Sobolev Spaces and Partial Differential Equations Differential Equations
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Distribution with compact support singular – Part 1

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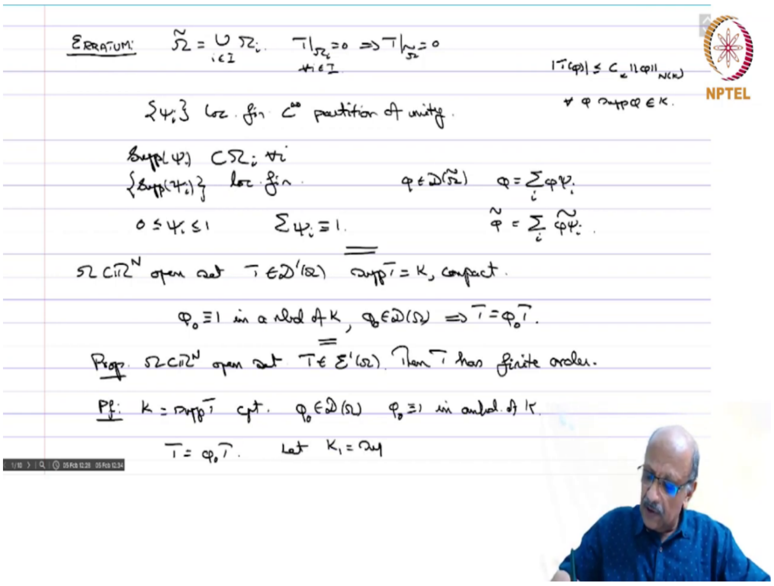

We continue our study of distributions and distributions of supports of distributions and distributions with compact support. Before that one ERRATUM. In the previous lecture, I was trying to prove the following: $\widetilde{\Omega} = \bigcup_{i \in I} \Omega_i$, $T|_{\Omega_i} = 0$, then $T_{\widetilde{\Omega}} = 0$. This is what I was trying to prove.

And in the process I used a partition of unity: $\{\psi_i\}_i$ −locally finite C^{∞} − partition of unity. So, in the subsequent things I made some confusion in the notation, namely, sometimes they use phi i, sometimes they use psi i. So, what we meant is that: $\mathrm{supp}(\psi_i) \subset \Omega_i,\ \forall i$. $\mathrm{supp}(\psi_i)$ is locally finite and $0 \leq \psi_i \leq 1,\ \forall i \in I,\ \sum_{i \in I} \psi_i = 1$.

So, some were written in terms of phi, sometimes in terms of psi and so on.

And if you took $\phi \in D(\widetilde{\Omega})$, then you have $\phi = \sum_{i \in I} \phi\psi_i$. So, some of these were written in terms of phi i, instead of psi i, so please correct it, sorry for the confusion.

So, now we continue with function, with distributions with compact support. So, if $\Omega \subset \mathbb{R}^N$ open set, $T \in D'(\Omega)$, supp(T)=K compact. Then we saw that if $\phi_0 = 1$ in a nbd. of K and $\phi_0 \in D(\Omega) \Rightarrow T = \phi_0 T$.

And we saw that distributions with compact support are precisely E prime of omega. So, namely the dual of the space of C infinity functions. So, these are what we have already seen.

So, now, we have the following proposition.

Proposition: Let $\Omega \subset \mathbb{R}^N$ open set, $T \in E'(\Omega)$. Then T has finite order.

proof: Recall the order of a distribution. If you have a distribution phi T, then mod T phi is less than equal to Ck times norm of k for all phi such that support a phi is contained in the fixed complex at k. And if N of k were the independent of this compact set then we say that it is that integer n which works for all compact sets is called the order of the distribution. So, and then we say T is a distribution of finite products. So, here we want to show that every distribution with compact support has in fact is in fact a distribution of finite order.

Let $\phi_0 \in D(\Omega)$, supp(T)=K compact, $\phi_0 = 1$ in a nbd. of K, $T = \phi_0 T$.

Let K_1 be the support to ϕ_0.

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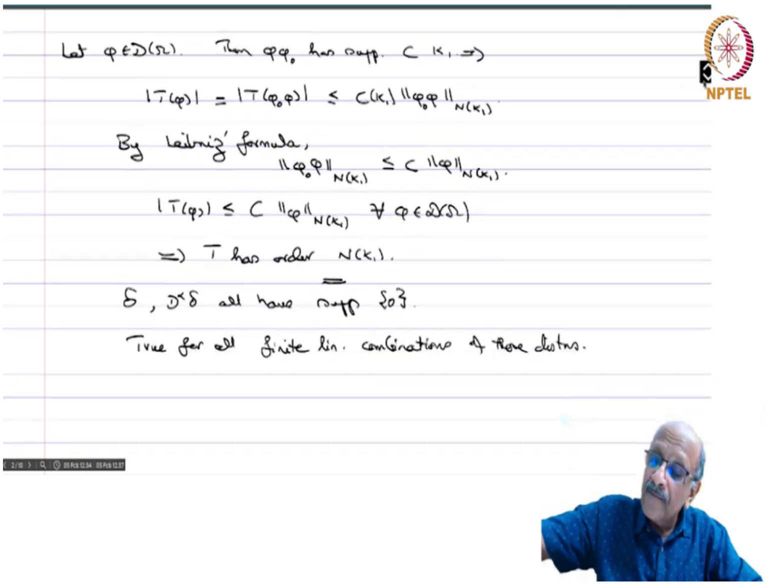

Now let $\phi \in D(\Omega)$, then $\phi\phi_0$ has supp. compact $\subset K$, and so, so this implies that

$$|T(\phi)| \leq |T(\phi_0\phi)| \leq C(K_1)||\phi_0\phi||_{N(K_1)}.$$

But my Leibniz's formula we have that

$$||\phi_0\phi||_{N(K_1)} \leq C||\phi||_{N(K_1)}$$

Because, what is the Leibniz formula? It just says that this derivative is all the derivatives of phi naught phi and linear combinations of derivatives of phi naught and derivatives of phi. Derivatives of phi naught are all uniformly bounded, so they can all be observed in the constant here, and therefore, you have just derivatives of phi which have to be estimated, and so, you get this.

Therefore, you have that

$$|T(\phi)| \leq C||\phi||_{N(K_1)}, \forall\phi \in D(\Omega).$$

So, this implies that T has order $N(K_1)$. So, this is every distribution in finite order. Now, we saw that the Dirac distribution δ and $D^{\alpha}\delta$ all for delta all have support singleton $\{0\}$. And so, true for all finite linear combinations of these distributions.

Now, we show that the converse is also true. Namely, if you have a distribution with is supported only at the origin, then it has to be a linear combination of Dirac and its derivatives.

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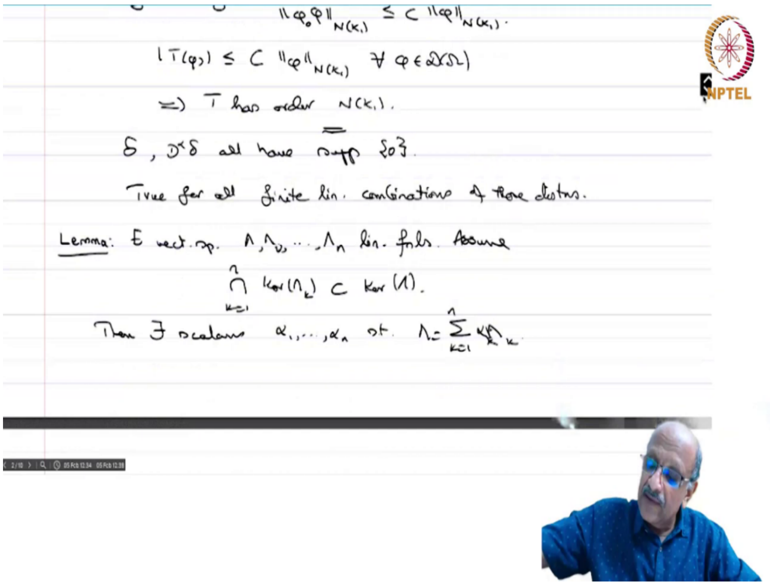

So, before we prove that we prove a very beautiful you might have seen it when you did the Find a find function scores. So, here is a Lemma.

Lemma: So, E vector space and Λ, Λ_1......, Λ_n linear functionals. So, assume

$$\bigcap_{k=1}^{n} ker(\Lambda_k) \subset \Lambda.$$

Then there exists scalars α_1, α_2,...., α_n such that $\Lambda = \sum_{k=1}^{n} \alpha_k \Lambda_k$.

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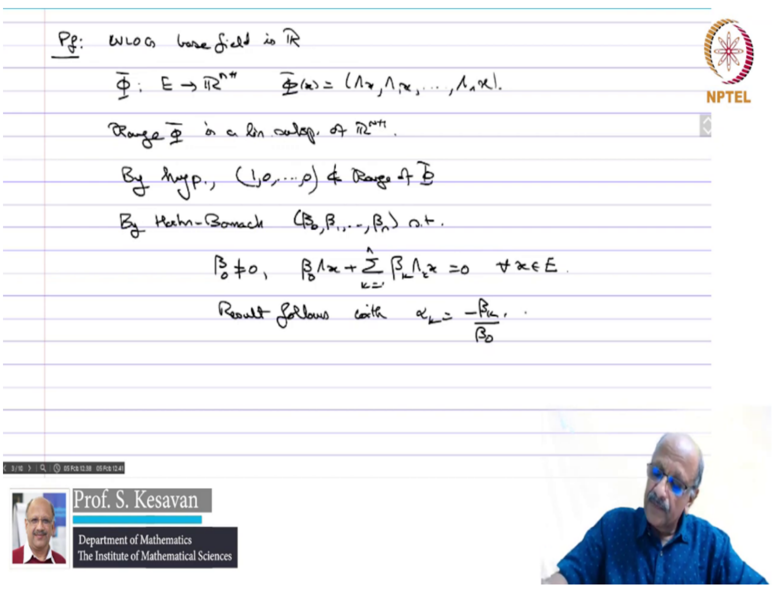

proof: So, we will of course, know without loss of generality, we will assume, the base field is R. We are throughout doing this, so. So, now you consider the map phi from E to Rn plus 1, so the phi of x is equal to lambda_x lambda 1x\ lambda nx. So, this, so range of phi is a linear subspace of Rn plus 1. And by hypothesis 1, 0, 0 does not belong to the range of phi. Why? Because if for some x you have 1, 0, 0 equal to this, then lambda 1x to lambda nx is equal to 0, so, x belongs to the intersection of the kernels and therefore, lambda of x must also be 0, but we are given that lambda of x equal to 1, so 1 is not there.

Therefore, by Hahn–Banach there is a regressor linear functional on Rn plus 1. But what is a linear functional on Rn plus 1, it is nothing but n plus 1 scalars, such that, its action on 1, 0, 0 namely beta naught is not equal to 0 and sigma action on the range beta naught lambda x plus sigma k equals 1 to n, beta k lambda k x equal to 0 for all x in E, so, this is nothing but a statement of Hahn–Banach theorem.

So, now, because beta naught is equal to 0, so, result follows with alpha i, k equals minus beta k by beta naught. So, that proves this lemma. So, we now use this lemma to prove the following proposition theorem.

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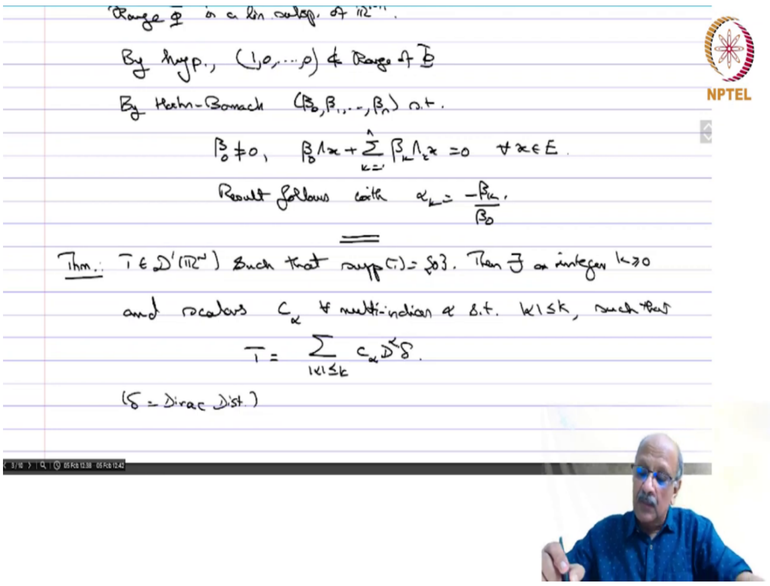

So, theorem

Theorem: $T \in D'(\mathbb{R}^N)$., such that supp(T)={0}. Then there exists an integer $k \geq 0$ and scalars c_α for all multi-index α s.t. $|\alpha| \leq k$, such that

$$T = \sum_{|\alpha| \leq k} c_{\alpha} D^{\alpha}\delta.$$

where delta is the Dirac Distribution.

So, we saw that linear combinations of the Dirac and its derivatives have support 0. Now, we are proving the converse, namely, if you have support equal to 0 single-times 0, then it must be only a linear combination of the Dirac and its derivatives.

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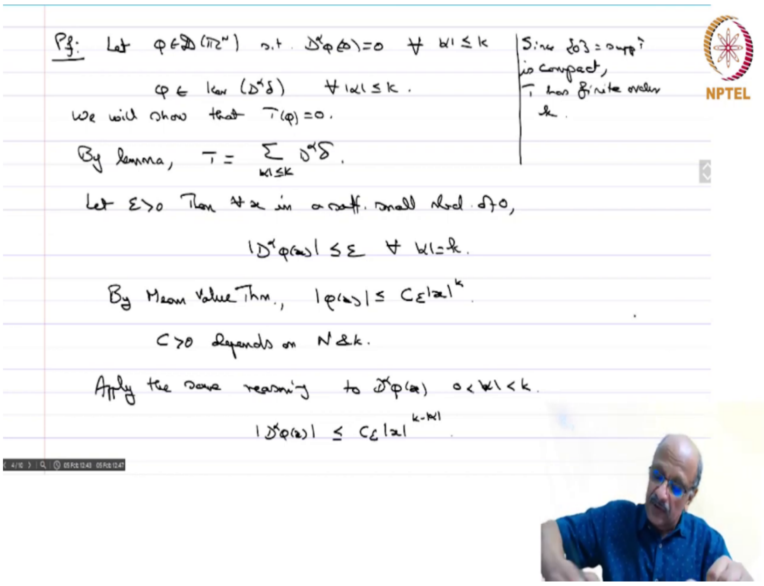

proof. So, we let $\phi \in D(\mathbb{R}^{N})$, s.t. $D^{\alpha}\phi = 0, \forall |\alpha| \leq k.$

That means that phi belongs to the kernel of D alpha delta for all mod alpha less than equal to k. So, we will show that T phi equal to 0. So, then by lemma

$$T = \sum_{|\alpha| \leq k} D^{\alpha}\delta.$$

Let $\epsilon > 0$. Then, for all x in a sufficiently small neighborhood of 0, we have

$$|D^{\alpha}\phi(x)| \leq \epsilon, \forall |\alpha| = k.$$

So, before. So, the first, so what is k? So, before we start the proof we say since singleton 0 equals support T is compact T has finite order k, so then, we will show the following things. So, this is, so k is the order of the distribution.

So, by the mean value theorem, $|\phi(x)| \leq C\epsilon|x|^{k}$, C>0 depends on N & k —------------ (1)

Now, apply the same, apply the same reasoning to $D^{\alpha}\phi(x)$:

$$|D^{\alpha}\phi(x)| \leq C\epsilon|x|^{k-|\alpha|} \quad \text{—------------ (2)}$$

I am writing D alpha phi x, as an expansion in the neighborhood of the origin, all derivatives up to k minus alpha of this function, namely, all derivatives up to order k of the original function are all 0, and therefore, by the mean value theorem once again you have this.

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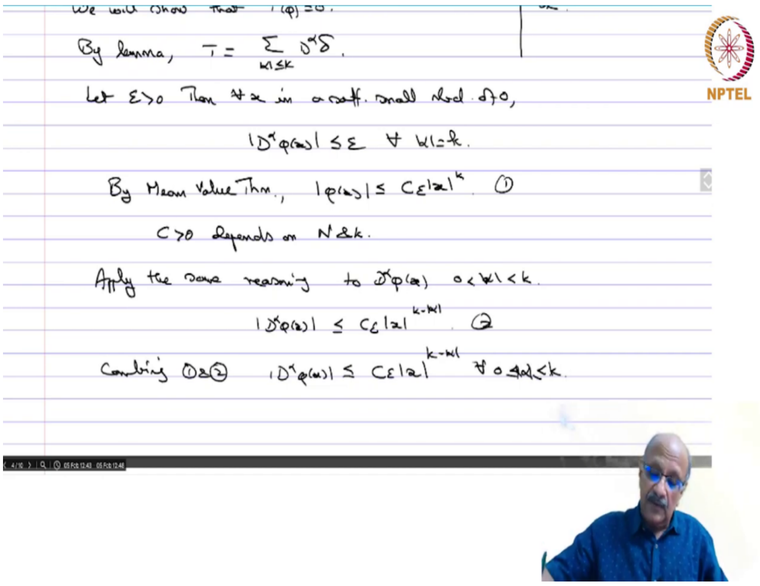

Therefore, combining these two we have that. So, combining (1) and (2) we have that

$$|D^{\alpha}\phi(x)| \leq C\epsilon|x|^{k-|\alpha|}, \forall 0 \leq |\alpha| \leq k.$$

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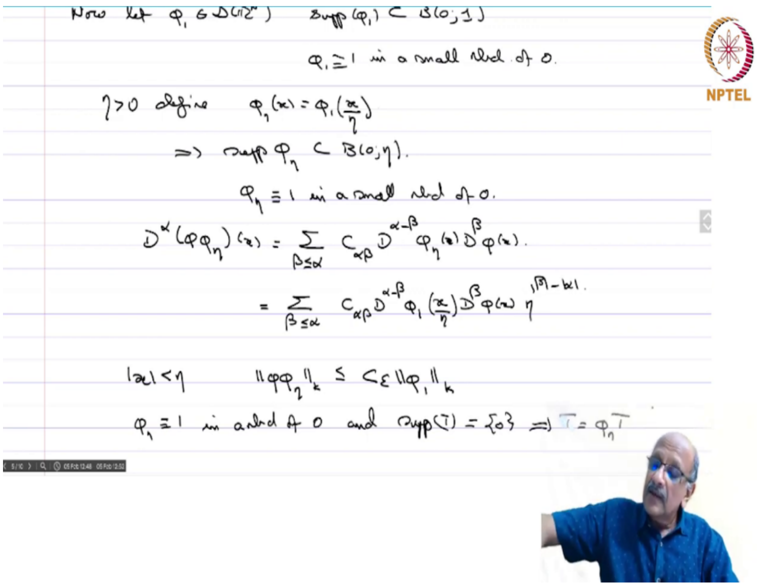

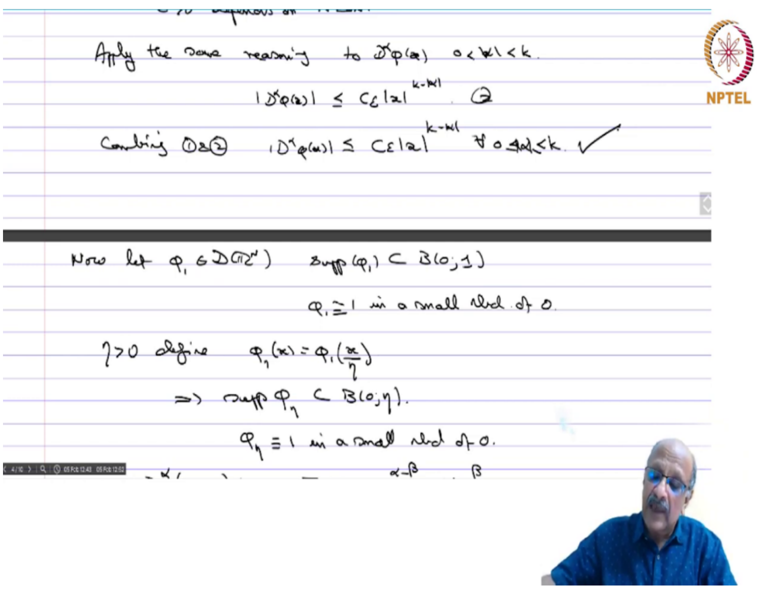

So, now, let phi 1 belong to D of Rn such that support of phi 1 is contained in the ball center origin and radius 1 and phi 1 identically equal to 1 in a small neighborhood of the orgin. Again this is the cut-off function which we have. Then for eta positive definite, phi eta of x equals phi 1 of x over eta. Then support of phi eta will be contained in the ball center origin and radius, and you have phi eta will still be equal to 1 in a small neighborhood of the origin.

Now, you take the D of alpha Phi phi eta of x is equal to sigma. I am writing their Leibniz formula. This is C alpha beta, these are the combinatorial coefficients alpha by alpha minus alpha factorial by alpha minus beta factorial times beta factorial and into D of alpha minus beta phi 1, phi eta of x times D beta of phi x, and phi eta is x by eta, and therefore, this will give you

$$D^{\alpha}(\phi\phi_0)(x) \ = \sum_{\beta\leq\alpha} C_{\alpha\beta}D^{\alpha-\beta}\phi_{\eta}(x)D^{\beta}\phi(x).$$

$$= \sum_{\beta\leq\alpha} C_{\alpha\beta}D^{\alpha-\beta}\phi_1(\tfrac{x}{\eta})D^{\beta}\phi(x)\eta^{|\beta|-|\alpha|}$$

So, if mod $|x| \ \leq \ \eta$, we have $||\phi\phi_{\eta}||_k \leq \ C\epsilon||\phi_1||_k$.

$\phi_{\eta} \ = \ 1$, in a nbd. of 0, and supp(T)={0} $\Rightarrow T \ = \phi_{\eta}T$.

and therefore, we have

$$|T(\phi)| \ = \ |T(\phi\phi_{\eta})| \ \leq \ C||\phi\phi_{\eta}||_k \leq \ C\epsilon||\phi_1||_k$$

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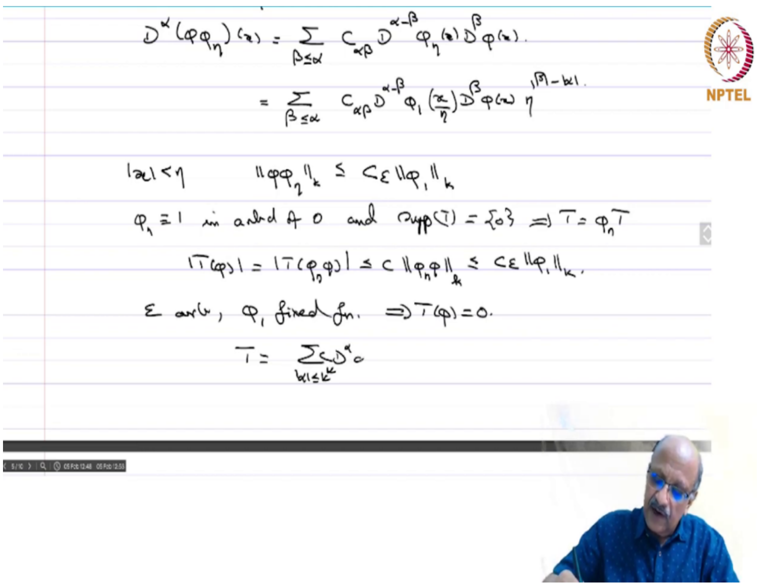

Now, epsilon is arbitrary, phi 1 is a fixed function and this implies that

$$T(\phi) \ = \ 0.$$

So, by lemma we have that

$$T \ = \sum_{|\alpha|\leq k} c_{\alpha}D^{\alpha}\delta.$$