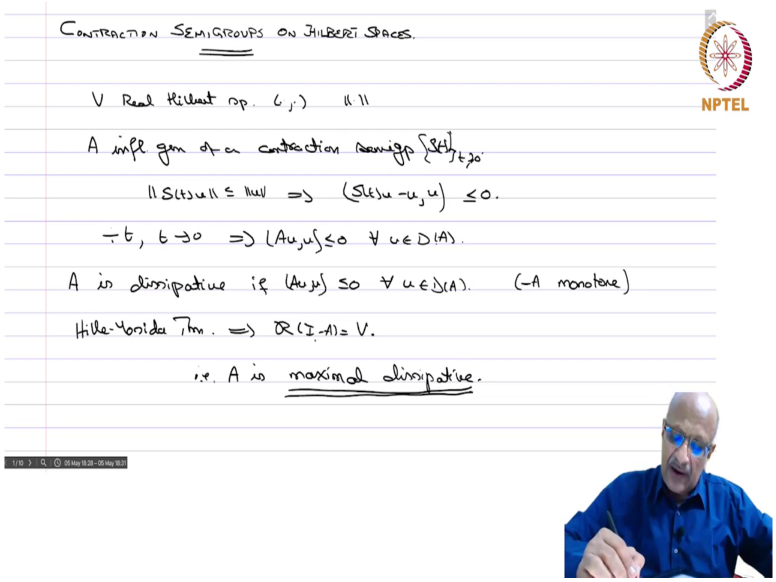
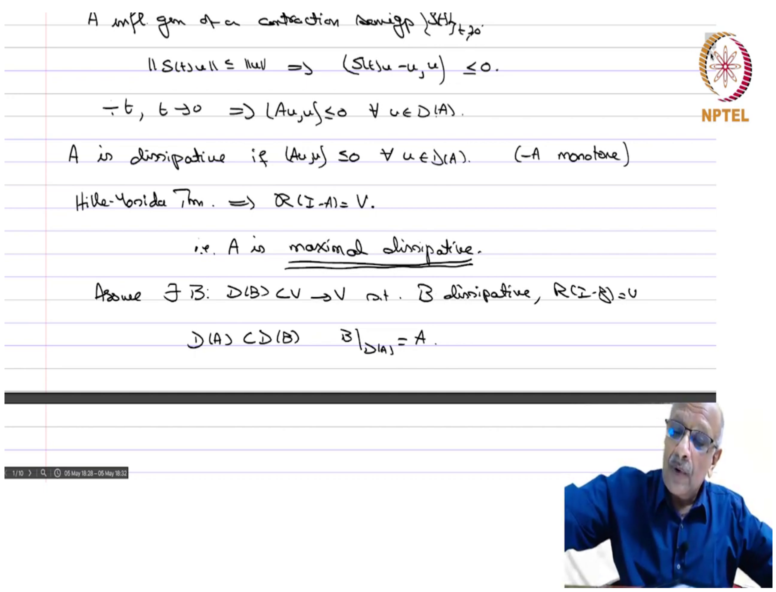
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Institute of Mathematical Science
Lecture 79
Contraction Semigroups on Hilbert Spaces

(Refer Slide Time: 0:17)





We know study **contraction semigroups on Hilbert spaces**. So, we saw the Hille Yosida theorem which characterizes the infinitesimal generator for contraction semigroup in a Banach space. Now, if you come to a Hilbert space it becomes even more present how to check these conditions. So, we assume that V is real Hilbert space. And the inner product will be denoted by this and the norm will be denoted by this.

And A infinitesimal generator of a contraction semigroup $\{S(t)\}$. So, $||S(t)u|| \leq ||u||$ and therefore, this implies that

$$(S(t)u - u, u) \leq 0.$$

So, the S(t) u inner product will by Cauchy Schwarz inequality be less than equal to norm u norm u square and therefore, this is less than or equal to 0. So, now if you will divide by t and t decreasing to 0.

So, this implies that

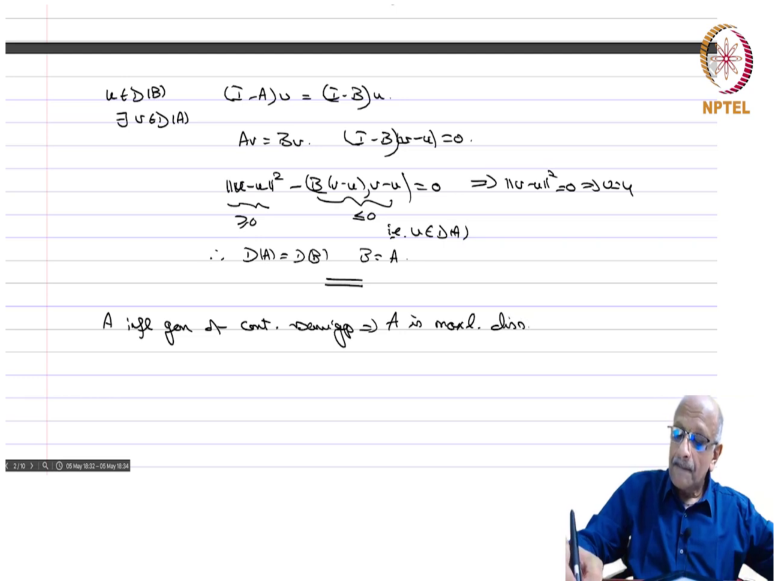
$$(Au, u) \leq 0, \ \forall u \in D(A).$$

Therefore, we say that A is dissipative if $(Au, u) \leq 0, \ \forall u \in D(A)$. So, we say minus A will be monotone that means the A u u will be greater $(-Au, u) \geq 0$. So, then we also know by the Hille Yosida theorem this implies that range of I minus A is equal to whole of v.

Because given any v you can invert $(I - A)$ and therefore, that is R of 1 and therefore, for every A range of I in fact lambda $I - A$ is equal to the range of that is equal to v for all v but anyway. So, then we say so, together these two that is A is maximal dissipate why it is called maximal dissipative that means it is dissipative and range of $I - A$ equal to v

Why is it called maximal? Because assume that there exists v from $B(D)$ contained in V to V such that B is dissipative range of $I - B = V$ and $D(A)$ contained in $D(B)$ and B restricted to $D(A)$ equal to A. So, assume that you have these conditions that means I am having a maximal dissipative operator I am extending it to another operator with the same properties.

(Refer Slide Time: 04:36)

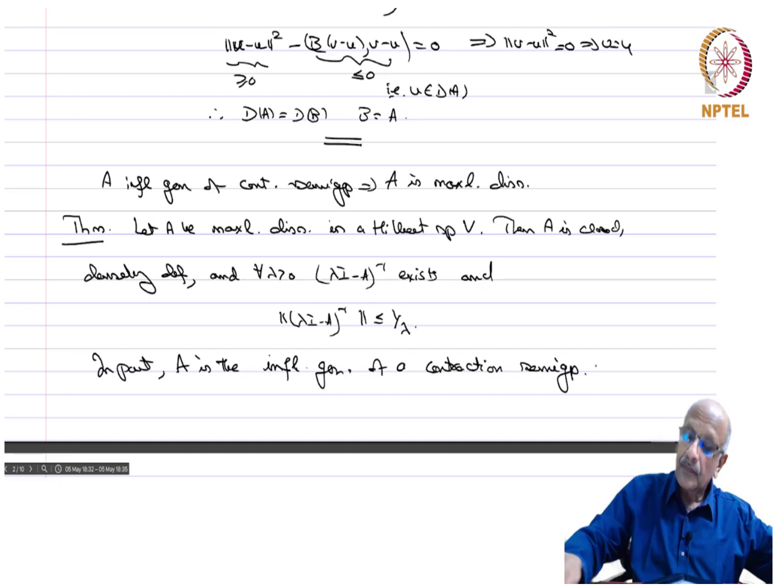


So, then if u belongs to $D(B)$ then I minus A so there exists v in $D(A)$ such that I minus A v equals I minus B u we have used this trick before when in the Hille Yosida theorem proof itself. When showing that the infinitesimal generator is in fact whatever we started with. So, then v is in $D(A)$ therefore, A v is the same as B v and therefore, I minus B v equal to I minus B v equal to I minus B u. So, I minus B v minus u equal to 0. Take the inner product with this you get norm of v minus u square minus B B minus u v minus u equal to 0. Now, this is greater or equal to 0 and this inner product is less than equal to 0.

So, with a minus sign the whole thing becomes greater equal to 0. So, this means norm of v minus u square equal to 0 that is v equal to u that is u belongs to $D(A)$. Therefore, $D(A) = D(B)$ and B equals A. So, that is why it is called maximal. So, if you have the dissipativity and the range is the whole space then automatically there is no possible extra extension with the same properties.

That is why it is called a maximal dissipative property. So, now so we have therefore so, A infinitesimal generator of contraction semigroup implies A is maximal dissipative. Now, we prove the converse.

(Refer Slide Time: 06:31)



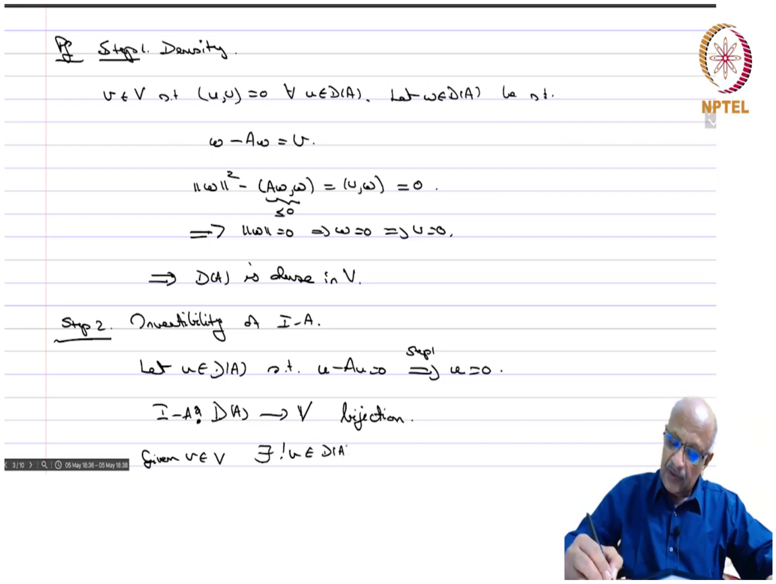
So,

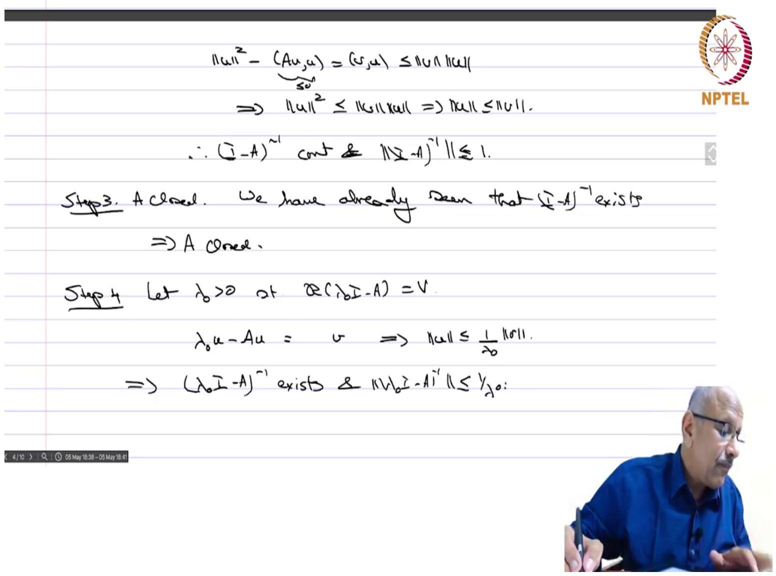
Theorem: let A be maximal dissipative in a Hilbert space real of course real Hilbert space always. Then A is closed densely defined and for every $\lambda > 0$, $(\lambda I - A)^{-1}$ exists. And

$$||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$$

in particular A is the infinitesimal generators of contraction semigroup. That is why the Hille Yosida theorem. So, it is now you see the conditions are very easy to check. So, you just have two conditions to check that (Av, v) is less than equal to 0 it is generally easy that is just a question of integration by parts and then you have check that $I - A$ has the range is whole V.

(Refer Slide Time: 7:49)





So,

Proof: step 1 we will improve the density. So, let $v \in V$ such that $(u, v) = 0$ for all $u \in D(A)$. So, let $w \in D(A)$ be such that $w - Aw = v$. This is possible because range of I -A is the whole of v. So, now you take the inner product with w. So,

$$||w||^2 - (Aw, w) = 0.$$

So, this again this is less than or equal to 0 given to you. So, that two non negative terms equal to 0 so, this implies that norm w equal to 0 implies w equal to 0 implies v equal to 0. And therefore this implies that $D(A)$ is then dense in V the han Banach theorem.

Step 2: invertibility of $I - A$.

So, let $u \in D(A)$ such that $u - Au = 0$. Then the same thing we just saw in step 1. So, this is the same argument is in step 1 this means that u equal to 0. So, this is so, I minus A is a bijection from $D(A)$ to whole of v bijection because it is on to and so there exists so given v in v there exists a unique u such in $D(A)$ such that u minus Au equal to v.

So, therefore, you have norm u square minus A u u is equal to v u less than equal to norm v times norm u. And therefore, this implies that norm u square this is of course less than equal to 0 so, with a minus is where is less than or equal to norm v norm u implies norm u is less than equal to norm v. Therefore, $(I - A)^{-1}$ is continuous and $||(I - A)^{-1}|| \leq 1$.

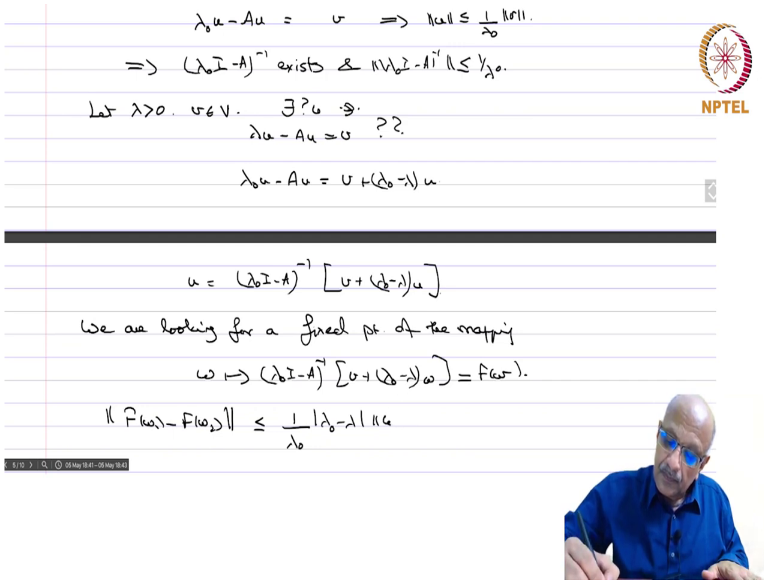
So, that proves the invertibility of $(I - A)$ then

Step 3 A closed. So, we are we have already seen that $(I - A)^{-1}$ exists implies A closed. So, we already have observed this fact So, that need not prove it again. So,

Step 4 let $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = V$. In particular, we have this for $\lambda_0 = 1$. So, let us assume that it is the thing.

So, then again you have for any v you have λ_0 u minus Au equal to v and therefore, this implies that norm u by the same argument take inner product with (Au, u) is negative and therefore, this is less than or equal to 1 by lambda naught norm v. So, this implies and also that if v 0 then the u has to be 0. So, this implies that $\lambda_0 I - A$ inverse exists and norm $(\lambda_0 I - A)$ inverse is less than or equal to 1 by λ_0.

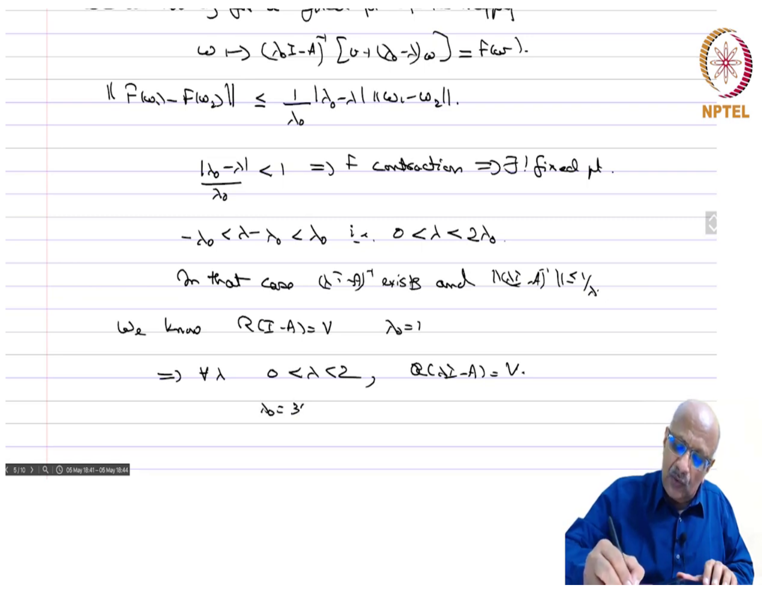
(Refer Slide Time: 12:49)



So, let now $\lambda \geq 0$ and v in V and we want to solve so, does there exist u such that lambda u minus u A u equal to v. So, let me rewrite this as lambda naught u minus A u equals v minus v plus lambda naught minus lambda u. So, then u is equal to $(\lambda_0 I - A)$ inverse of v plus λ_0 minus lambda u. So, we are looking for a fixed point of the mapping w going to lambda naught $(I - A)$ inverse v plus λ_0 minus lambda w. So, if we have a fixed-point u that is precisely the solution of this.

And then therefore, we can solve it. So, now, call this F of u F of w and therefore, F of w 1 minus F of w 2 in norm is less than or equal to so, we have to subtract the v gets cancelled out and $\lambda_0 I - A$ is 1 over lambda naught into mod lambda naught minus lambda times norm w sorry w 1 minus w 2.

(Refer Slide Time: 15:03)



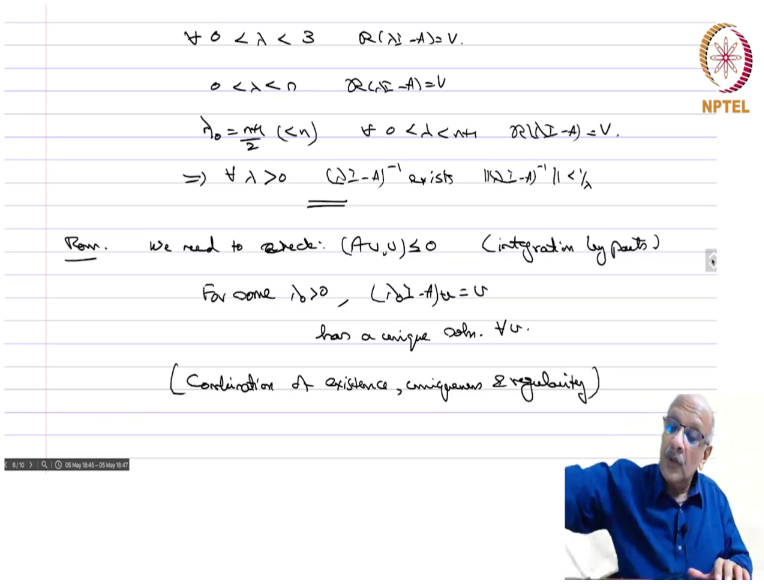
So, then if more $\frac{|\lambda_0 - \lambda|}{\lambda_0} < 1 \Rightarrow$ F is contraction in the sense of metric spaces implies there exists unique fixed point. So, when does this happen? So, this means that minus λ_0 less than or equal to lambda minus λ_0 less than λ_0 and that is we have the 0 less than lambda less than 2 λ_0.

And of course, in that case I $(\lambda\ I\ -\ A)^{-1}$ exists as usual and

$$||(\lambda\ I\ -\ A)^{-1}|| \leq \frac{1}{\lambda}$$

all this we have seen already in terms in the case of λ_0 and it is no different for any other number. So, now we know $R(\lambda\ I\ -\ A)$ equal to B that is λ_0 equal to 1 implies for all lambda such that 0 less than lambda less than 2 we have range of $(\lambda\ I\ -\ A)$ equal to v. Now, you take 3 by 2. So, take λ_0 equal to 3 by 2 which is in this situation and apply the previous step.

(Refer Slide Time: 16:39)



And therefore, for all $0 < \lambda < 3$, we have $R(\lambda\ I - A) = V$. So, in general if you have it for any n, 0 less than lambda less than n you have a $R(\lambda\ I - A) = V$. Then you can take n plus 1 by 2 and therefore, you have take lambda naught equal to n plus 1 by 2 which is less than n of course because 2 n minus n is n n is greater than 1.

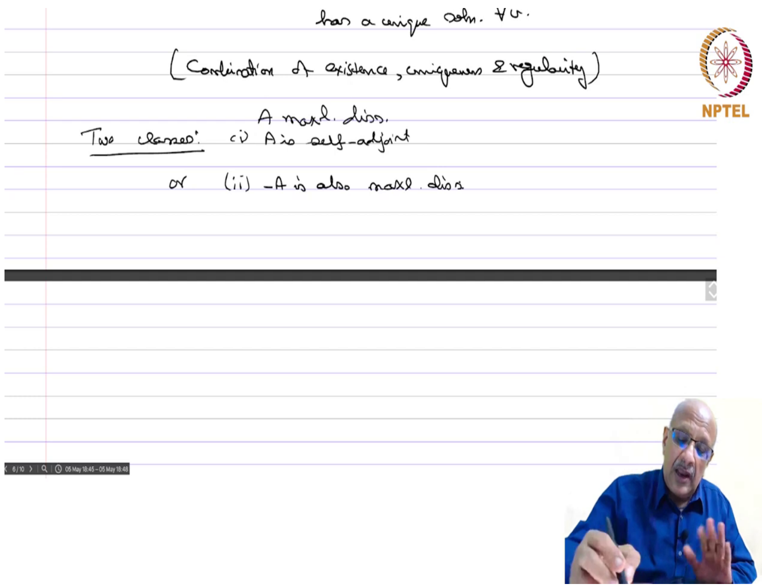
And therefore, this implies that for all 0 less than lambda less than n plus 1 we have $R(\lambda\ I - A)$. So, this implies $\lambda \in \mathbb{R}, \lambda > 0$ for every lambda positive we have $(\lambda\ I - A)^{-1}$ exists and $||(\lambda\ I - A)^{-1}|| \leq \frac{1}{\lambda}$. Because $R(\lambda\ I - A) = V$. So, that therefore, we have proved this.

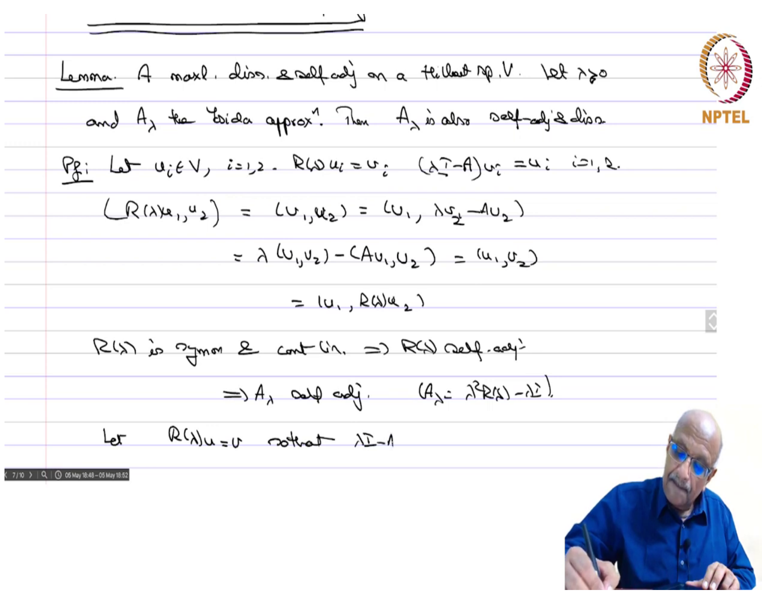
So, this is the proof of the theorem. So,

Remark. So, the importance of this theorem comes from the fact we have to only check two things. So, we need to check if $(Av, v) \leq 0$ this is usually integration by parts normally done that way. And for some $\lambda_0 > 0$ we want to show that $(\lambda_0 I - A)v = u$ has a unique solution for every v. So, this is a combination of existence uniqueness and regularity theorem.

So, this is a single equation which we have received which is the type we will see in the application this of the type which we have seen in the previous chapter the elliptic equations and so, on. And therefore, we have you need to check only for that and then it becomes obvious that you have that it becomes infinitesimal generator of a C_0 semigroup.

(Refer Slide Time: 19:31)





So, the two important classes. one so, A maximal dissipative then 1 A is self adjoint or two is minus A is also maximal dissipative. So, there are two important classes of maximal dissipative operators which we like to study because they are very nice properties. And one is that A is self adjoint and the other is when minus A also maximal dissipative that means A and minus A are maximal dissipative.

So, we will take the case A is maximal dissipative and self-adjoint. So, this is the case which we would like to consider now. So, we start with

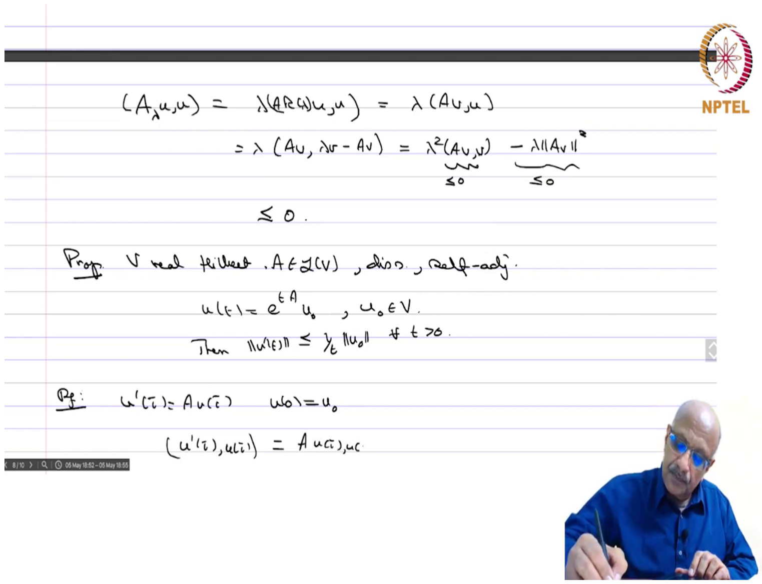
Lemma so, A is maximal dissipative and self-adjoint on a Hilbert space let $\lambda \geq 0$ and A_λ the Yosida approximation. Then A_λ is also self-adjoint and dissipative. So,

Proof: let $u_i \in V,\ i = 1, 2$ and let $R(\lambda)(u_i) = v_i$.

That means lambda I minus Av_i equal to u_i i equals 1, 2. Now, you take R lambda u_1, u_2 want this is equal to $R(\lambda)(u_i) = v_i$. So, this is $v_1 v_2$ which is equal to $v_1\ u_2$ sorry but u_2 is what lambda I minus A v_2. So, this is v_1 into lambda v_1 minus lambda v_2 minus Av_2. So, that is equal to lambda $v_1\ v_2$ minus by self adjointness A$v_1\ v_2$ which is u 1 v_2 which is equal to u 1 $R(\lambda)$ u.

So, this says that $R(\lambda)$ is symmetric. And of course, it is continuous linear operator implies $R(\lambda)$ is self-adjoint. This implies that A lambda is also self-adjoint because A lambda is nothing but lambda square $R(\lambda)$ minus lambda I. So, this therefore this no need it is definitely self-adjoint operator. So, now, let $R(\lambda)$ u equal to v. So, that again lambda I minus A v is same as u.

(Refer Slide Time: 24:20)



Now, A lambda u u equals lambda A $R(\lambda)$ u u $R(\lambda)$ u is equal to v. So, this is lambda Av u which equal to lambda Av. And then what is u? lambda u minus lambda v minus Av and that is equal to lambda square Av v minus lambda norm Av squared. Now, $(Av,\ v) \leq 0$ is dissipative And this is also less than equal to 0. So, this is also less than or equal to 0. So, this proves this lemma.

So, this one more result before with which we want to conclude. So, proposition which we will use in our calculation in our in the main theorem next time. So,

Proposition: V Hilbert real Hilbert $A \in L(V)$ A dissipative and self-adjoint. So, it is everything A lambda satisfies all these conditions we will ultimately apply it to that. So,

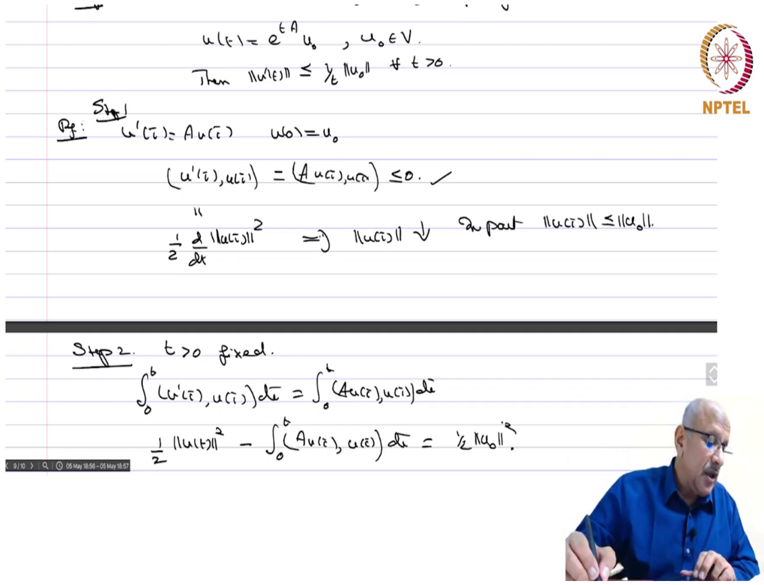
$u(t) = e^{tAu_0},\ u_0 \in V.$

So, then $||u^{'}(t)|| \leq \frac{1}{t}||u_0||,\ \ t > 0.$

So, this we have an estimate for the norm of the derivative. So,

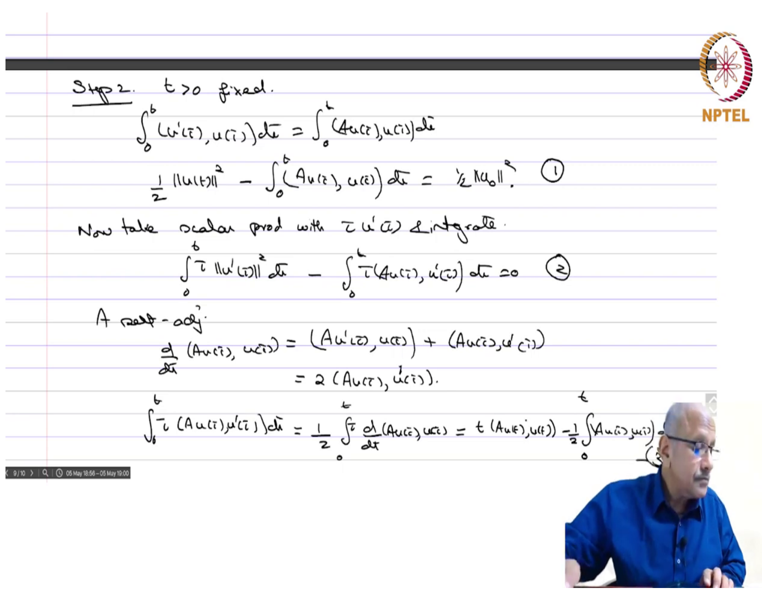
Proof so, you have $u^{'}(\tau) = Au(\tau)$ and $u(0) = u_0$ that is the definition because e power t Au_0 is precisely the solution of this equation here. So, if I now take the inner product. So, you dash τ uτ is equal to A of uτ uτ and that is less than or equal to 0. Because you have dissipativity.

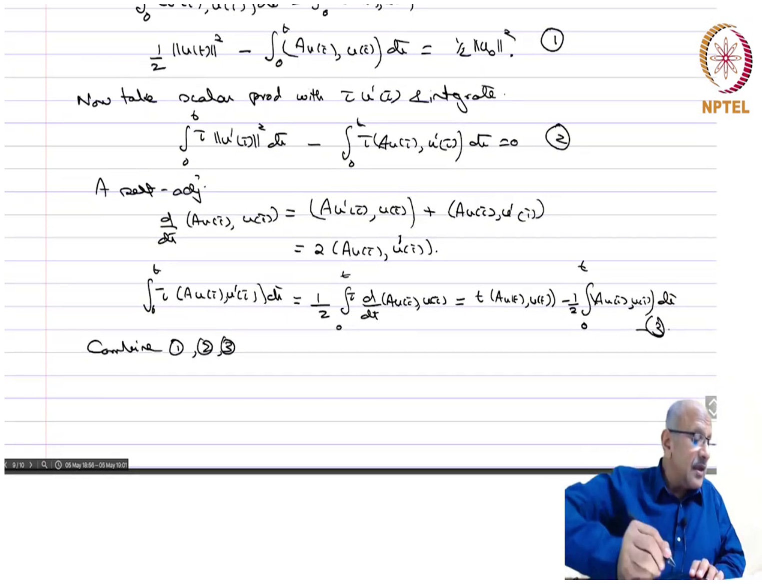
(Refer Slide Time: 27:17)

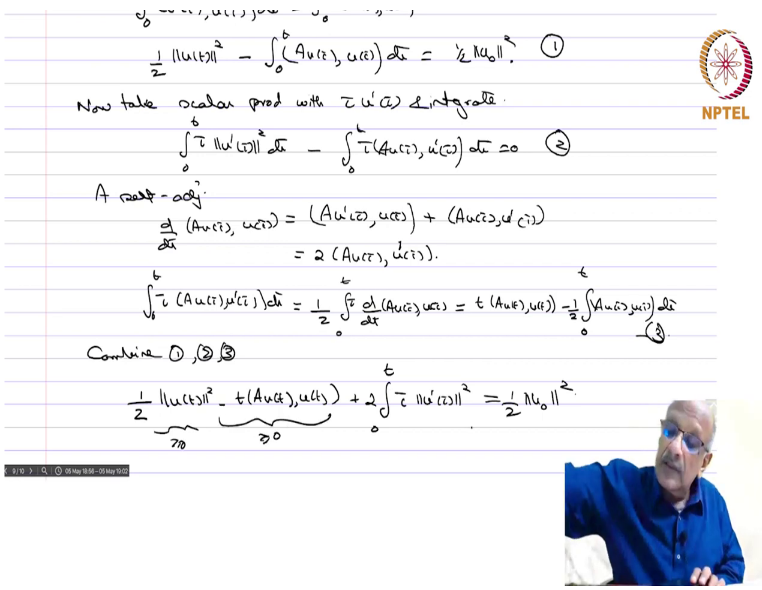


But this is equal to minus one half just one-half d by dt of norm u of tau squared. Therefore, this means that norm of u tau is decreasing as tau increases because its derivative is less than or equal to 0. So, in particular norm of u tau is always less than or equal to norm of u_0. So, call this step 1.

(Refer Slide Time: 28:08)







So, now

step 2. So, let t be positive fixed. So,

integral 0 to t u dash tau u tau d tau equals integral 0 to Au tau u tau d tau. So, this I have just integrated this equation here over 0 to t. And therefore, this says this is minus d by dt of u tau square. So, you get norm u t square minus one half minus integral 0 to t Au tau u tau d tau equal to one half norm u 0 square. So, I have just taken it to the other side brought this to the side and taken minus half u naught square to the other side. So, let us call this relationship as 1.

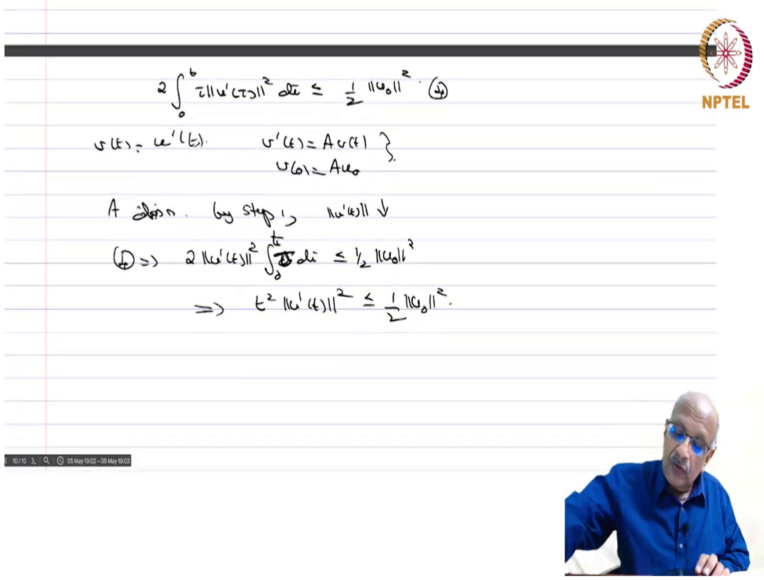
Now, take scalar product with tau u dash tau and integrate. So, then you get if you take the scalar product with this equation with tau u dash tau and integrate you get 0 to t tau norm u dash tau squared d tau minus integral 0 to t tau Au tau u dash tau d tau equal to 0 let me call this as 2. Now, A is self-adjoint we are going to use that we are used the dissipativity already we have to use the self-adjoint.

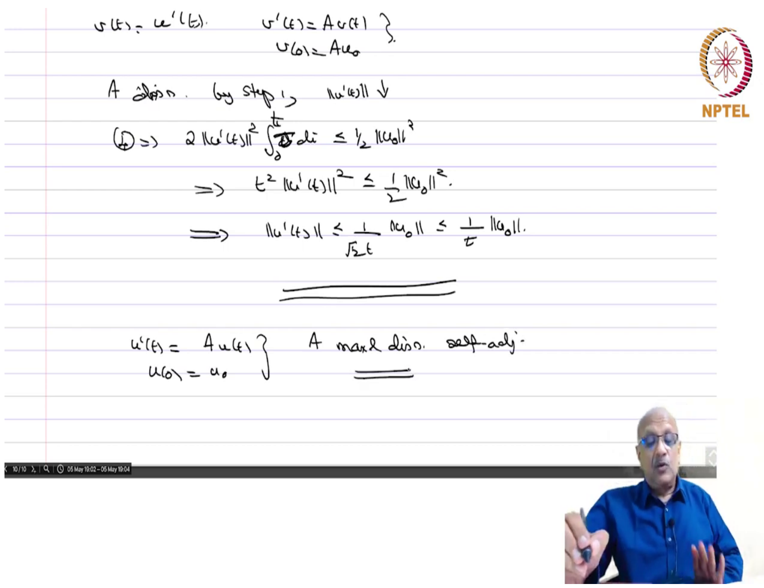
So, d by d tau of Au tau u tau if I use the product for product rule this is Au dash tau u tau plus Au tau u dash tau. Now, the A can go anywhere because it is self-adjoint. So, this is equal to two times Au tau u dash tau. So, again we have the integral 0 to t tau Au tau u dash tau d tau is equal to one half integral d by dt of Au tau u tau in the tau of course because I am just using this thing and that by integration by parts.

So, 0 to t so, when I integrate by parts I have two terms on the boundary the lower term with tau equals 0 will vanish. And therefore, you have t Au t u t minus one half integral 0 to t of then derivative with respect to tau is just 1 Au tau u tau d tau call this 3. So, now you combine 1, 2, and 3 so, 1, 2 and 3 if you combine so, 1 you have integral Au tau u tau. And here you have integral Au tau u tau equal to something else.

And then t Au tau u dash tau in terms of tau u dash tau. So, you have all these connections and therefore, you have if you combine all the three and you just check the algebra. So, you get half norm u t square minus t Au t u t plus 2 integral tau norm u dash tau square 0 to t is equal to one half norm u 0 square. Now, this is great equal to 0 minus something is also great equal to 0.

(Refer Slide Time: 33:30)





And therefore, from these two you get that

$$2\int_{0}^{t} \tau||u'(\tau)||^2 d\tau \leq \frac{1}{2}||u_0||^2$$

Now, if you take $v'(t) = u'(t)$, then what is this equation it says $v'(t) = Av(t)$. And then $v(0) = Au(0)$. Now, A is dissipative. So, by step 1

$$||u'(t)|| \leq ||u_0|| \text{ decreases.}$$

That is less important. So, now it decreases. So, if you tried this from this condition so, this is (4). So, (4) implies that

$$2||u'(t)||^2 \int_0^t \tau d\tau \leq \frac{1}{2}||u_0||^2$$

Now, this is nothing but

$$t^2||u'(t)||^2 \leq \frac{1}{2}||u_0||^2.$$

So, from this you get

$$||u'(t)|| \leq \frac{1}{\sqrt{2}t}||u_0|| .$$

The root 2 we do not bother about it and therefore this. So, we will remember this. So, next time we will look at the problem

$$u'(t) = Au(t)$$

$$u(0) = u_0.$$

And where A is maximal dissipative and self-adjoint. And then see what conclusions we can especially extra conclusions we can draw about this equation.