Sobolev Spaces and Partial Differential Equations
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Lecture 76
Infinitesimal Generators of Contraction Semigroups

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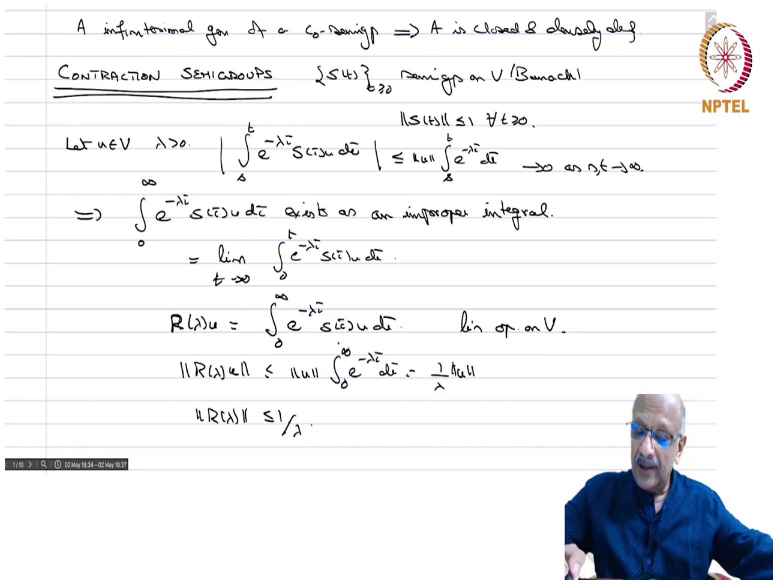

So, we are looking at the characterization of infinitesimal generators of C_0 semigroups. So, A infinitesimal generator of C_0 semigroup we have seen now the first set of conditions namely A is closed and densely defined. So, our aim is to produce some necessary and sufficient conditions to for this to be A infinitesimal generator. So, we will now look at contraction semigroups. Here everything will be very clear and so and from this we can deduce for the general case.

So, this means what? So,

Contraction semigroup:

$\{S(t)\}$ is the semigroup on V which is a Banach space and norm of $||S(t)|| \leq 1, \quad t \geq 0$. So, that is the thing. So, let $u \in V$ and $\lambda > 0$. So, then you look at the

$$\left|\int_s^t e^{-\lambda\tau} S(\tau)u\ d\tau\right| \leq ||u||\int_s^t e^{-\lambda\tau}\, d\tau \to 0,\ \ s, t\ \to \infty$$

Therefore, this implies that the

$$\int_0^\infty e^{-\lambda\tau} S(\tau)u\ d\tau$$

exists as an improper integral. So, this is well defined and because forever whatever so you can define it as the limit. So, this is in fact is nothing but the

$$lim_{t\to\infty}\int_0^t e^{-\lambda\tau} S(\tau)u\ d\tau.$$

So, now we define

$$R(\lambda)u\ = \int_0^\infty e^{-\lambda\tau} S(\tau)u\ d\tau.$$

So, this is of course, a linear operator on V and also it is a bounded linear operator because

$$||R(\lambda)u||\ \leq\ ||u||\int_0^\infty e^{-\lambda\tau}\ \ d\tau\ = \frac{1}{\lambda}||u||.$$

So, you have that

$$||R(\lambda)||\ \leq \frac{1}{\lambda}.$$

So, this is you may recognize from this expression for R lambda it looks like what is called the Laplace transform which you are probably familiar in your study of ordinary differential equations.

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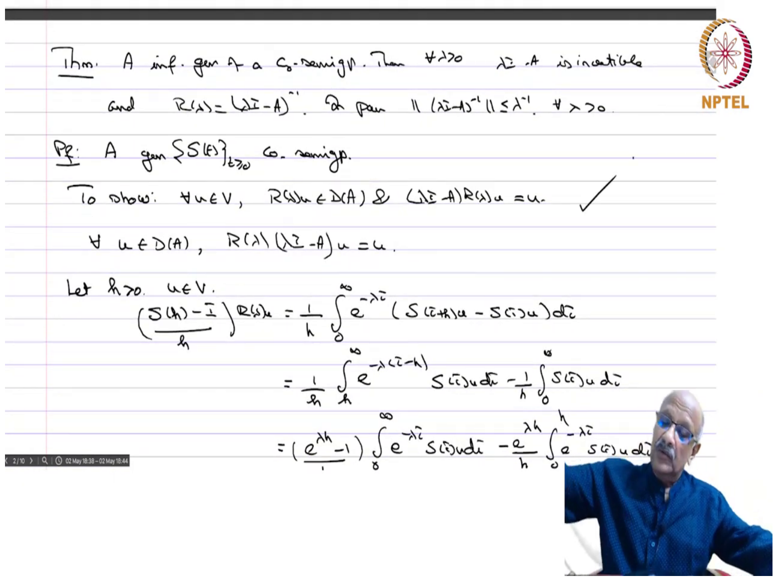

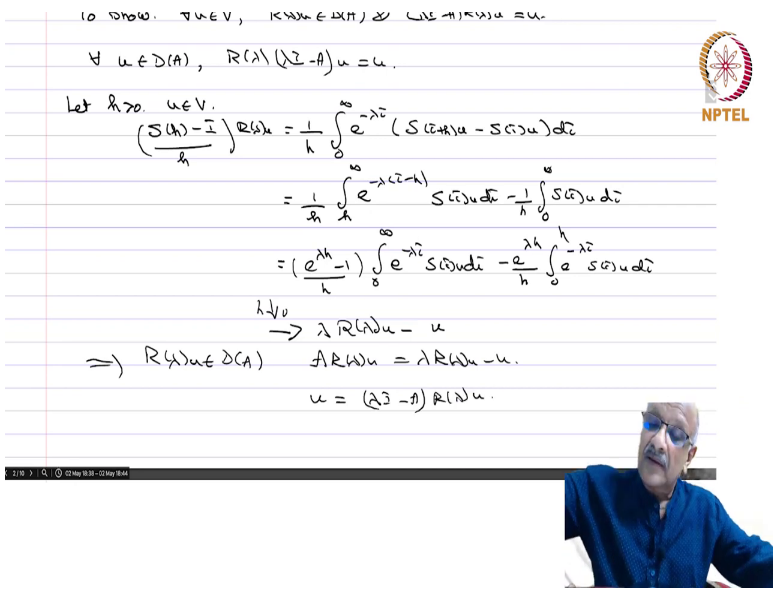

So, then it is normal, we are trying to solve some ordinary differential equations of a special form and it is normal that the Laplace transform comes in somewhere or the other. So, now we have a

Theorem: A infinitesimal generator of a C_0 semigroup then for every $\lambda > 0$, $\lambda I - A$ is invertible and $||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$ and $R(\lambda) = (\lambda I - A)^{-1}$.

So, remember that $\lambda I - A$ is an unbounded operator. But it has a bounded inverse this always happens very often happens. So, it has a bounded inverse and it has a norm. So, which is equal to norm of the inverse is in fact less than one way. So, in particular norm of $||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}, \quad \lambda > 0$.

Proof: so we have an unbounded operator. So, A generates $\{S(t)\}_{t\geq 0}$ C_0 semigroup.

So, now we want to show that $(\lambda I - A)^{-1} = R(\lambda)$, so what do we need to show? We need to show the following for every $u \in V$, $R(\lambda)u \in D(A)$ and

$(\lambda I - A)R(\lambda)u = u$.

Because if you want it to be the inverse now and for every $u \in D(A)$, $R(\lambda)(\lambda I - A)u = u$.

So, these see $(\lambda I - A)u$ it makes sense only for Du in $D(A)$ and $R(\lambda)$ makes sense for any vector and here of course $R(\lambda)u$ make sense for any vector but then it should be in $D(A)$ for $\lambda I - A$ to act on it. So, this is what we mean by saying that $\lambda I - A$ has a bounded inverse. So, $R(\lambda)$ is a bounded linear operator and these two things have to be shown so, this will prove the theorem.

So, let $h > 0$ and $u \in V$. So, we have to show $R(\lambda)u$ is in the domain so, you have to take

$$\left(\frac{S(h)-I}{h}\right)R(\lambda)u = \frac{1}{h}\int_0^{\infty} e^{-\lambda\tau}(S(\tau + h)u - S(\tau)u)\ d\tau$$

$$= \frac{1}{h}\int_h^{\infty} e^{-\lambda(\tau-h)}S(\tau)u\ d\tau - \frac{1}{h}\int_0^{\infty} e^{-\lambda\tau}S(\tau)u\ d\tau$$

$$= \frac{e^{\lambda h}-1}{h}\int_0^{\infty} e^{-\lambda(\tau)}S(\tau)u\ d\tau - \frac{e^{\lambda h}}{h}\int_0^{h} e^{-\lambda\tau}S(\tau)u\ d\tau \rightarrow \lambda R(\lambda)u - u.$$

So, I have combined these two terms. And therefore, I have to since I have taken from h to infinity I have replaced by 0 to infinity. So, I have to subtract a 0 to h so minus e power lambda h by h of integrals 0 to h e power minus lambda tau s tau of u d tau. So, this is just trafficking with this making this 0 to infinity and then rewriting everything neatly.

So, now we know how to find the limit of these things. So, this is e power lambda h by minus 1 by h goes to lambda this a standard limit which you probably learned in your first class in calculus and then this is nothing but R lambda of u minus e power lambda h is lambda tends to infinity oh sorry S h goes to 0. We are looking not this lambda this is lambda is fixed sorry excuse me. So as h goes to 0 1 by h integral 0 to h will go to the value at tau equal to 0 and that will just give you u and e power lambda 0 is what.

So, this is just this and therefore this implies that $R(\lambda)u \in D(A)$ and $AR(\lambda)u = \lambda R(\lambda)u - u$. So, this is exactly saying what we whatever we wanted to say. So, we have shown that u equals $\lambda I - R(\lambda)$ and that is exactly this statement which we wanted to.

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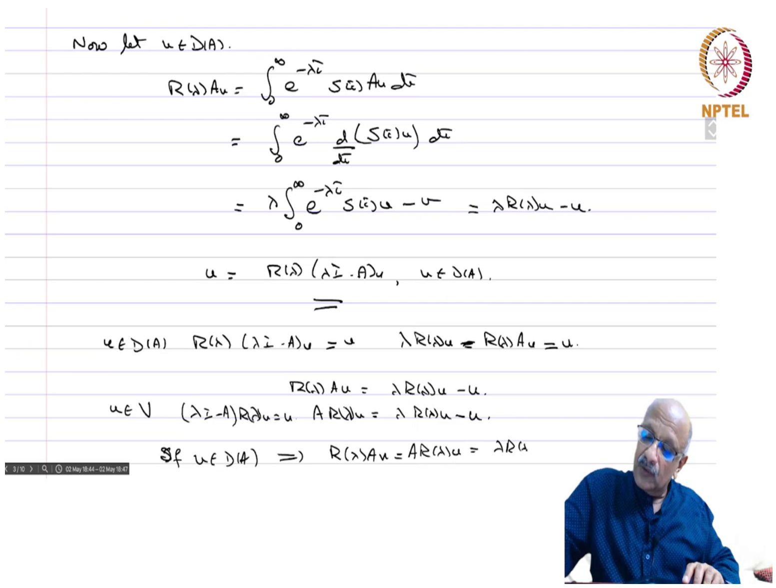

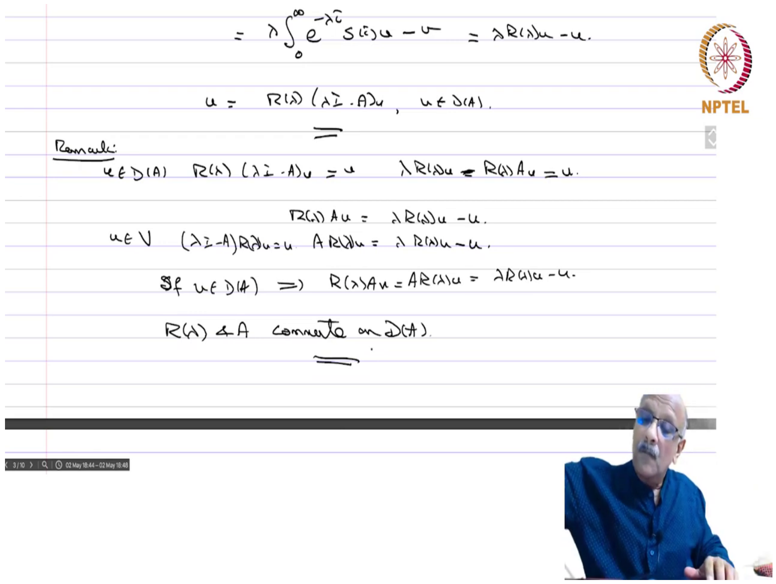

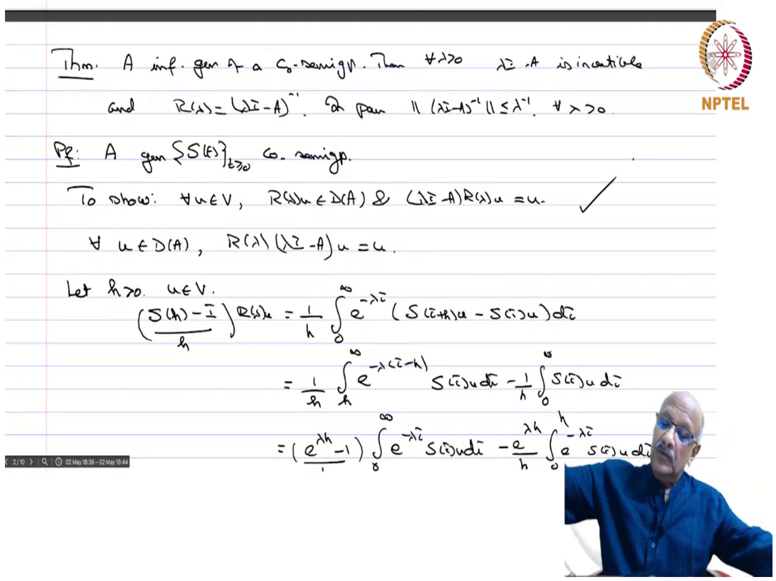

Now about so, we have that so, now let $u \in D(A)$. So, $R(\lambda) - Au$ equals integral 0 to infinity e power minus lambda tau S tau of A u d tau. Which is equal to integral 0 to infinity e power minus lambda tau d by d tau of S tau of u that we know. Because u is in $u \in D(A)$ so S tau of A u is nothing but d by d tau S tau of u d tau. And now, we can integrate by parts. So, the equal to minus integral with the derivative coming here. So, lambda integrals 0 to infinity e power minus lambda tau S tau of u. And then the boundary terms at infinity there is this go to 0 and this is bounded by norm u.

And therefore, this whole thing will go to 0 and at 0 we have minus e power lambda 0 is 1 and then your S 0 of u is just u. And therefore, this equal to $\lambda R(\lambda)u - u$. So, you have that again u is equal to R lambda times lambda I minus A of u or $u \in D(A)$. And that is the second statement which we wanted to show of $(R(\lambda) - \lambda A - A)u = u$ and therefore, that proves the theorem completely.

So, now if you look at this expression

Remark:

If $u \in D(A)$ you have $R(\lambda)(\lambda I - A)u = u$. And therefore

$(\lambda I - A)R(\lambda)u = u$. So, you have $R(\lambda)Au = \lambda R(\lambda)u - u$ and if $u \in V$ you have that $(\lambda I - A)R(\lambda)u = u$. So, A of $R(\lambda)(\lambda I - A)u = u$. So, the right hand side is the same. So, if $u \in D(A)$ then both these statements are true this implies that $R(\lambda)(\lambda I - A)u = u$. So, $R(\lambda)$ and A.

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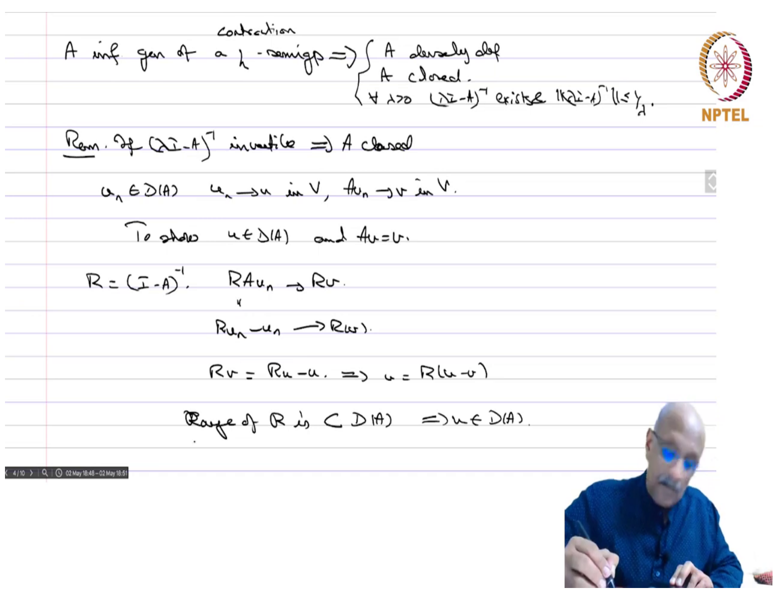

So, $R(\lambda)$ and A commute on $D(A)$ so this is a remark. So, A infinitesimal generator of a C_0 semigroup so this implies now we have three things A is densely defined A is closed and for every lambda greater than 0 $(\lambda I - A)$ inverse exists. And norm $(\lambda I - A)^{-1}$ is less than or equal to 1 over lambda. And this is now we these are all necessary conditions and we will show that these are also sufficient of a contraction semigroup.

So, let me correct this. So, it is not semi group it is contraction semigroup. So, we would like to show that these are also sufficient. So, before that one another

Remark. If $(\lambda I - A)^{-1}$ then it automatically implies that A is close. So, this is if you have the third condition the second one is redundant you do not need to see that. So, let us show that So, let $u_n \in D(A)$. And $u_n \rightarrow u$ in $V(A)$ $u_n \rightarrow u$ in V. So, to show u belongs to the domain of A and A u equals so this is what we need to prove.

So, let us take $R = (I - A)^{-1}$ which exists of course so, then Au_n converges to u So, RAu_n converges to R of but RAu_n is what $RAu_n - u_n$. We have seen if you is in domain of A. So, your R is nothing but lambda equals 1. So, you have r u n minus u n so, we have just used that condition here. So, this converges to R of. So, $Rv = Ru - u$.

So, this is same. So, this implies that $Ru - Rv = u$ but range of R is in contained in $D(A)$ because we know that are of any element is contained in domain of A that was the content of the previous theorem. And therefore, this implies that $u \in D(A)$. So, I f now you apply $I - A$ to this so $(I - A)u = I - AR(u) - v$ and that we know is equal to this u minus v because $I - A$ is the inverse of R. And therefore, if you take u gets cancelled and therefore you have minus A u equals minus v or a u equals. So, therefore, A is close. So this so, if $I - A$ is invertible then automatically A has to be a closed operator.

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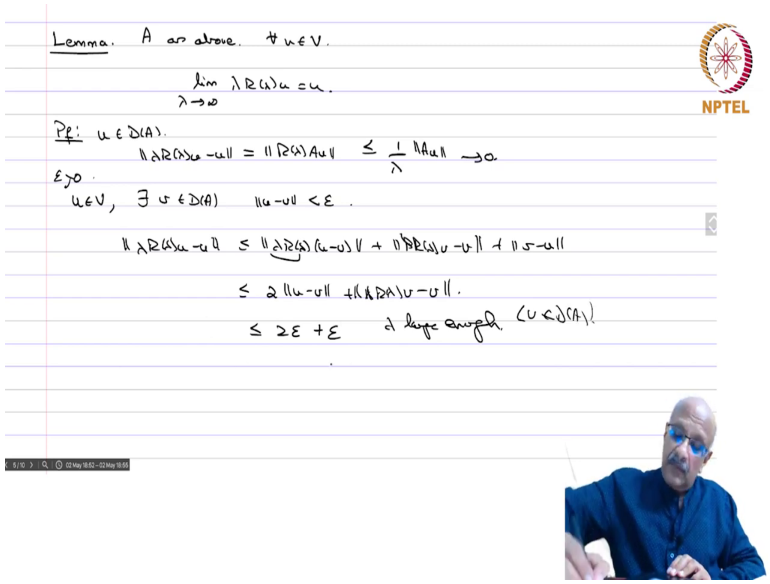

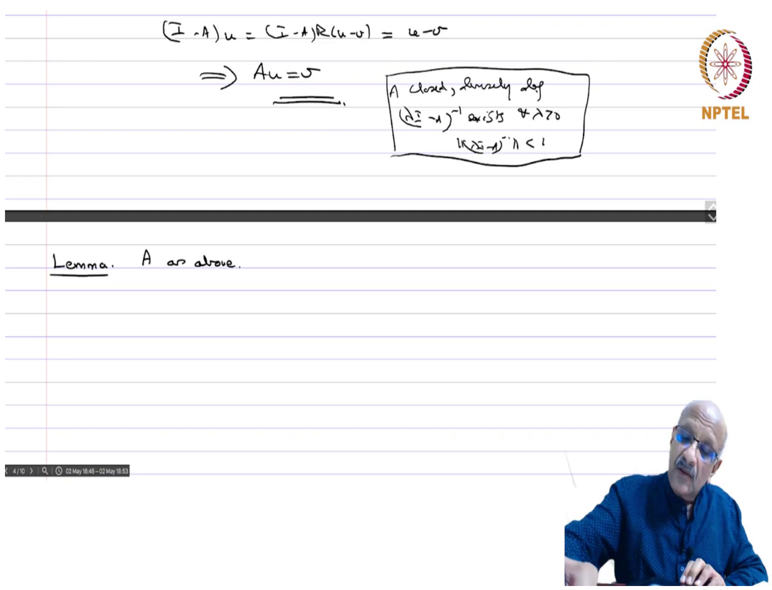

So, now, we before we can proceed to the characterization of the infinitesimal semigroup generators of contraction semigroup we need a couple of technical results. So, let A as above. So, what is as above? So, henceforth we assume this even so closed densely defined and $(\lambda I - A)^{-1}$ inverse exists for all lambda positive norm of $(\lambda I - A)^{-1}$ is less than or equal to what.

So, this is the condition on A which we are going to work with and that we want to show is sufficient to generate a contraction semigroup. So,

Lemma: A as above then for every $u \in V$ we have a

$lim_{\lambda \to \infty} \lambda R(\lambda)u = u$.

Proof: so, let us first start with $u \in D(A)$ then you have

$||\lambda R(\lambda)u - u|| = ||R(\lambda)Au|| \leq \frac{1}{\lambda}||Au||$

and therefore goes to 0 therefore, this is true for all $u \in D(A)$.

Now, if u belongs to v there exists $v \in D(A)$ such that norm u minus v is less than epsilon whatever may be epsilon greater than 0. So, now you take norm of lambda R lambda u minus u is less than or equal to norm of lambda R lambda of u minus v plus norm of R lambda R lambda v minus v plus norm of v minus u. Now, that is less than or equal to 2 times lambda R lambda norm is less than or equal to 1 and therefore you have twice norm u minus v plus norm of R lambda lambda R lambda v minus v.

So, you choose. So, this will be less than or equal to 2 epsilon and then this is less than or equal to whatever you want epsilon for lambda large enough. So, you first choose a v and then you choose a lambda this is going to because $v \in D(A)$ and $v \in D(A)$. And therefore, you have the lemma is proved.

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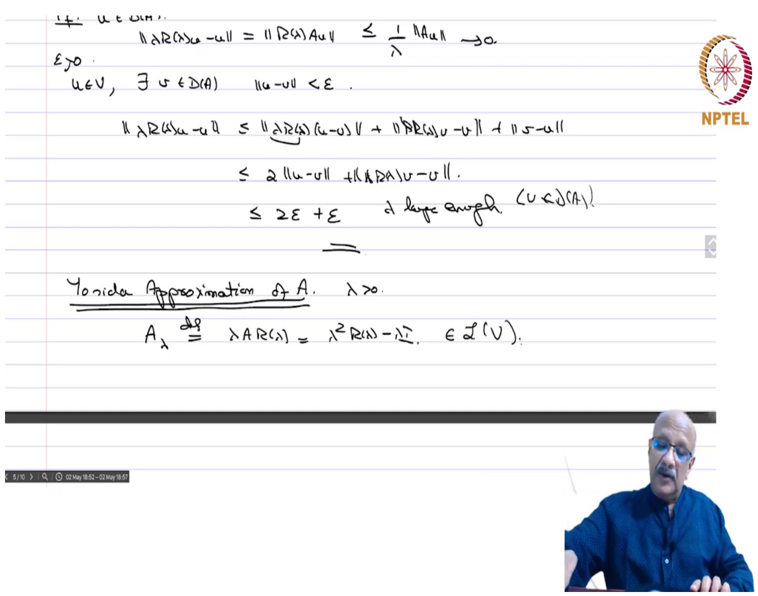

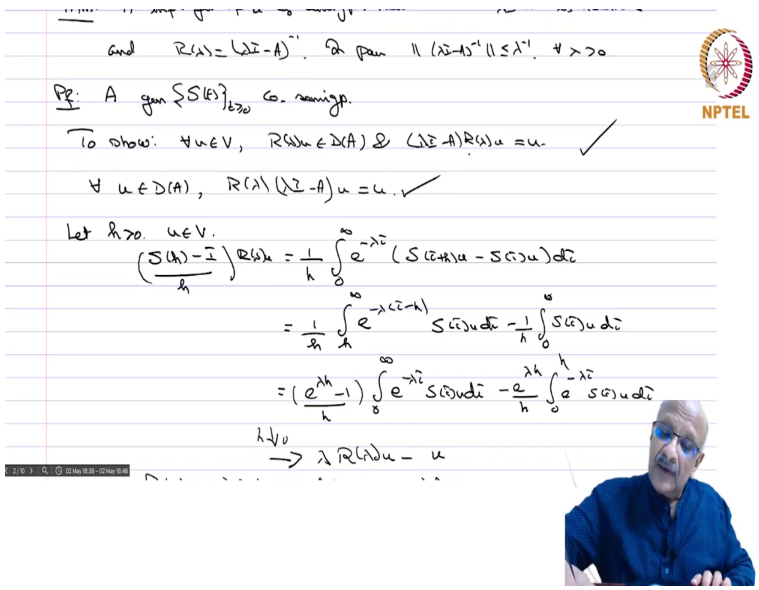

So, now we are going to introduce a very important approximation. So, this is called the **Yosida approximation** of A. So, $\lambda > 0$ and then we define

$$A_\lambda = \lambda A R(\lambda) = \lambda^2 R(\lambda) - \lambda I.$$

you know for every $u \in V,\ R(\lambda)u \in D(A)$. So, $AR(\lambda)$ is well defined and we also know for any u for any u you have lambda so lambda A R lambda is equal to lambda R lambda minus u.

So, we have another lambda in addition here and therefore, you have lambda square R lambda minus lambda I. So, this lambda A R lambda is lambda R lambda minus identity. And therefore, you have lambda square so this of course is belongs to $L(v)$ it is a bounded linear operator and therefore it is in $L(v)$.

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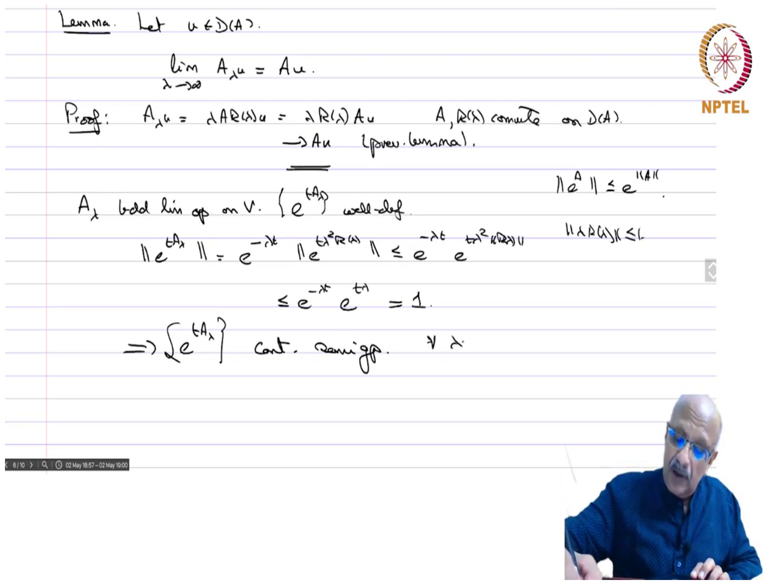

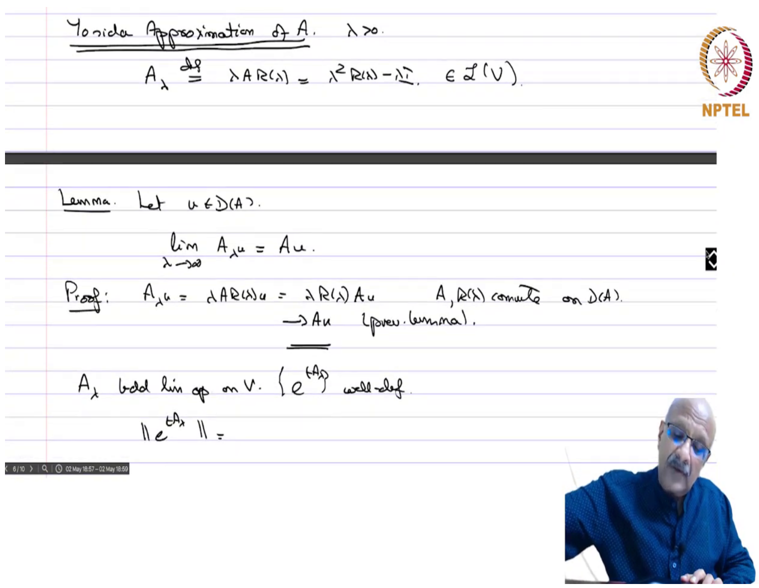

So, another lemma why do we call it an approximation?

Lemma: Let $u \in D(A)$ then

$lim_{\lambda \to \infty} A_{\lambda} u = Au$

that is why we call it an approximation we have an unbounded operator. And then we are replacing it by a bounded linear operator and then we are going to use that. So,

Proof:

$A_\lambda u = \lambda AR(\lambda)u = \lambda R(\lambda)Au.$

Because you know that A and $R(\lambda)$ compute on $D(A)$. Now, $\lambda R(\lambda) - A$ this of course goes to Au by the previous lemma. So, we have this. So, we have this nice property so, another so A lambda is a bounded linear operator on v. So, $e^{tA\lambda}$ well define. And then so it is the C_0 group in fact. Now, norm e^{tA_λ}. So, A lambda is what? A lambda is lambda square R lambda minus lambda I.

So, you have

$$||e^{tA_\lambda}|| = e^{-t\lambda}||e^{t\lambda^2 R(\lambda)}|| \leq e^{-t\lambda}e^{t\lambda^2||R(\lambda)||} \leq 1.$$

Because it is $||e^{A}|| \leq e^{||A||}$ always so we have that. So, we have this. But then $\lambda||R(\lambda)|| \leq 1$, and that is equal to 1. Therefore, implies $\{e^{tA_\lambda}\}$ is a contraction semigroup. So, this is another important point for every lambda positive.

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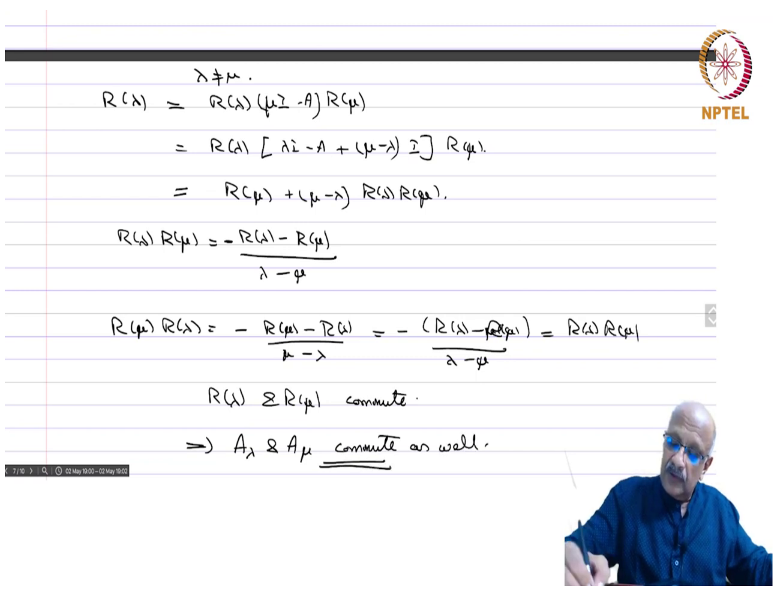

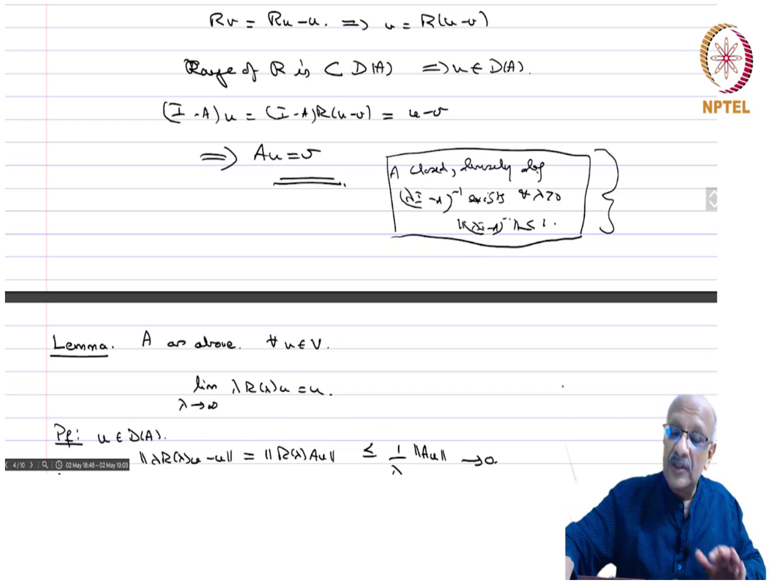

Now, finally before we can proceed to the important theorem we need one more remark. So, let us take $R(\lambda)$. So, let us take $\lambda \neq \mu$,

I am going to write as

$$R(\lambda) = R(\lambda)(\mu I - A)R(\mu)$$

this is well defined. Because for any u R mu is in the domain of A and $(\mu I - A)R(\mu)$ will give you just u again. So, this is equal to

$$= R(\lambda)[\lambda I - A + (\mu - \lambda)I - A]R(\mu)$$

You might have seen this kind of calculation when studying the spectrum of an operator also. Since a similar resolvent equation which we wrote it the which one writes at that time. So, if now $\lambda R(\lambda)$ times $\lambda I - A$ is of course the identity map and therefore, you multiply by $R(\mu)$ so, you get

$$= R(\mu) + (\mu - \lambda)R(\lambda)R(\mu)$$

So, $R(\mu)R(\lambda) = \frac{R(\lambda) - R(\mu)}{\mu - \lambda} = -\left(\frac{R(\lambda) - R(\mu)}{-\mu + \lambda}\right) = R(\lambda)R(\mu)$.

So, $R(\lambda)$ and $R(\mu)$ commute. And this implies therefore that $A(\lambda)$ and $A(\mu)$ commute.

So, these are the important things which we must remember and next step now is to prove that the three conditions which we wrote down close densely defined and the inverse exists and the norm is less than or equal to 1. So, these three conditions are form less are both of which are now necessary for the contraction semigroup are also sufficient for A to generate a contraction symbol that is the content of the Hille Yosida theorem which we will prove next time.