Sobolev Spaces and Partial Differential Equations
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Lecture 78
Regularity

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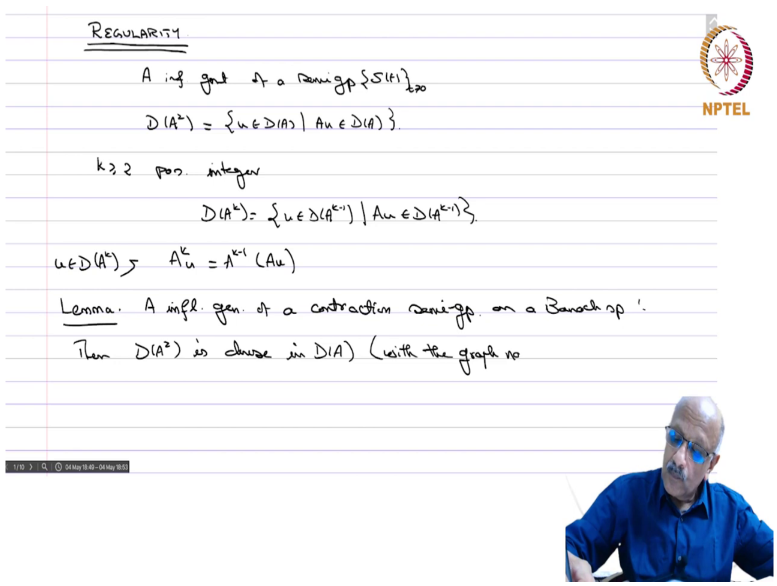

We will now talk about Regularity. Regularity means when the data is more the solution is smarter than what we normally expect it to be. So, if you look at A infinitesimal generator is $\{A(t)\}$ then in look at think of the 2 examples which we saw. We saw that the space was l ,2 and in case of u Au equals u dash the domain was h 1 0. And in case of Au equals Laplacian u the domain was h 2 intersections h 1 0.

So, the domain of the operator is normally a space of smoother functions than the o channel the space ambient space where we are working. Now, if we look at A applied to itself again A^2 that will be an under unbounded operator whose domain will be even smoother for instance if you have you dash if you apply it twice you get u double dash. So, you would need at least h 2 functions to make sense.

Similarly, if you have Laplacian and then you apply again Laplacian square then you get h 4 should be the space where these functions will be ultimately coming into l 2 and therefore, the higher the domain or power of the operator is the domain will become smoother and smoother functions. And therefore, if your the initial data belongs to those then you can expect the solution to be smooth also. So, that is the principle on which we are going to work today.

So, A is infinitesimal generators of a semigroup and so, we define

$$D(A^2) = \{u \in D(A): \ Au \in D(A)\}.$$

And if more generally if $k \geq 2$ is a positive integer. Then we define

$$D(A^k) = \{u \in D(A^{k-1}): \ A^{k-1}u \in D(A)\}.$$

and then we define $A^k(u) = A^{k-1}(Au)$. So, for $u \in D(A^k)$, $A^k(u) = A^{k-1}(Au)$ because it is in the domain of Au.

Au is in the domain and therefore, $A^{k-1}(u)$ and Au is well defined. So, this is how we define the higher order higher powers of the generator of an infinitesimal semi group. So, then we have the following

Lemma: A infinitesimal generator of a contraction semigroup on a Banach space. Then $D(A^2)$ is dense in $D(A)$ remember whenever we want $D(A)$ to be a independent Banach space so we need to put it with the graph norm. So, this is the lemma which we want to do

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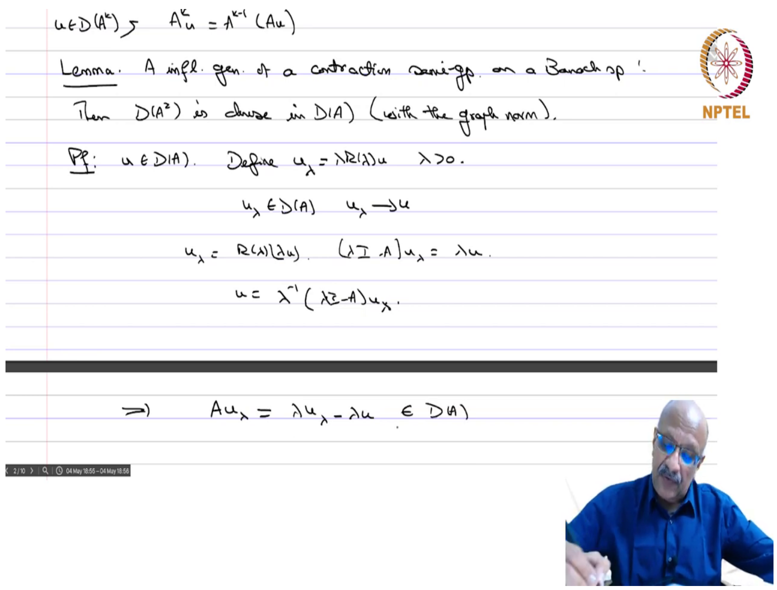

Proof. So, what do you want to show? You want to show that $u \in D(A)$. So, we want to produce a sequence or a set of approximations which are in $D(A^2)$. And such that they approximate u in the sense of the graph norm. So, let us define

$$u_\lambda = \lambda R(\lambda)u, \quad \lambda > 0.$$

Then $u_\lambda \in D(A)$ we know that because $R(\lambda)$ has its range in $D(A)$.

And further you know that $R(\lambda)u$ always converges to u. And therefore, u_λ converges to u. So, this is the one first lemma we proved.

Therefore, we know this so this we know. Now, what about so, u_λ is nothing but $R(\lambda)u_\lambda$. So, that means $(\lambda I - A)u_\lambda = \lambda u$. So,

$$u = \frac{1}{\lambda}(\lambda I - A)u_\lambda.$$

So, what is Au_λ? $Au_\lambda = \lambda u_\lambda - \lambda u$ from this equation here. And that we know belongs to $D(A)$ because u lambda belongs to $D(A)$ u also belongs to $D(A)$. So, the $Au_\lambda \in D(A)$.

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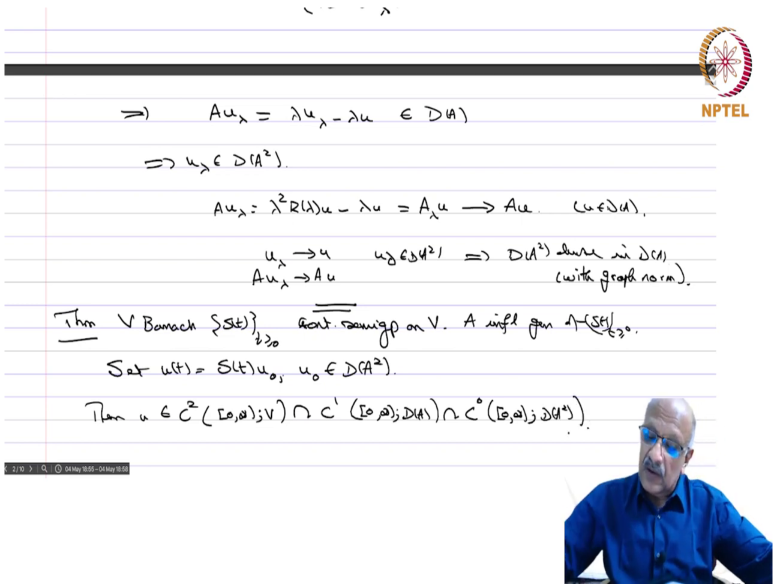

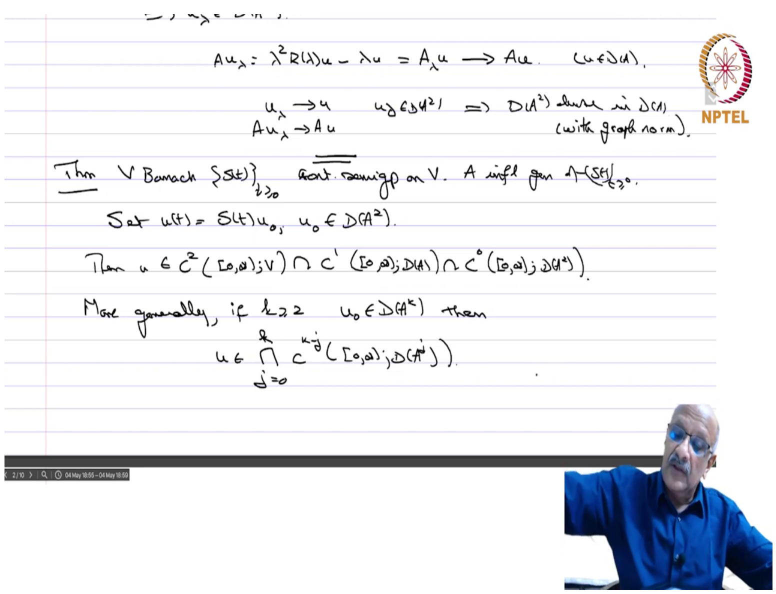

So, therefore, you have that u lambda belongs to $u_\lambda \in D(A^2)$ further $Au_\lambda = \lambda u_\lambda$ which is lambda square R lambda of u minus lambda of u from this equation u and the definition of lambda. But this is nothing but A lambda of u and A lambda of u of course converges to A of u we know because $u \in D(A)$. So, you have u lambda converges to u Au lambda converges to A of u and $u_\lambda \in D(A^2)$. And therefore, this implies $D(A^2)$ square dense in $D(A)$ with graph norm. So, that proves that lemma.

So, now we have a nice theorem. So,

Theorem: V Banach space $\{S(t)\}_{t\geq 0}$ contraction semigroup on V and A infinitesimal generator of $\{S(t)\}_{t\geq 0}$. Set $u(t) = S(t)u_0$, where $u_0 \in D(A^2)$. So, we are assuming now u_0 usually we can solve the differential equation if $u_0 \in D(A\ )$ we are now assuming further smoothness as I explained earlier we are assuming it is in $D(A^2)$.

So, we expect the solution to be smooth then u of belongs to C^2. So, it is usually it was in C^1 now it is in $u \in C^2([0,\infty): V) \cap C^1([0,\infty]: D(A)) \cap C([0,\infty]: D(A^2))$. So, the solution is very smooth I mean it is twice differentiable.

And the u_t itself is very smooth it belongs to $D(A^2)$. For more generally if k greater or equal to 2 it is a positive integer and $u_0 \in D(A^k)$. Then

$$u \in \bigcap_{j=0}^{k} C^{k-j}([0,\infty); D(A^j)).$$

So, this is the general theorem. So, we have a lot of smoothness in the case of the contraction semigroup.

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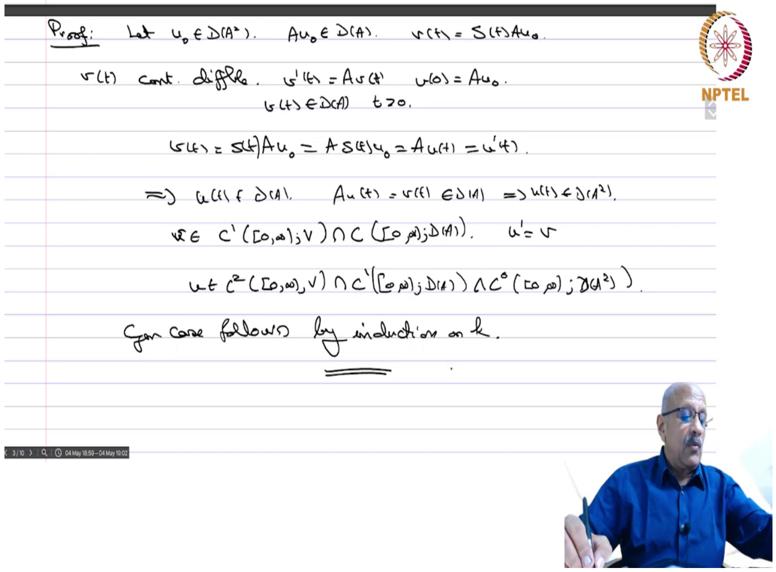

Proof: the smoother the data the smoother the solution. So, let us assume so, let $u_0 \in D(A^2)$ then Au_0 belongs to D(A) that is the definition. Now, you said

$$v(t) = S(t)Au_0 = AS(t)u_0 = Au(t) = u^{'}(t).$$

Then v(t) is differentiable and v dash t is equal to A of v(t). And v of 0 equals A of u_0. And v (t) of course belongs to $D(A\ )$ for every t positive is all standard stuff. So, now what is v(t)? v(t) is S(t) of $A(u_0)$ which is A of S(t) of u_0 which is A of u t and A of u(t) is u dash t. So, v(t) is equal to $u^{'}(t)$.

Therefore, this belongs means so, $v(t) \in D(A\ )$. $u^{'}(t)$ is A of u(t) so, with u t belongs to $D(A)$. And A of u t is equal to v t also belongs to $D(A)$. So, this implies that u t belongs to $D(A)$ square. And u itself v itself belongs to C^1 of 0 infinity with values in v intersection C of 0 infinity with values in $D(A)$. And therefore, the u and u dashed equal to v.

And therefore, you have u belongs to C^2 intersection 0 infinity with values in v intersection C^1 of 0 infinity with values in $D(A)$. And of course it is C of 0 infinity it is a continuous function and its values are in $D(A^2)$ so, this we have. Now general case follows by induction on k. So, that is the theorem about the regularity. Now, we want to prove one more theorem which will be useful later on probably.

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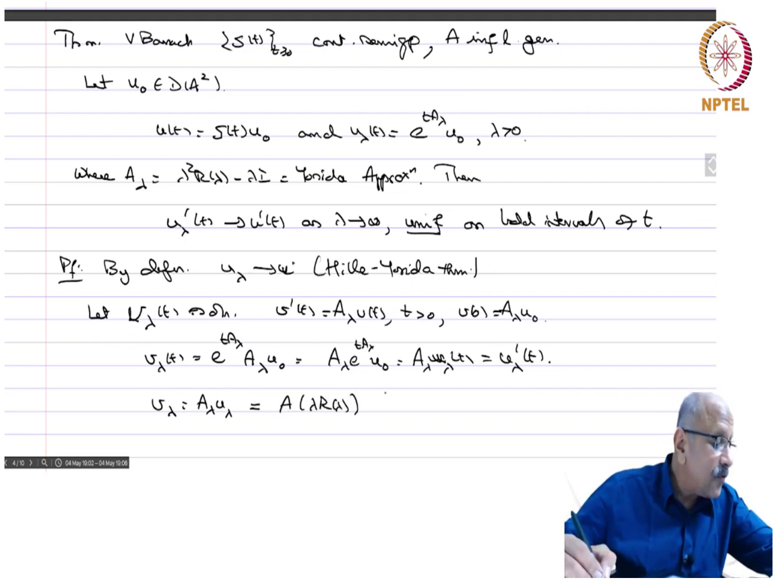

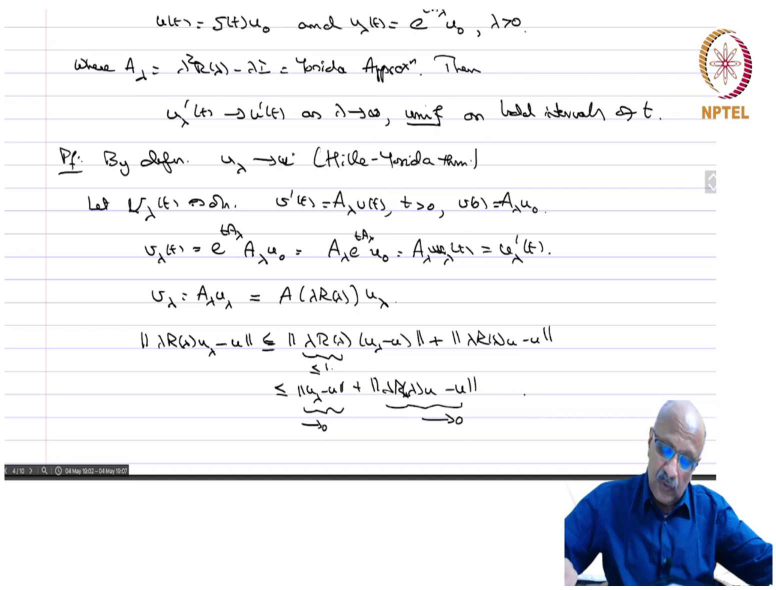

Theorem: so V Banach $\{S(t)\}_{t\geq 0}$ contraction semigroup and A infinitesimal generator let $u_0 \in D(A^2)$. And you take

$$u(t) = S(t)u_0 \ and \ u_\lambda(t) = e^{tA_\lambda u_0}, \ \lambda > 0,$$

that is the usual solution of the differential equations. So, where $A_\lambda = \lambda^2 R(\lambda) - \lambda I$ equals equals **Yosida approximation.**

Then $u_\lambda^{'}(t) \to u^{'}(t)$ as $\lambda \to \infty$ uniformly on bounded intervals of t. So, this is the theorem. So, we how did we define the semigroup at all we define the semigroup in fact so

Proof: by definition u_λ goes to u(t) is the Hille Yosida theorem. So, how did we produce the semigroup we simply took it as the limit of S(t) of A lambda u S(t) of u_0 is nothing but the limit of e power t A to lambda u_0 as lambda tends to infinity.

That was the definition that is how we constructed the semigroup and therefore u lambda goes to u is just straightforward solutions. Now, let so we want to show that Au lambda goes to Au also and the 2 uniformly unbounded interval so that is what we want to show. So, let us take v lambda t equal to solution of v dash t equals A_λv(t) , t positive and v of 0 equals $A_\lambda(u_0)$. So, then what is the solution v lambda of t.

v lambda t is e power t A lambda times the initial condition A lambda u 0 it is equal to A lambda e power t $A_\lambda u_0$ which is A lambda v lambda. Sorry A_λ u lambda t which is nothing but u lambda dash of t that is how because it is the solution of the differential equation. e power t $A_\lambda u_0$ is the solutions of this differential equation. So, A_λ u lambda t is nothing but u lambda dash of t. So, we have v lambda equals A_λ u lambda. And what is A lambda? Which is A_λ, $R(\lambda)$ of u lambda.

So,

$$||\lambda R(\lambda)u_\lambda - u|| \le ||\lambda R(\lambda)(u_\lambda - u)|| + ||\lambda R(\lambda)u - u||$$

$$\le ||u_\lambda - u|| + ||\lambda R(\lambda)u - u||.$$

Now, $||\lambda R(\lambda)|| \le 1$ and therefore, this is less than equal to norm of u lambda minus u plus norm of lambda $R(\lambda)$ u minus u. Now, we know for all u this goes to 0 and then this goes to 0. Because we saw by the Hille Yosida thoerem u goes to 0.

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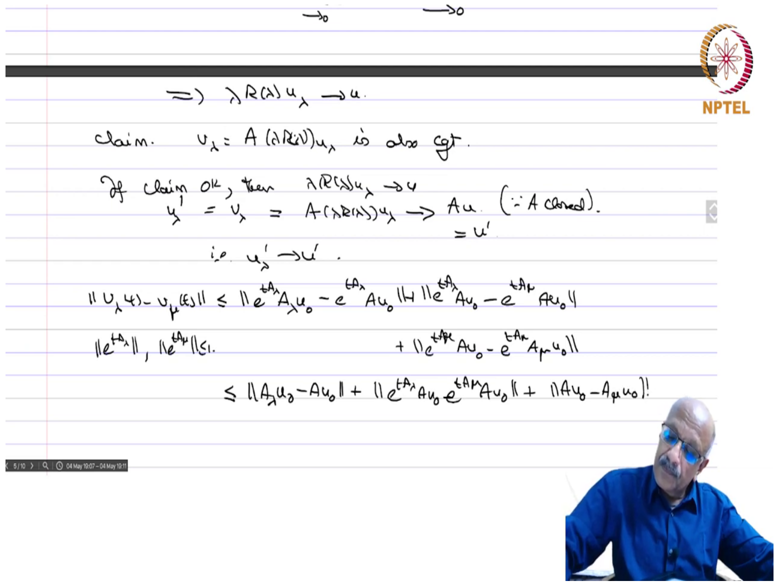

So, we have that so this implies that $\Rightarrow \lambda R_{\lambda}(u_{\lambda}) \to u$ and what is

$$v_{\lambda} = A(\lambda R(\lambda)\ u_{\lambda}).$$

And we claim is also convergent suppose we prove this claim so, you

$\lambda R_{\lambda} u_{\lambda} \to u$ and v_{λ} is A of lambda R_{λ} is also convergent A is a closed operator. So, v_{λ} must go to v. So, and therefore, you will have by the closeness you will have it will go to u of t v lambda will v lambda is u lambda dash t which will go which will converge to u dash of t. So, that is what it will go to A of u which is u dashed of t by the closeness of the operator and therefore, you have so, if claim then lambda R lambda u lambda goes to u and $A_{\lambda}\ R(\lambda)$ u lambda must converge to something which must converge to A of u since A is closed. And that is but lambda $R(\lambda)$ is v lambda which is u lambda dash and that goes to A u which is equal to u dash.

And we will show that this happens uniformly unbounded introverts that is $u_{\lambda}^{'} \to u^{'}$. So, that is what we want. So, we want to show the claim so now so we will show that $\{v_{\lambda}\}$ is a Cauchy sequence. So,

$$||v_{\lambda}(t) - v_{\mu}(t)|| \le ||e^{tA_{\lambda}} A_{\lambda} u_0 - e^{tA_{\lambda}} A u_0|| + ||e^{tA_{\lambda}} A\ u_0 - e^{tA_{\mu}} A u_0|| + ||e^{tA_{\mu}} A\ u_0 - e^{tA_{\mu}} A_{\mu} u_0||.$$

That is the definition of $v_{\mu}(t)$. So, we have all these things. So, now

$$||e^{tA_{\lambda}}|| \leq 1 \text{ and } ||e^{tA_{\nu}}|| \leq 1$$

So, we call that so, this will be less than or equal to first one is nothing but

$$||e^{tA_{\lambda}}A\ u_{0} - e^{tA_{\mu}}Au_{0}|| \leq t||A_{\lambda}Au_{0} - A_{\mu}Au_{0}||$$

$$\leq t||A_{\lambda}(Au_{0}) - A\ (Au_{0})|| + t||A_{\mu}(Au_{0}) - A(Au_{0})||$$

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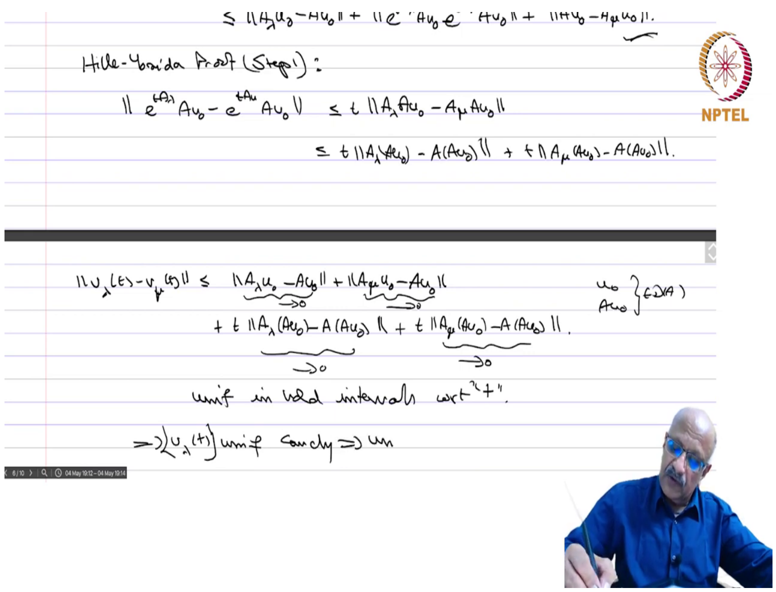

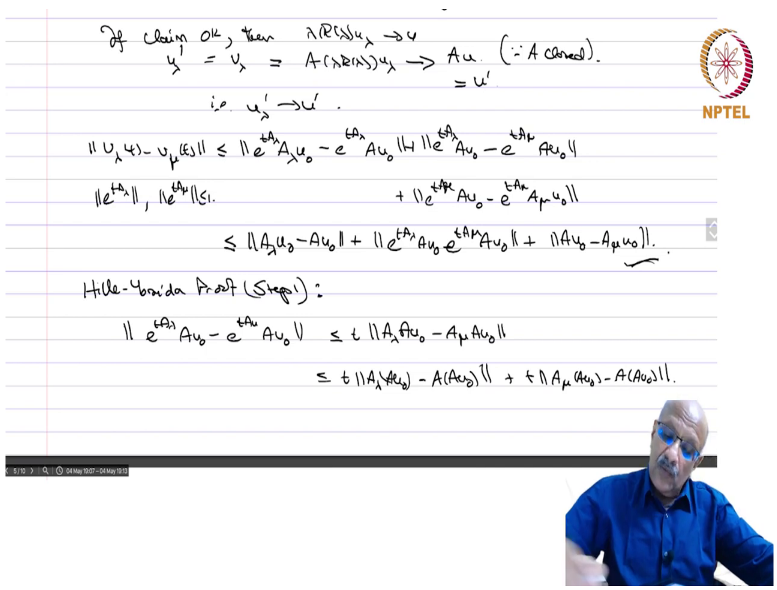

So, now let us see what is the middle term. So again, from the Hille Yosida proof and that is step 1. We have already seen this e power Now, that is well defined because $Au_0 \in D(A)$ because $Au_0 \in D(A^2)$.

That is why we are using this plus t of norm of A_μ A(u_0) minus A of Au_0. I have added and subtracted the whole thing. So now, combine these two. So we have that

$$||v_\lambda(t) - v_\mu(t)|| \le ||A_\lambda u_0 - Au_0|| + ||A\ u_0 - Au_0|| + t||A_\lambda(Au_0) - A\ (Au_0)|| + t||A_\mu(Au_0) - A(Au_0)||.$$

And then all these terms go to 0 because u_0 is in the domain. So, by early lemma this goes to 0 for the same reason this also goes to 0 as $\lambda, \mu \to \infty$ tends to infinity. And then this again goes to 0 and once more because Au_0 so $u_0 Au_0$ are both in $D(A)$. And by the lemma all these goes to 0 and of course uniformly in bounded t intervals.

If t is in a bounded interval you can replace this space of capital T which is fixed. So, independent of T you can choose lambda mu sufficiently large and then it will be a Cauchy sequence. So, uniformly Cauchy implies uniformly convergent. And then by whatever we said earlier we have that if the claim is true and because of the closeness the theorem remains proved.

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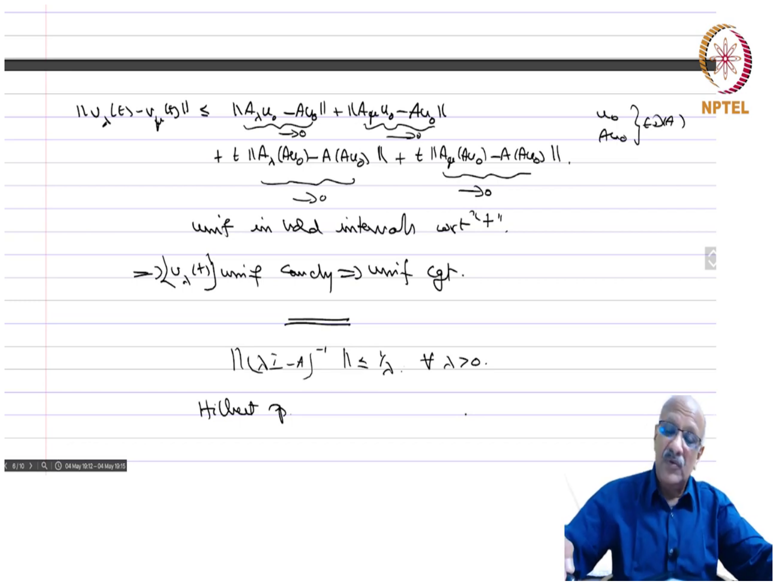

So, our next thing we will see is in the context of Hilbert space, the Hille Yosida theorem becomes even more beautiful. So, for what did we need to know for the Hille Yosida theorem we needed to show that lambda I minus A inverse this was the crucial thing exists and it is norm is less than or equal $\frac{1}{\lambda}$. Now, this can be it is enough to say for all $\lambda > 0$. In the case of a Hilbert space we will show that it is enough to check just for 1 lambda. Then it will automatically be true for all lambdas that makes our life even more pleasant. And then we will see some special cases and that will lead to various applications to the standard PDEs which we will then see. So, we have we will stop here.