Sobolev Spaces and Partial Differential Equations
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Lecture 77
Hille-Yosida Theorem

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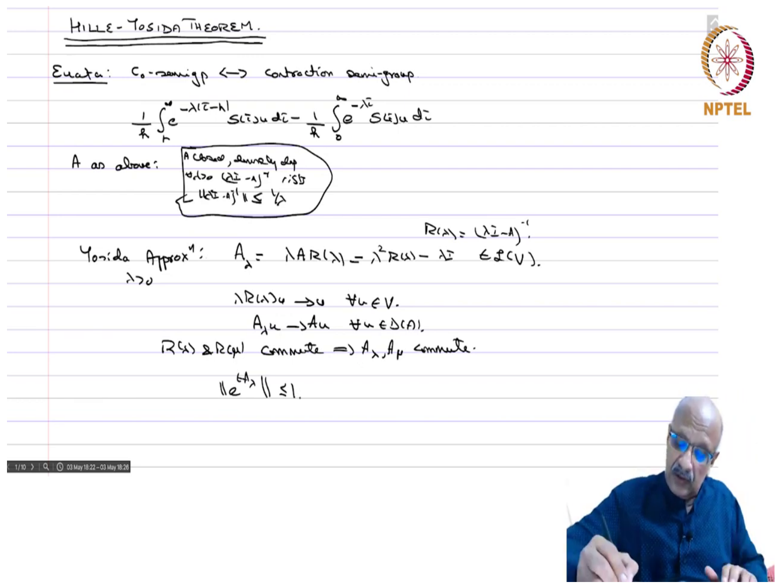

Today we will look at the famous **Hille Yosida theorem**. Which characterizes the infinitesimal generator of a C_0 semigroup. So, before I start some errata as usual. So, in yesterday's the previous lecture in the very first theorem both in the statement and the first sentence of the proof I wrote C_0 semigroup so replace it by contraction semigroup you would have worked it out because in fact we were talking that section itself was devoted to contraction semigroups.

Now, and then in the course of the proof I am trying to prove that r lambda u is in d omega I had the following expression

$$\frac{1}{h}\int_{h}^{\infty} e^{-\lambda(\tau-h)}S(\tau)u\,d\tau = \frac{1}{h}\int_{0}^{\infty} e^{-\lambda(\tau)}S(\tau)u\,d\tau.$$

Now, this $e^{-\lambda(\tau)}$ which is missing so you please add that all that. And then in front of the lemma I said A as above. And then in a little square I wrote down various things.

And so, I wrote that A closed densely defined and for every $\lambda > 0$, $(\lambda I - A)^{-1}$ exists. And then norm of $(\lambda I - A)^{-1}$ I wrote it as 1 that is wrong you might again you would you known from the rest of the proof that it was actually 1 over lambda. So, these are the corrections which we wanted to do. So, now, we want to so what did we do last time we introduced the Yosida approximation.

So, for $\lambda > 0$, you have

$$A_\lambda = \lambda A R(\lambda)$$

we call is nothing but lambda A minus A inverse. Which was given by the Laplace transform. So, it is equal to lambda A lambda A R lambda which makes sense because R lambda u for any u belongs to the domain of A and therefore this makes sense. And then this is if you recall since lambda A minus A inverse is R lambda you get this

$$= \lambda^2 R(\lambda) - \lambda I \in L(V).$$

So, this is a bounded linear operator and we had two lemmas and that was lambda $R(\lambda)u$ goes to u for all $u \in V$. And then and you have that $A_\lambda u$ goes to A u for all u in $D(A)$. And then we also saw that $R(\lambda)$ and $R(\mu)$ commute which implies that A_λ and A_μ commute. And also, the $\{e^{tA_\lambda}\}$ which is a contraction semigroup. And therefore, that is equal to less than or equal to 1. And so, these are the this is where we stood last time. So, now we have the following theorem.

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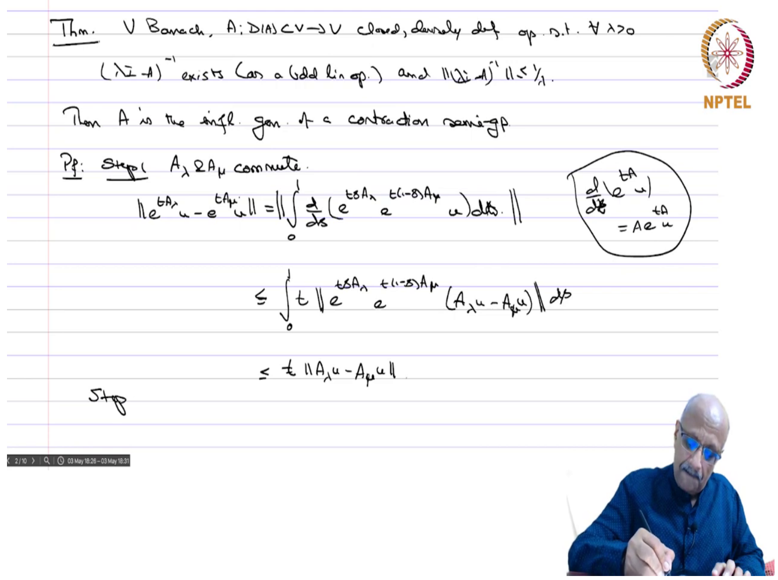

Theorem: V Banach space $A: D(A) \subset V \to V$ closed densely defined operator such that for every $\lambda > 0$, $(\lambda I - A)^{-1}$ exists as a bounded linear operator. And

$||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$.

Then A is the infinitesimal generator of contraction semigroup. So, this is the converse of what we have been doing up to now.

Proof:

Up to now we have shown that all these properties are hold for the infinitesimal generator contraction semigroup now we are saying if A is unbounded operator with all these properties then it is also the infinitesimal generator for contraction semigroup. So, this is a converse. So, therefore, we will have an if and only if statements at the end of this theorem. So,

step 1 so we have already seen that A_λ and A_μ commute.

And therefore it follows that

$$||e^{tA_\lambda}u - e^{tA_\mu}u|| = ||\int_0^1 \frac{d}{ds}\left(e^{tsA_\lambda}e^{t(1-s)A_\mu}u\right)ds||$$

$$\leq \int_0^1 t||\left(e^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda u - A_\mu u)\right)||ds$$

$$\leq t||A_\lambda u - A_\mu u||.$$

that is the reason why I am using the fact that A_λ and A_μ are commuting. So, therefore, that is equal to this norm.

So, d by dx of this when S is equal to 0 you will get A_μ and when S is equal to 1 you will get A lambda and therefore, you get these two expressions. Now, you differentiate inside so that is equal to that is less than or equal to take the norm so inside integrals 0 to 1 t times norm e power t s A lambda e power t into 1 minus s A_μ. And into A lambda u minus A_μu. So, if you differentiate d by dx of this expression this is what you will get.

Because remember d by dx of e power t A lambda u is nothing but A lambda times e power t lambda u so that is the v naught. So, dt ds sorry because d by ds of e power s A t s A sorry d by dt of e power t Au is nothing but A e power t A u which is e power t Au for the exponential operator. So, you have this. So, they know that is less than or equal to t times now e power t s A lambda is norm is 1 this also has norm 1. So, you will just get norm A lambda u minus A mu u. So, t times norm of A lambda u minus A mu integral 0 to 1 ds is just 1. So, this is the first estimate.

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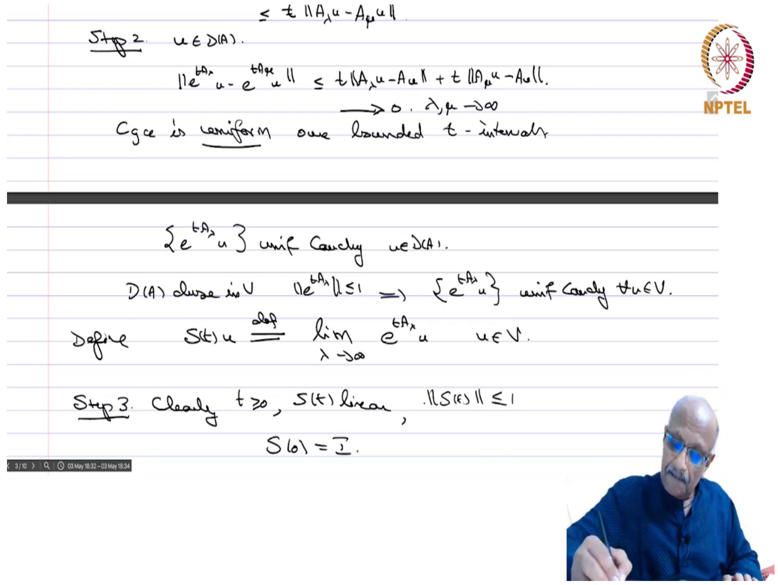

Step-2:So, now you let $u \in D(A)$ then you have

$$||e^{tA_\lambda}u - e^{tA_\mu}u|| \leq t||A_\lambda u - Au|| + t||A_\mu u - Au||.$$

I am using the triangle inequality A lambda u minus A u plus t times norm of A mu u minus A. Now, this can be made as lambda mu go to infinity by the lemma because A for u is in $D(A)$, A lambda u goes to A u and therefore, A mu u also goes to A u.

That is why it is called the Yoshida approximation. So, this goes to 0 and further because this is just t in the front this convergence is uniform over bounded t intervals. So, if t belongs to bounded interval then of course you can choose lambda mu sufficiently large independent of t and therefore that will give you the uniform convergence. Therefore, this means that e power t A lambda u uniformly Cauchy for u in $D(A)$.

But $D(A)$ is dense in V and norm of e power t A lambda is less than or equal to 1 and therefore the shows this is a limit when you have uniform convergence and therefore this implies e power t A lambda u is uniformly Cauchy for all u in v u in $D(A)$ we said but because $D(A)$ is dense and this is uniformly bounded therefore it is also uniformly Cauchy in for all u. So, please you can check that.

So, then we define

$$S(t)u = \lim_{\lambda \to \infty} e^{tA_\lambda}u, \quad u \in V.$$

So, we have a candidate for the semigroup. So, we have to check now several things we have to check that $\{S(t)\}$ is in fact a contraction semigroup and that its domain is in fact infinitesimal generator is in fact t A. So, let us do that. So,

Step-3: So, clearly for $\{S(t)\}_{t \geq 0}$ is linear and because it is the limit of a linear operator point wise and $||S(t)|| \leq 1$, . Since $||e^{tA_\lambda}|| \leq 1$, so we and further $S(0) = I$.

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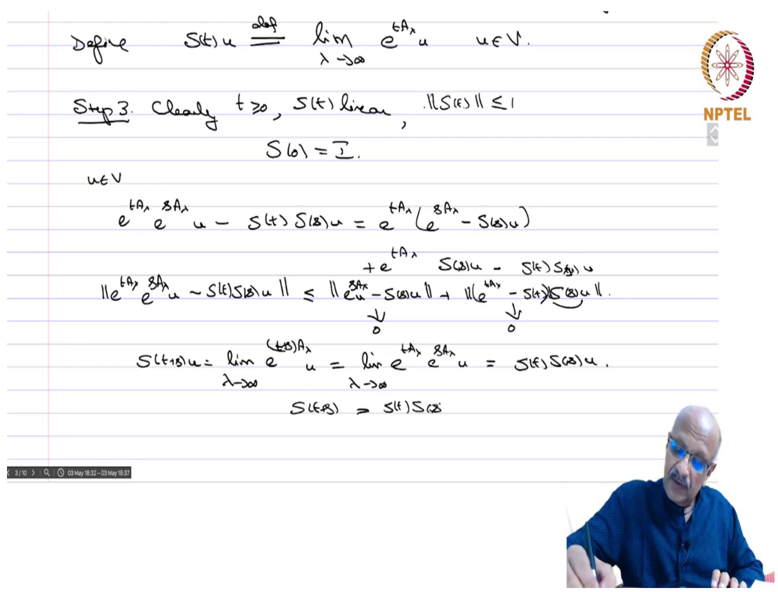

So, these are obvious properties from the definitions here. So, now we want to check the Semigroup property. So, let us take $u \in V$ and

$$e^{tA_\lambda}e^{sA_\lambda} - S(t)S(s)u = e^{tA_\lambda}\left(e^{sA_\lambda} - S(s)\right)u + e^{tA_\lambda}S(s)u - S(t)S(s)u.$$

So, you have that

$$||e^{tA_\lambda}e^{sA_\lambda} - S(t)S(s)u|| \leq ||e^{tA_\lambda}\left(e^{sA_\lambda} - S(s)\right)u|| + ||e^{tA_\lambda}S(s)u - S(t)S(s)u||.$$

And then this $e^{tA_\lambda}\left(e^{sA_\lambda} - S(s)\right)u$ may now this goes to 0 by definition. And this also $e^{tA_\lambda}S(s)u - S(t)S(s)u$ goes to 0 by definition S s of u is some vector and therefore this also goes to 0 of the definition.

Therefore,

$$S(t + s)u = \lim_{\lambda \to \infty} e^{(t+s)A_\lambda}u = \lim_{\lambda \to \infty} e^{tA_\lambda}. e^{sA_\lambda}u = S(t)S(s)u = S(t + s)u.$$

And therefore, this shows $S(t)S(s) = S(t + s)$. So, that shows the semigroup property.

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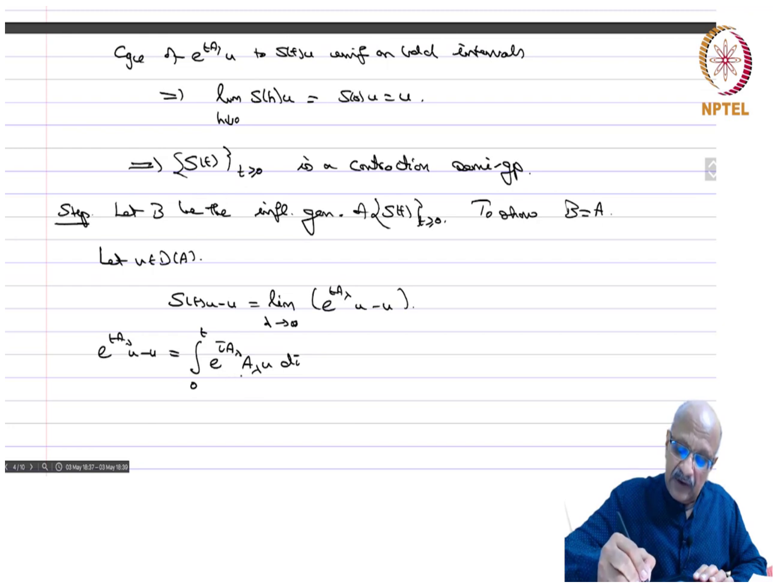

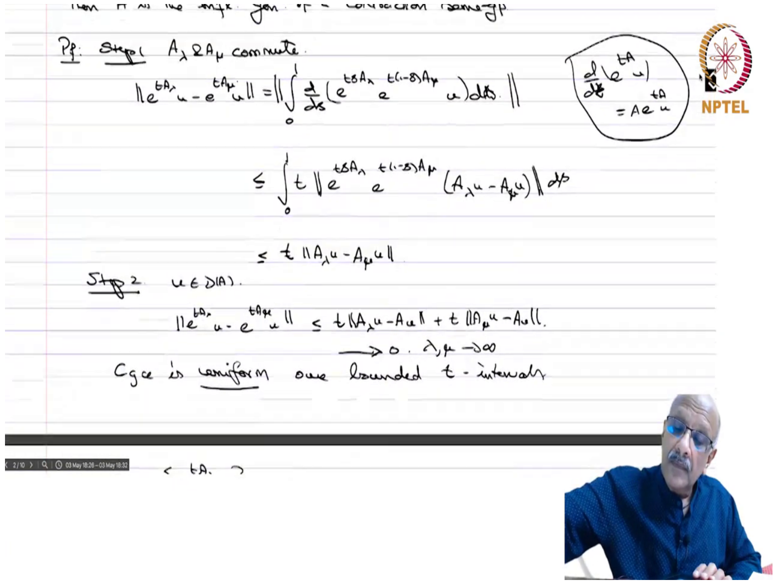

Finally, the C_0 property we have to show. And now the convergence of $e^{tA_\lambda}u$ to $S(t)u$ is uniform on bounded intervals. And this implies that limit that h tending to 0 S h of u is nothing but is S 0 u. So, this shows so this implies that $\{S(t)\}_{t\geq 0}$ is a contraction semigroup. So, step four so let B be the infinitesimal generator of $\{S(t)\}$. So, to show $B = A$.

So, that means, we have to show that the domains coincide and on the domains the operators also coincide. So, now, let u belong to $D(A)$. So, $S(t)u - u$ is nothing but limit as lambda tends to infinity e power t A lambda u minus u but what is e power t A lambda u minus u is equal to integral 0 to t e power tau A lambda A lambda u d tau. Because this is nothing but the derivative of e power tau A lambda. So, I am evaluating the derivative. Derivative is nothing that we have already seen that d by dt of we have that here. So, I am just quoting that so I am writing it like this.

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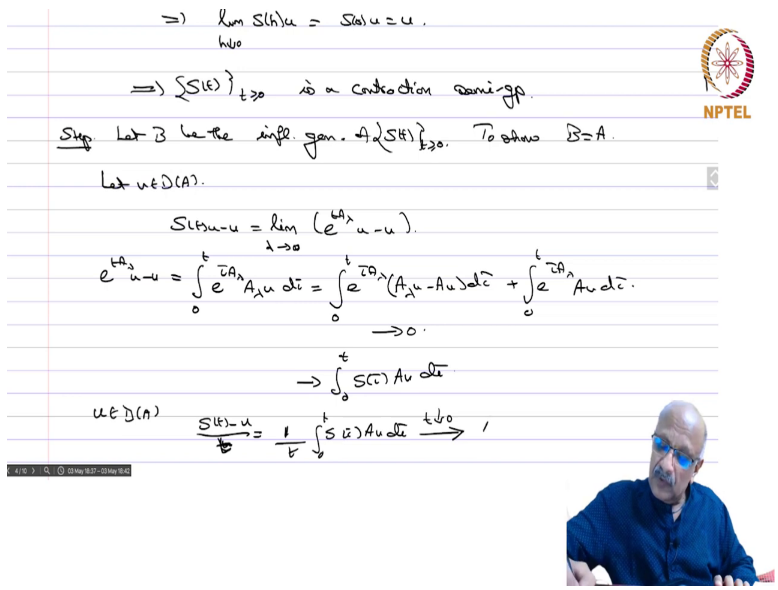

Now, that is equal to integral 0 to t of e power tau A lambda into A lambda u minus A u d tau plus integral 0 to t e power tau A lambda A u d tau. So, as lambda goes to infinity the first term this norm is less than or equal to 1 and this norm is less than or equal to 1 A lambda u goes to A u because u is in $D(A)$ and therefore this goes to 0. A lambda u goes to A u and which is a norm of tau A lambda is less than or equal to 1.

So, the first term goes to 0. The limit of the second term is what? It is e power tau A lambda as lambda goes to infinity A u uniform on bounded intervals therefore this is a integration of a uniformly convergent sequence goes to the integral in the limit. So, this will converge to integral 0 to t of S tau of A u. So, if u is belongs to $D(A)$ you have S t of u minus u by t is equal to limit of e power t A lambda u minus u by t which is equal to 1 by t integral 0 to t S tau Au d tau. And we know this goes as t decreases to 0 up to the value at 0. So, 0 1 time is just goes to Au.

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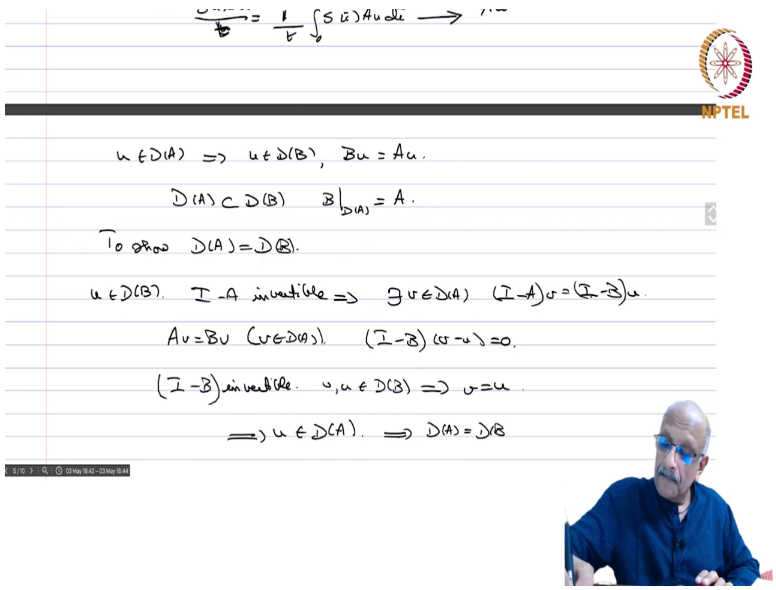

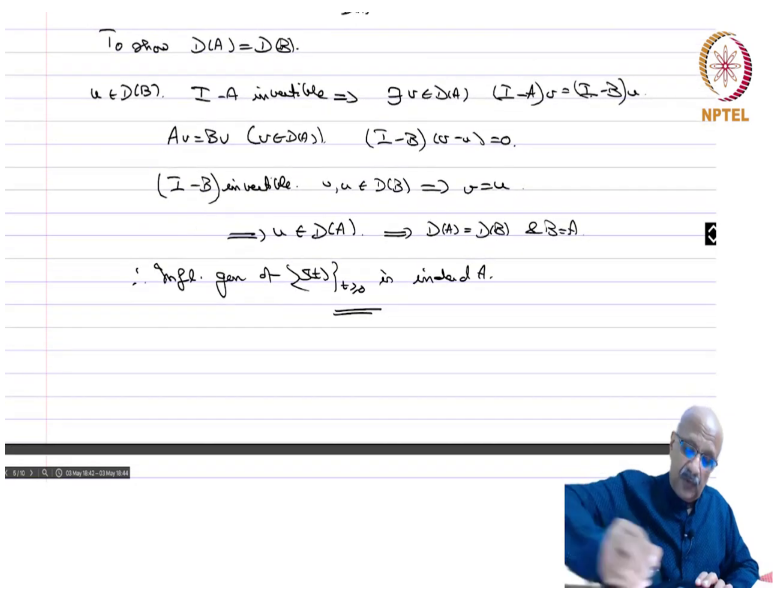

Therefore, if $u \in D(A)$ this implies that $u \in D(B)$ and B u equals A u. Therefore, $D(A) \subset D(B)$ and B restricted to $D(A)$ is the same as A. So, we have now to show. So, to show $D(A) = D(B)$. So, let u belong to $D(B)$. Now, I minus A is invertible implies there exists v in the domain of A such that I minus A v equal to B sorry I minus B u. So, this is some vector any vector you can invert it.

So, I am going to say this since it is invertible and therefore I have this that is it mean v equals R 1 of this quantity that is all I am saying. But then A v equal to B v since v belongs to D of A and therefore I minus B of v minus u equal to 0. But I minus B is invertible. And v and u belong to v belongs to $D(A)$ therefore, it is in $D(B)$ and this is in $D(B)$. And therefore, this implies that v equal to u and therefore this implies u belongs to D of A. So, this implies that $D(A) = D(B)$ and B equals. So, the infinitesimal generator therefore infinitesimal generator of S t is in d b A and that completely proves the theorem. So, now we combine all the theorems which we have proved.

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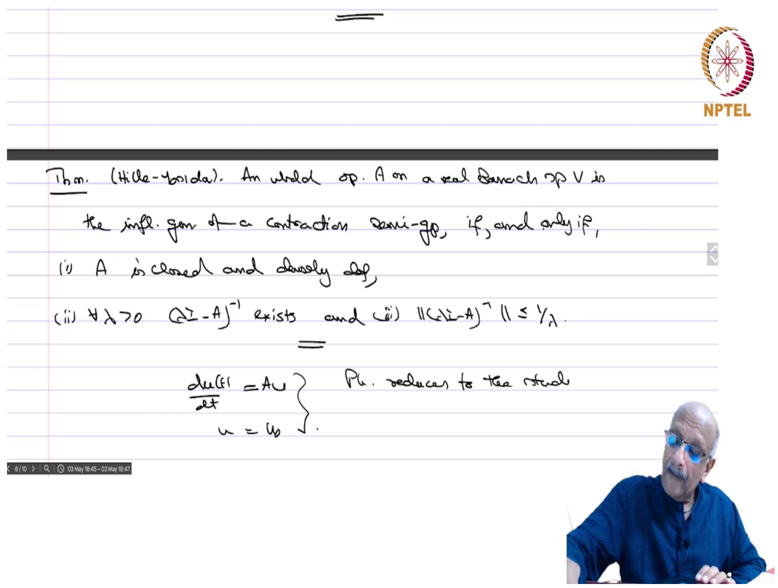

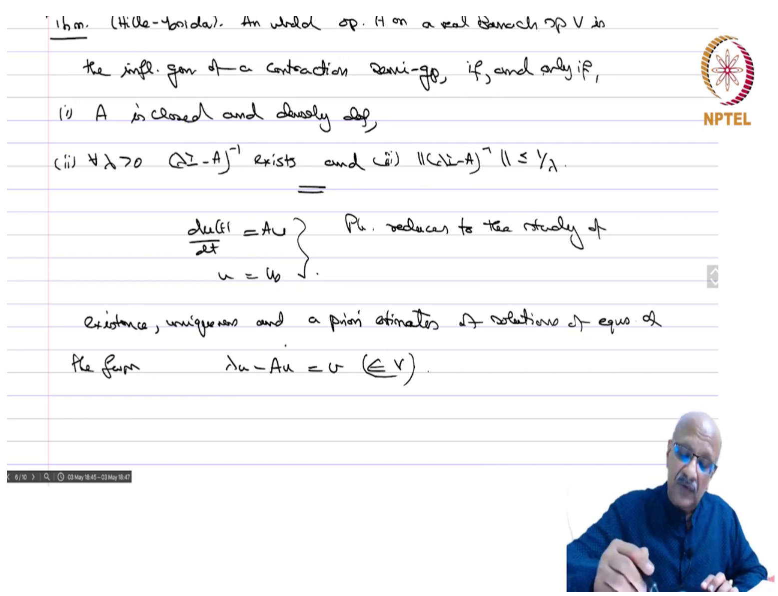

So, we have the following statement. So, this theorem is the Hille Yosida

Theorem(Hille Yosida) : an unbounded operator A on a real Banach space V is the infinitesimal generator of a contraction semigroup if and only if A is closed and densely defined for every $\lambda > 0$, $(\lambda I - A)^{-1}$ exists. And so, this is 1 this is 2 and 3 norm of $(\lambda I - A)^{-1}$ is less than or equal to 1 over λ.

So, these are the necessary and sufficient conditions for an unbounded operator to be the infinitesimal generator of a same contraction semigroups. What is the importance of this theorem? So, we wanted to solve du t by dt equals A u. And u equals u naught so we wanted to solve this differential equation. Now, we want to so A is an unbounded operator. So, it is closed and densely defined and the important thing to verify is a condition 2 and 3. And therefore this lead is so this problem reduces to the study of existence.

Uniqueness and a priori estimates of solutions of the form offer solutions of equations of the form you have that lambda u minus A u equal to v with only to be arbitrary. So, this so this is not evolution problem this is the kind of problem you will see in fact is the kind of problem we been looking at the previous in this previous chapter the elliptic equations type of problems. So, these are the stationary problems and solutions of this and proper estimation will give you the see if you can solve this problem or not.

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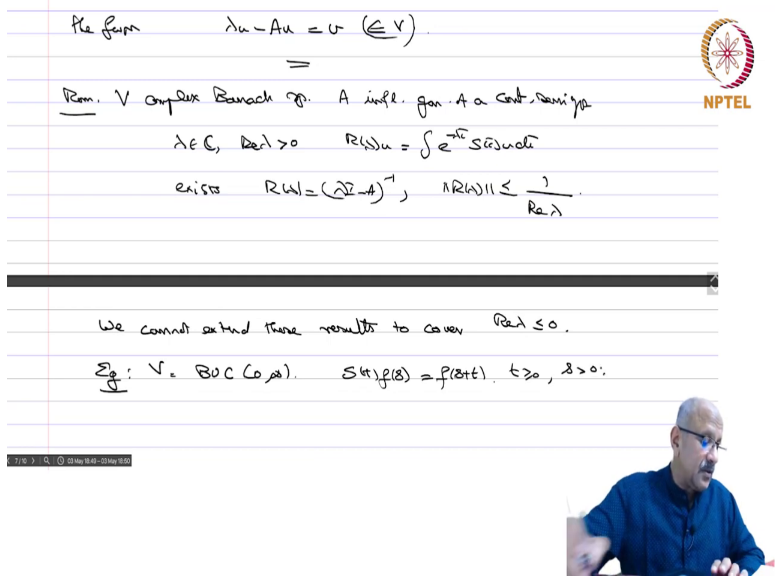

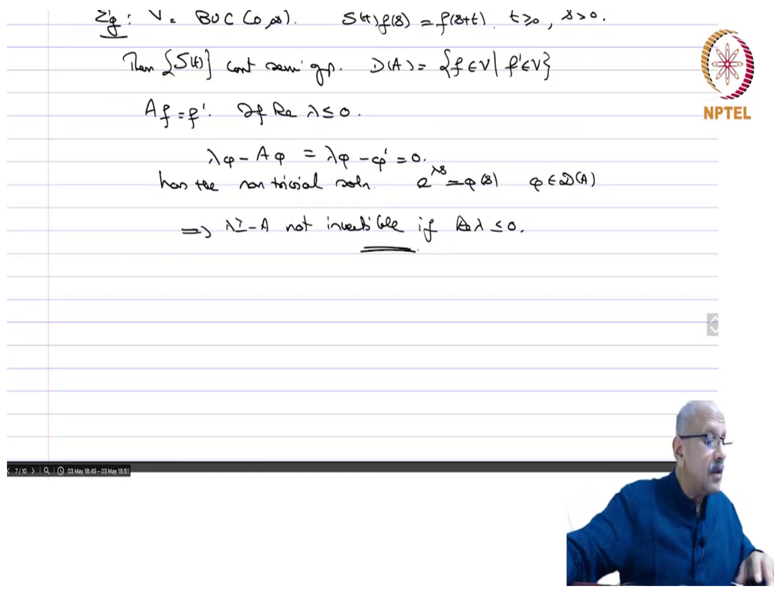

So, up to now we worked in a real Banach space to

Remark: If V is a complex Banach space. And A is a infinitesimal generator of a contraction semigroup. Then for $\lambda \in C$ with real $\lambda > 0$ you will have that $R(\lambda)u$ equals integral e power minus lambda tau s tau of u d tau exists and R lambda equals lambda I minus A inverse and norm of R lambda is less than or equal to 1 over real part of lambda.

So, that will be the thing and this will also be a sufficient condition. So, these three conditions. Now, we cannot extend these results to cover real lambda less than or equal to 0 even so let us say take the following example. So, let us v equals bounded uniformly continuous functions on 0 infinity. And $S(t)s$ complex valued functions you can take $S(t)fs = f(s + t) + t$ the translation semigroup which we have already seen. So, $s, t \geq 0$.

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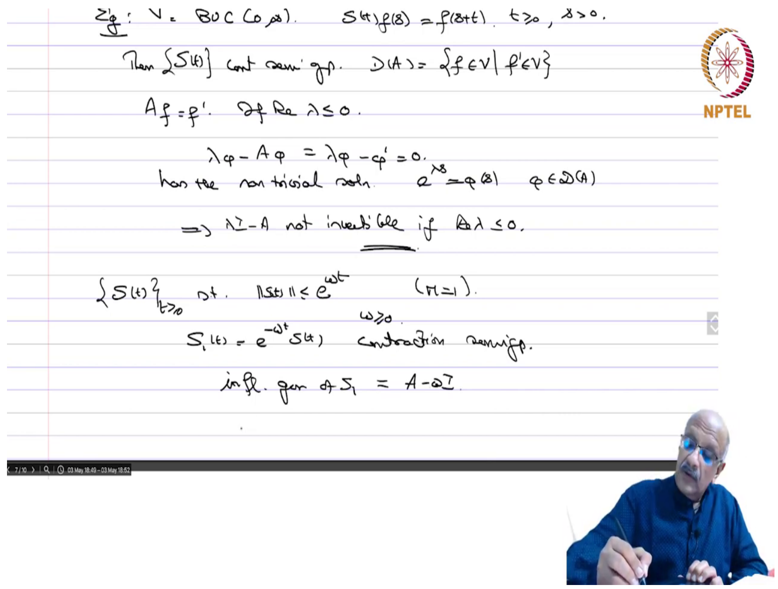

So, then $\{S(t)\}$ is a contraction semigroup and $D(A)$ will be equal to set of all $f \in V$ such that $f' \in V$. So, this is exactly as we did before. There is no difference we have already seen this in the real case you can do it for the complex case also. So, and $Af = f'$. So, now, if real $\lambda \leq 0$. Now, if you look at the equation $A(\lambda)\varphi - a\varphi$ this is lambda phi minus phi dash equal to 0 has the non-trivial solution. Namely e power lambda s and this equal to phi of s. And $\varphi \in D(A)$.

Because its derivative is just lambda times e power lambda s and that is also a bounded uniformly continuous function. Therefore, implies that $(\lambda I - A)$ not invertible if real lambda is less than or equal to 0. So, you cannot expect anything better than that. So, now assume so we looked at contraction semigroups. Now, if $S(t) \geq 0$ is such that

$||S(t)|| \leq M\omega t,$ let us say $M = 1$.

So, $M = 1, \{e^{\omega t}\}$, $\omega \geq 0$. So, then you look at $S_1(t) = e^{-\omega t}S(t)$ this is a contraction semigroup. And it is infinitesimal generator of S_1 you can check is nothing but $A - \omega I$. And therefore, you can deduce from the **Hille Yosida Theorem** or following theorem.

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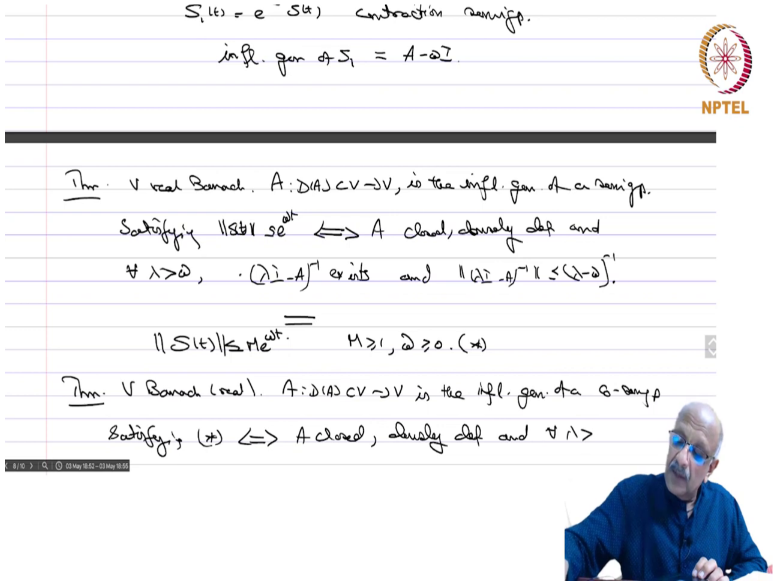

So,

Theorem: V real Banach space. $A: D(A) \subset V \rightarrow V$ is the infinitesimal generator of a semi group satisfying $||S(t)|| \leq e^{\omega t} \Leftrightarrow A$ is closed densely defined and for every $\lambda > \omega$, $(\lambda I - A)^{-1}$ inverse exists and $||(\lambda I - A)^{-1}|| \leq (\lambda - \omega)^{-1}$.

So, you have to just translate the origin by omega and you get this theorem.

Now, what about general theorems? So, characterization of the infinitesimal semigroup for the general case M $e^{\omega t}$. So, I will not tell you so $M \geq 1$ and $\omega \geq 0$. So, then we have the following

Theorem: V Banach real of course $A: D(A) \subset V \rightarrow V$ is the infinitesimal generator of a C_0 semigroup satisfying * if and only if A is closed densely defined.

And for every $\lambda > \omega$, $(\lambda I - A)^{-1}$ exists. And now the condition is a little more stringent

$||(\lambda I - A)^{-n}|| \leq M(\lambda - \omega)^{-n}$.

N is a positive integer. So, this is the genuine Hille Yosida Theorem for an arbitrary the C_0semigroup. So, we will continue further. So, this is the complete completion of the proof of the **Hille Yosida Theorem**. So, we have done the whole thing in detail for contraction semigroups. And we have sent how we can modify it for the general case. So, if you want to see details see the book by Passi.