adjoint maximal dissipative case in a Hilbert space you can solve the problem classically even for any data in the entire space.

The only price you pay for that is that you lose continuity at t equal to 0 that was the only difference. Previously we had $C^1[0, \infty)$ with values in C $[0, \infty)$ with values in $D(A)$. But now we have $C\ [0,\infty)$ values in v but $C^1(0,\infty)$ with values in v and $C\ (0,\infty)$ with values in $D(A)$. Obviously at 0 it is not in $D(A)$ necessarily.

So, this is a very beautiful result namely you can solve for any initial data irrespective you do not have to worry about it being in the domain. And you lose continuity at t equal to 0 which is not a big deal. So,

Proof.

Step-1: So, first step is uniqueness. So, let $u_1 and\ u_2$ be two solutions. And you write

$\varphi(t) = ||u_1(t) - u_2(t)||^2$.

Then $\varphi(t) \geq 0,\ \ \varphi(0) = 0$. Because the initial value is the same.

And also, you have that

$u_1'(t) - u_2'(t) = A(u_1(t) - u_2(t))$.

. Now, you take the inner product with the $u_1 - u_2$ so that will be

$\frac{1}{2}\varphi'(t) = (A(u_1(t) - u_2(t)),\ u_1(t) - u_2(t) \leq 0$

And that by the maximal by the dissipativity is less than or equal to 0. So, φ is non zero $\varphi(0) = 0$ and φ is decreasing because norm of d by dt is less than or equal to 0. So, this implies that $\varphi = 0$. That is $u_1(t) = u_2(t)$ for all t positive. So, that proves the uniqueness of the solution in this case.

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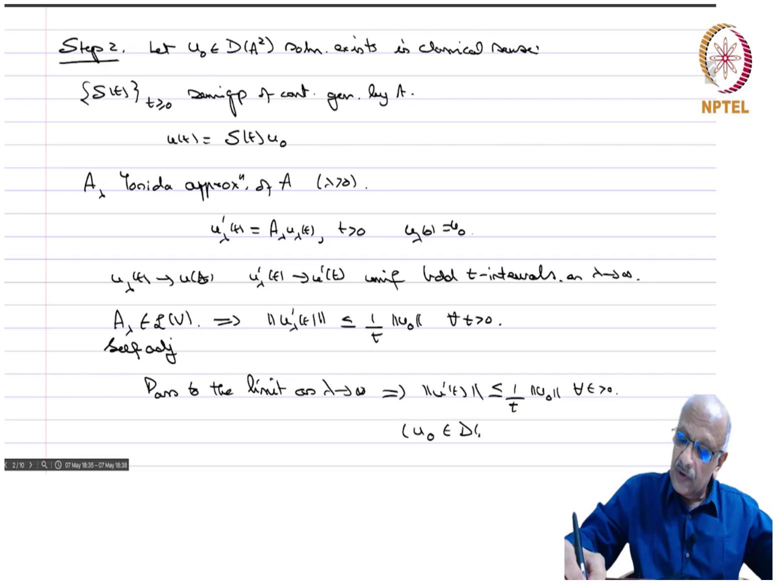

Step 2. Let $u_0 \in D(A^2)$. So, we are going we said we are going to prove things in for $u_0 \in V$. But we start with $u_0 \in D(A^2)$ which is very very smooth. So, that solution exists classical sense that means that even you have even continuity the origin much more than that. So, $\{S(t)\}$ greater or equals 0 semigroup of contractions is generated by A then of course you know that u(t) is nothing but $S(t)u_0$.

So, A_λ Yosida approximation of A_λ positive. Then you have let us take

$$u_\lambda'(t) = A_\lambda u_\lambda(t),\ t > 0,\ u_\lambda(0) = u_0.$$

Then we have seen that of course $u_\lambda(t)$ goes to $u\ (t)$. And also, u_λ dashed of t goes to u dashed of t. This was one of the last theorem which we proved when in the general Hille Yosida case uniformly unbounded intervals as lambda tends to infinity.

Now, $A_\lambda \in L(V)$. It is a bounded linear operator. And therefore, we can apply the theorem which we proved the proposition which I proved last in the previous lecture namely. So, this implies that norm of u_λ dash t and this is self-adjoint this also be proved. So, the norm of

$||u_{\lambda}^{'}(t)|| \leq \frac{1}{t}||u_{0}||, \ t > 0.$

This we proved last time. So, this is of course you assume that your u is in u naught is in $D(A)$ square. So, now we have to do a density argument.

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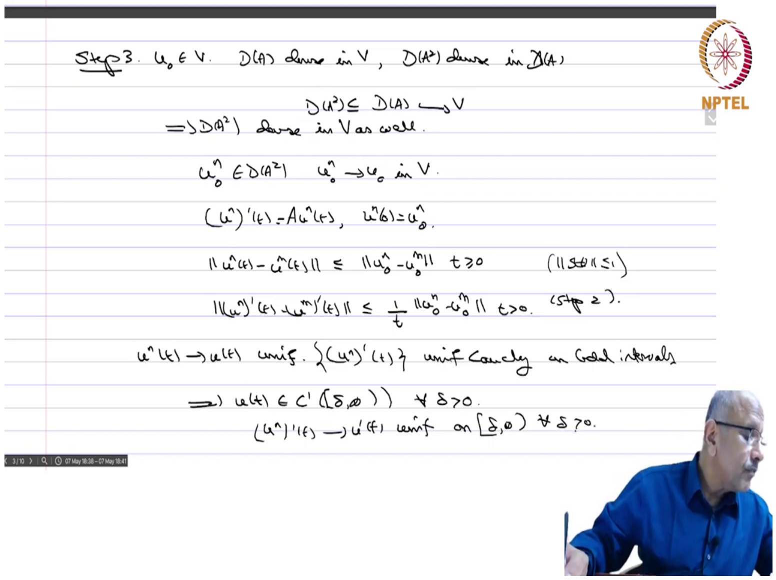

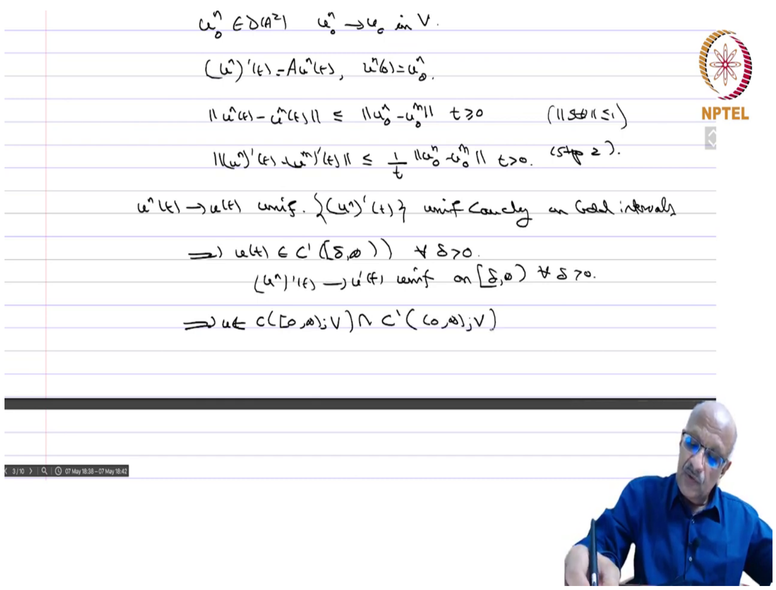

Step 3. So, we have $u_{0} \in V$ then $D(A)$ dense in V and $D(A^{2})$ dense in $D(A)$ this also solve. So, now what is that we have $D(A)$ has the graph norm therefore it is continuously

included in v and $D(A^2)$ is dense here. Therefore, by the continuity of this, this implies that $D(A^2)$ is dense in v as well. Consequently, let us take $u_0^n \in D(A)$ square $u_0^n \to u_0$ *in* V.

So, u^nt equals A of u^n t u n 0 equals u n 0. And then we know that norm of u n t minus u m t is because you have a dissipative operator we know that the norm is always less than equal to the initial value. So, you have u^n_0 minus u m 0 t greater or equal to 0. And norm of u^n dash t minus u m is again less than or equal to 1 by t times norm of u^n 0 minus u m 0 t positive. Now, the first one is just comes from the fact you have this contraction semigroup and also s norm f t is less than or equal to 1.

Norm of S t is less than equal to 1. And the second 1 comes from step 2. So, then you have that u n t converges to u of t uniformly. Because there is uniformly Cauchy and u n dash t uniformly Cauchy unbounded intervals. And therefore, you know if the derivative converges uniformly and the function even converges at one point you know that limit of the should be differentiable and it should be the limit of this implies that u of t is C^1 of delta infinity for all delta positive. And u of u n dash t goes to u dash t uniformly on delta infinity for all delta positive. So, what does this? So, this implies that u of t where u belongs to C of 0 infinity with values in v intersection $C^1(0, \infty)$ with values in V.

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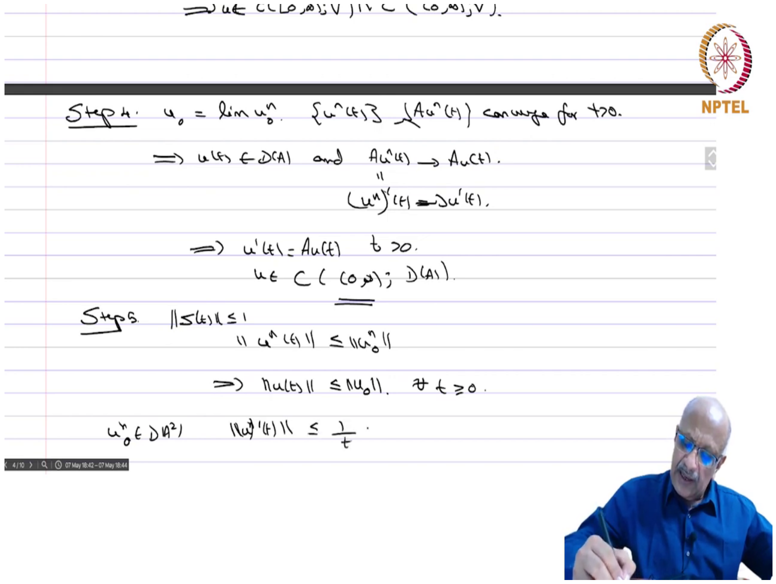

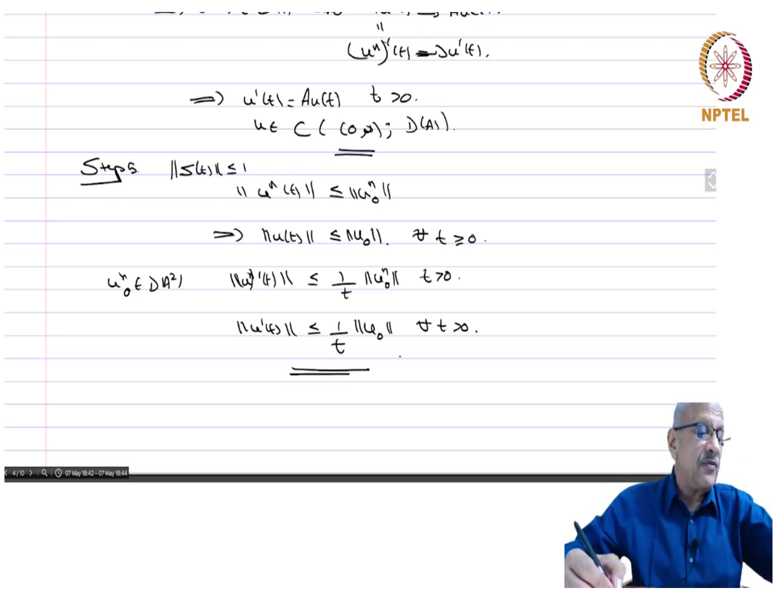

So, now

Step 4: you have to show that u is the solution of the equation which we want. So,

$$u(0) = \lim_{n \to \infty} u^n{}_0 = u_0.$$

that we know and $\{u^n(t)\}$ and $\{Au^n(t)\}$ converge for t positive. This implies that $u(t)$ belongs to $D(A)$ and $\{Au^n(t)\} \to Au(t)$. But your $\{Au^n(t)\} = u^{n'}(t)$ and the t converges to u dash t. Therefore, we have that u dash t equals $Au(t)$ for all t positive u of t belongs to the domain of A.

And therefore, you have u belongs to $C(0,\infty)$ with values in $D(A)$. So, that proves on the continuity properties that u is the solution we have produced a solution for the equation. So, finally step 5. So, u^n S(t) is a contraction semigroup. So, $u^n(t)$ in norm is less than or equal to the norm of $u^n(0)$. And this implies that norm of u(t) is less than equal to norm of u_0. And for all t greater equal to 0. And you have that

$$||(u^n)^{'}(t)|| \leq \frac{1}{t}||u_0^{\,n}||.$$

And therefore, for if you pass to the limit that norm of u dash t is less than or equal to 1 by t times norm of u(0) for all t positive. And this completes the proof of the step. So, we will conclude the self-adjoint case that actually we have much more to say in this.

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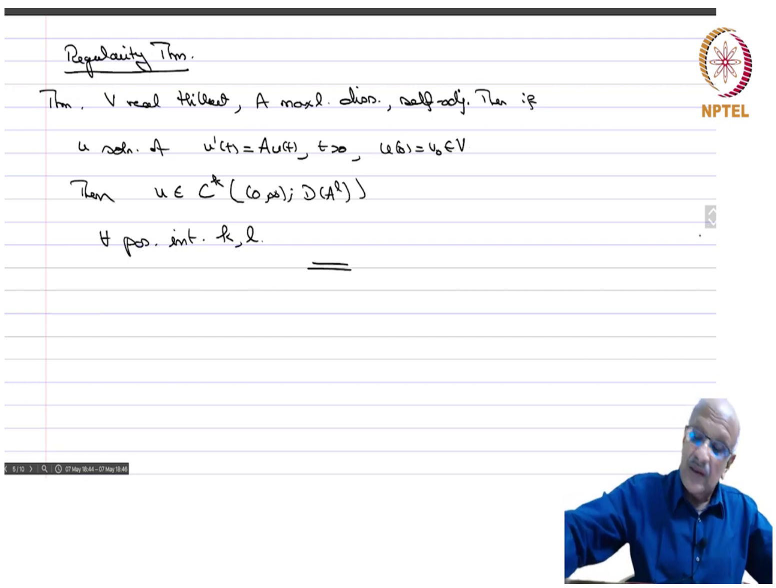

So, the self-adjoint case is a very, very smooth, it is a regularizing effect. So, regularity result so

Theorem: Vreal Hilbert A maximal dissipative self-adjoint. Then if u solution of

$$u^{'}(t) = Au(t)$$

$$u(o) = u_0.$$

Then $u \in C^k((0, \infty); D(A^l)), \quad for\ all\ k, l > 0$. So, this is really wonderful result I will not prove it you can find the proof in the book topics in function analysis applications not a very long proof but I do not want to get into it here.

And so, you see whatever however rough the initial data is namely it is just in v u I told you $D(A)$ is a space of very smooth functions. So, u belongs to $D(A)$ l for any l so it is in fact like if you think of Sobolev spaces this will be something by the Sobolev inclusion theorem something like C^∞. And in time also it is C^∞ as long as you do not go to the origin t equal to 0. So, it is $C^k(0, \infty)$ for all k. So, the solution is extremely smooth and in the case of the self-adjoint operators. So, that is about self-adjoint case. So, now we will I will not as I send and give a proof of this. So, let us save some time.

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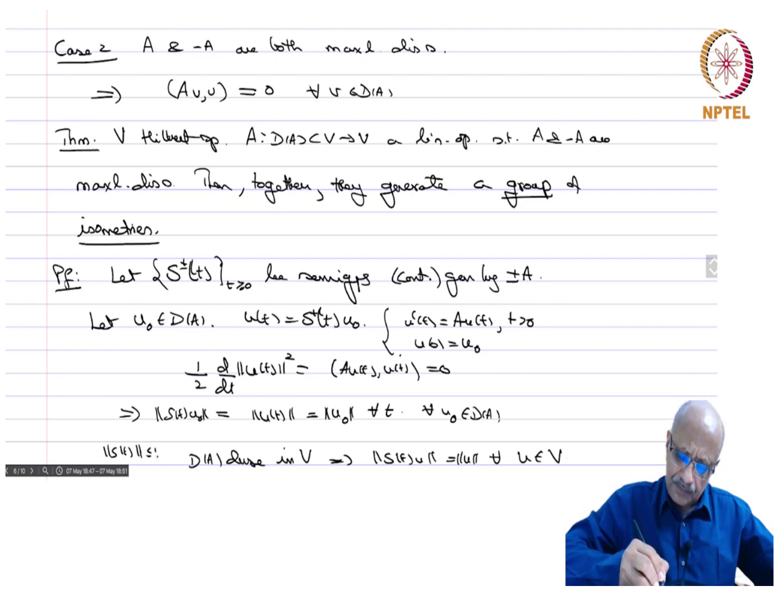

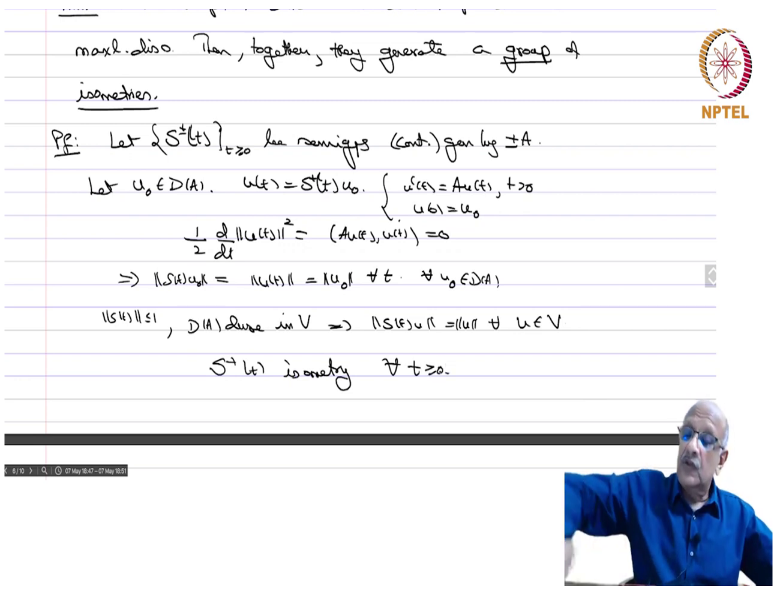

So, now I want to go to so, now we want to look at the case when so, case 2 A and minus A are both maximum dissipative. So, in this case you have $(Av, v) \geq 0$ and $(Av, v) \leq 0$ should also be less than or equal to 0. Because my both A and minus A are dissipative that means this $(Av, v) = 0, \; \forall v \in D(A)$. So, now we have a theorem. So, this is a different kind of theorem.

Theorem: V Hilbert space real of course $A: D(A) \subset V \to V$ a linear operator. Such that A and $-A$ are maximal dissipative. Then together they generate a **group** of isometries. So,

Proof: so the norm of the solution will not change at all. So, that is why you have an isometry. So, let us let $\{S^{+,-}(t)\}_{t\geq 0}$ be the semigroups contraction of course generated by plus or minus A. So, let $u_0 \in D(A)$ and $u(t) = S^{+}(t)u_0$.

So, then you have that

$$u'(t) = Au(t)$$

$$u(o) = u_0.$$

This is the solution to this equation. Now, norm of so, you take the scalar product of this equation with u of t and then you get one half of d by dt norm u(t) square we have seen this before A of u(t) u(t) and that is 0 and this means that norm of u t equals norm of u 0 for all t.

Now, this is true for all u naught $D(A)$ but $D(A)$ is dense in V and therefore this implies that norm u(t). So, but what is norm of S(t) of u naught is this and this implies that norm of S t of u equal to norm u for all u in V. So, the norm of S(t) is less than or equal to 1. And $D(A)$ is dense V and therefore, you have this. And therefore, S plus t isometry for all t greater or equal to 0 for t equals 0 it is just identity map.

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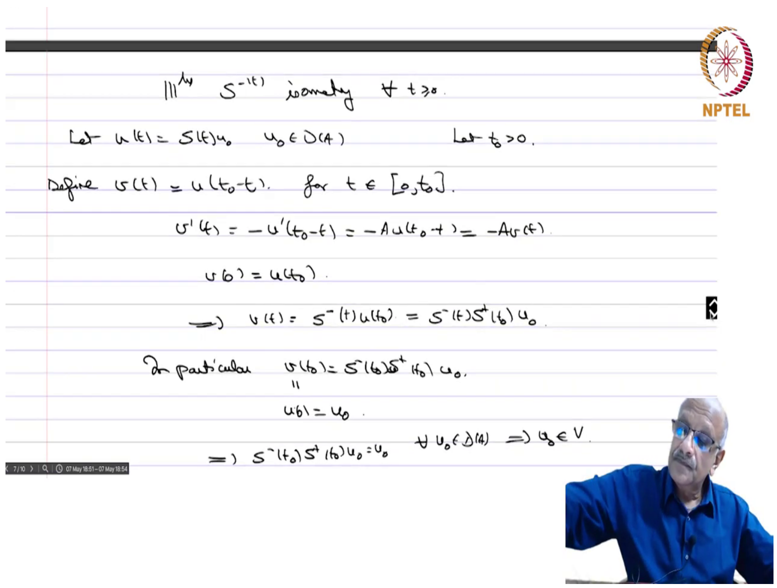

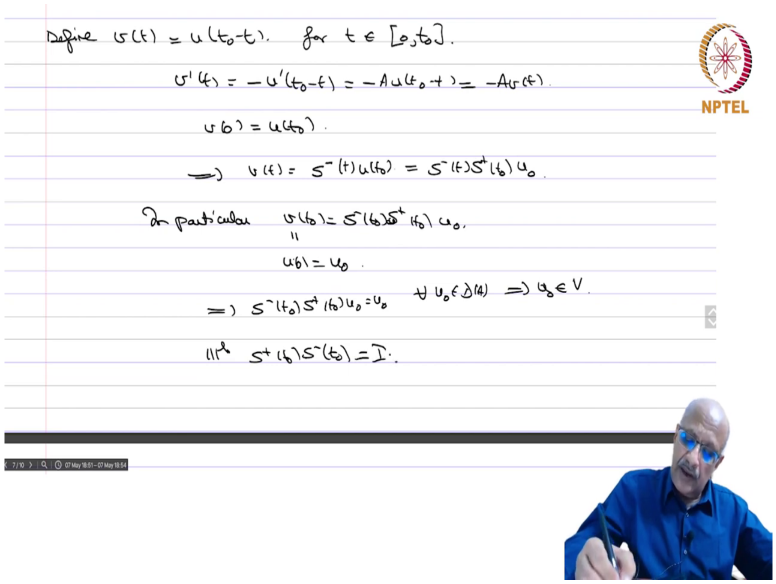

So, in the same way similarly, $S^{-}(t)$ is the isometry for all $t \geq 0$. So, now let u t equals S(t) of u naught. u naught in $D(A)$ and you define v(t) equals u($t_0 - t$). For t belonging to 0,t_0 t naught is some fixed. So, let t naught that is equal to 0. So, then you do this and then what do you get you have that v dash t if you differentiate this is equal to minus u dash t naught means t which is equal to minus A of u of t naught minus t by definition. And therefore, this is equal to minus A of v(t). So, v dash t equals minus A of v(t). v(0) is equal to what? v(0) is equal to u of t naught. This implies that v of t equals S(t) S minus t of u of t naught. And that is equal to S minus t of S plus t a t naught of u 0.

So, in particular $v(0) = u(t_0) = S^{+}(t_0)u_0$ but what is v of t naught v of t naught from the definition is nothing but u of 0. So, this implies that

$$S^{-}(t_0)S^{+}(t_0)u_0 = v(t_0) = u(0) = u_0.$$

When all u naught in $D(A)$ implies by density again for all $u_0 \in V$. Similarly, $S^{-}(t\ )S^{+}(t\ )u = u$ to the identity map.

Therefore, if u define. So, define

$$S(t) = S^{+}(t), \quad t \geq 0,$$

$S^{-}(t), \quad t < 0.$

Then you have $S(t\ )^{-1} = S(-\ t),\ \ t \in \mathbb{R}$. And so, we have a group of isometries. So, in this case we can solve the equations backwards and forwards. Because you have a group of isometries and also you have a conservation property namely the norm of the initial value initial data is preserved throughout the flow. So, these are the important properties of this. So, we have these two particular cases. And now, our next aim is to give examples from partial differential equations of these situations namely the self-adjoint and the case where you have a group of isometries.