Sobolev Spaces and Partial Differential Equations

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Lecture 75

C_0 Semigroups- Part 2

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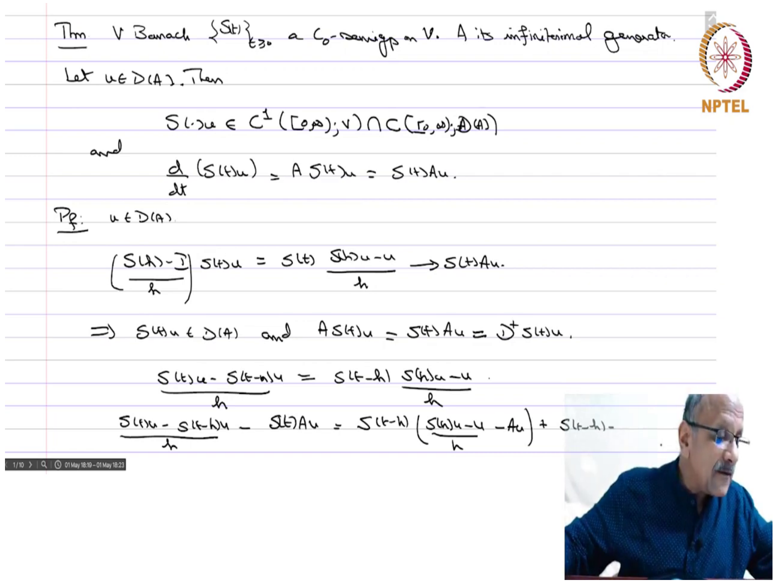

We were looking at the infinite decimal generators of a C_0 semigroup and our next result is crucial for the study of evolution equations in the framework of C_0 semigroups. So,

Theorem: V Banach $\{S(t)\}_{t\geq 0}$ *be a* C_0 semigroup on V. A it is infinitesimal generate. Let $u \in D(A)$ then $S(.)u \in C^1([0,\infty):V) \cap C([0,\infty):D(A))$.

Remember $D(A)$ is a Banach space with the graph now so you think of that that way. So, if you take t going to $S\ (t)u$ it is continuously differentiable when you think of it as values in V and it is continuous it means this $S\ (t)u$ actually belongs to $D(A)$ for every t and mapping going to t going to $S\ (t)u$ is continuous into the $D(A)$ that means with the graph norm. And

$\frac{d}{dt}(S\ (t)u) = AS(t)u = S\ (t)Au$ in $D(A)$ remember that only that.

So,

Proof: $u \in D(A)$ so we want to show that $S(t)u \in D(A)$ because $D(A)$ that is part of the problem and then we want because we also have it in the statement of the theorem $AS\ (t)u$ et cetera. And therefore, we compute see if it so

$$\left(\frac{S(h)-I}{h}\right)S(t)u = S(t)\frac{S(h)u-u}{h} \to S(t)Au.$$

So, this goes to $S(t)Au$ Therefore, this implies that means this limit always exists. So, this means that $S(t)u$ belongs to $D(A)$. And $AS(t)u$ which is the limit of this is nothing but

$$S(t)Au = S(t)Au = D^{+}S(t)u$$

and that is equal to of course since we are taking

$$\frac{S(t)u-S(t-h)u}{h} = S(t-h)\frac{S(h)u-u}{h}.$$

So, this forward derivative so now you consider

So, therefore you have

$$\frac{S(t)u-S(t-h)u}{h} - S(t)Au = S(t-h)\left(\frac{S(h)u-u}{h} - Au\right) + (S(t-h) - S(t))Au.$$

simply added and subtracted something.

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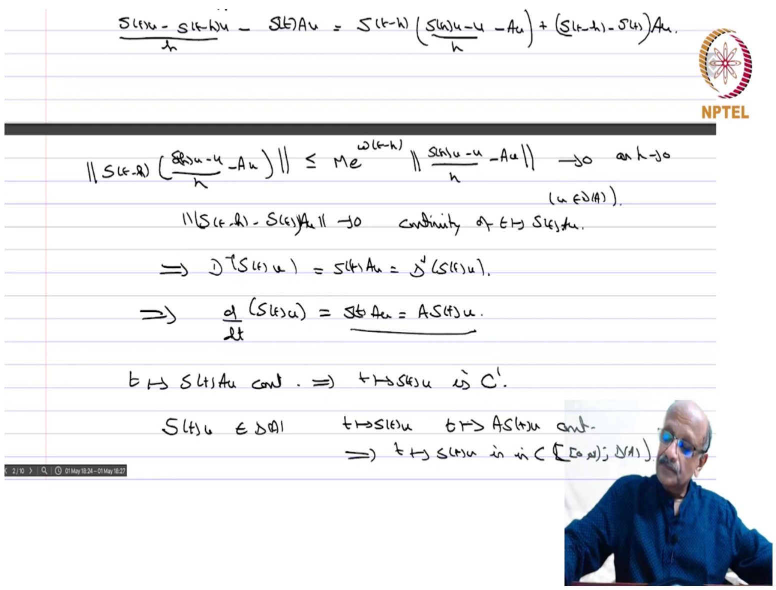

So, now let us look at the first term

$$||S(t-h)\left(\frac{S(h)u-u}{h} - Au\right)|| \le Me^{\omega(t-h)}||\left(\frac{S(h)u-u}{h} - Au\right)|| \to 0.$$

Because u is in domain of A and A u is precisely this limit. So, this goes to 0 and this can be bounded and therefore this goes to 0. And so this is because $u \in D(A)$ again.

And

$$||(S(t-h) - S(t))Au|| \to 0$$

because of the continuity of t going to $S(t)Au$ that we have already seen the continuity for any vector and therefore this. Therefore, this means that this limit here exists and the limit is equal to $S(t)Au$. So,

$$D^{-}S(t)u = S(t)Au = D^{+}S(t)u.$$

Therefore, so this implies d by dt of $S(t)u$ exists and is equal to $S(t)Au$ which also we saw is a of $S(t)u$ so that proves. Now, from we know that $t \to S(t)Au$ is continuous this implies $t \to S(t)u\ in\ C^{1}$. It is already continuous and this derivative is $t \to S(t)Au$ and therefore that is also continuous therefore it is equal to C^{1}. And you have $S(t)Au$ belongs to $D(A)$ and you have that $t \to S(t)Au$ is continuous. And therefore, and S t derivative is nothing but $AS(t)u$ and therefore so t going to S t of u and t going to $AS(t)u$ continuous because of this relationship here. And therefore, that implies t going to S t of u is in C of 0 infinity with values in $D(A)$.

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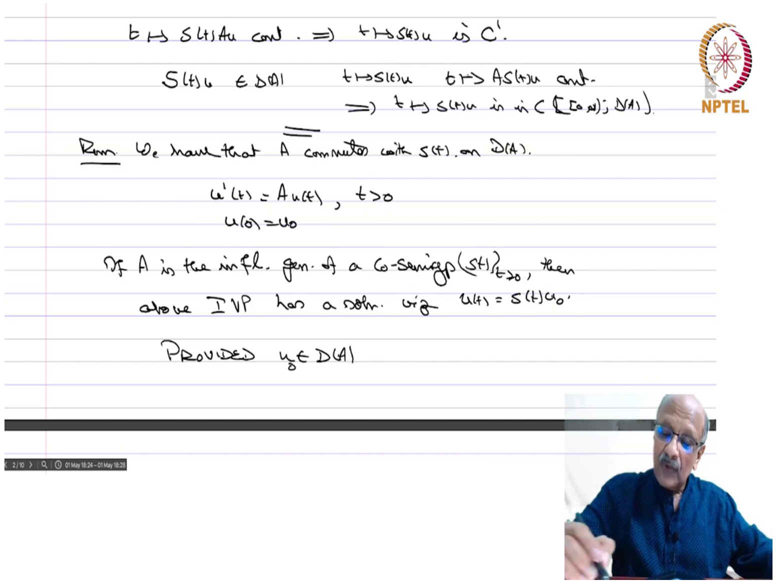

So, therefore you have because the graph norm has been that. So,

Remark: we have that A commutes with S(t) on the D(A) that is important. So, what have we shown we have shown that

$$u'(t) = Au(t)\ ,\ t > 0$$

$$u(0) = u_0$$

Remark: if A is the infinite decimal generator of a C_0 semigroup S(t) then about initial value problem has a solution namely u(t)=S(t)u_0.

It satisfies the thing provided u belongs to D of u $u_0 \in D(A)$. So, you can solve this equation if A if it is multiple generator of a C_0 semigroup then this initial value problem where the initial data belongs to the domain of the infinitesimal generator can be solved and you have a classical solution of this initial value problem namely you have a C^1 solution here which satisfies this equation.

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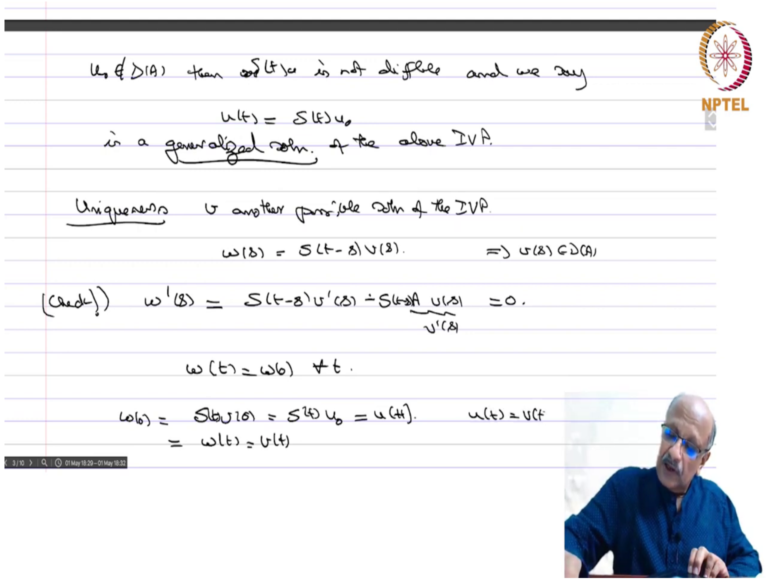

So, if $u_0 \notin D(A)$ then V then S(t)u is not differentiable. And we say u t equals S t of u naught is a generalized solution of the above initial value problem. So, we also have uniqueness. So, you like suppose I have V is another solution another solution possible solution of the IVP.

So, then we put $w(s) = S(t - s)V(s)$. Then what is $w'(s)$? Then you can check so check you have to so very easy checking so we have we can do it anyway. So, if we apply the product rule so when I apply the product rule so we have that means $V(s)$. So, this implies that $V(s)$ is in $D(A)$ because otherwise u cannot v cannot be a solution for the initial value problem because $\frac{du}{dt} = AV(t)$.

So, V must be in domain A so V is a domain A. So, if you differentiate with respect to S so you will get S of t minus s V dashed of s plus you have V of s is here differentiate S of t minus s you get minus S of t minus s AV dash V of s. Now, AV of s is V dash of s and therefore this equal to 0. So, that means w(t)=$w(0)$ for all t and what is $w(0)$ that is nothing but $S(t)$. So, $w(0) = S(t)V(0)$ which is equal to S of t u naught which is equal to u of t. And that is equal to W t which is equal to if it is equals t it is just V of t. So, u of t equals v of t for all t. So, that means the solution is unique.

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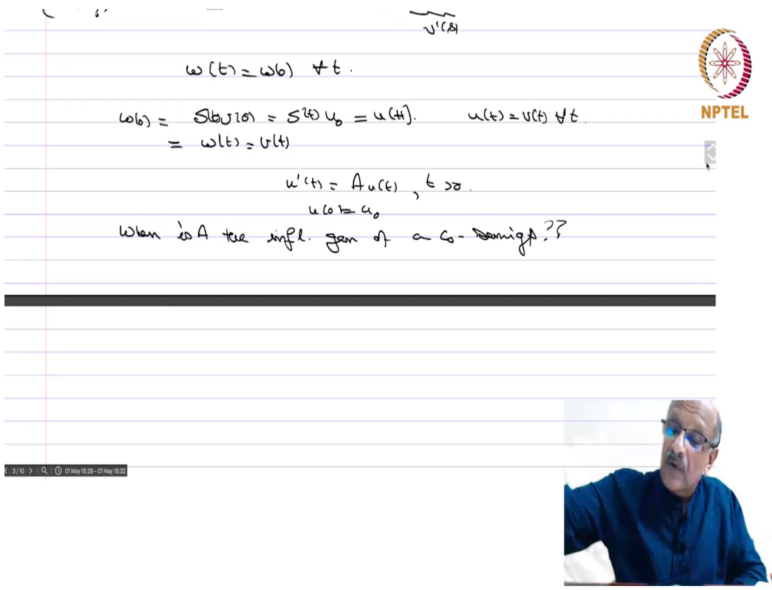

So, we have a unique solution provided the domain so, if we want to investigate initial value problems of the form

$$u'(t) = Au(t),\ t > 0$$

$$u(0) = u_0$$

So, then we must know when is A the infinitesimal generator of a C_0 semigroup. So, we want to recognize those things. So, if we know that A is infinitesimal generator then we can write down the solution at least when u naught is in domain of A explicitly. And then we can write down generally a solution if u naught is not in the domain of A and therefore we have we can solve this problem in some way or the other. Therefore, we have to know when the CA the infinitesimal generator.

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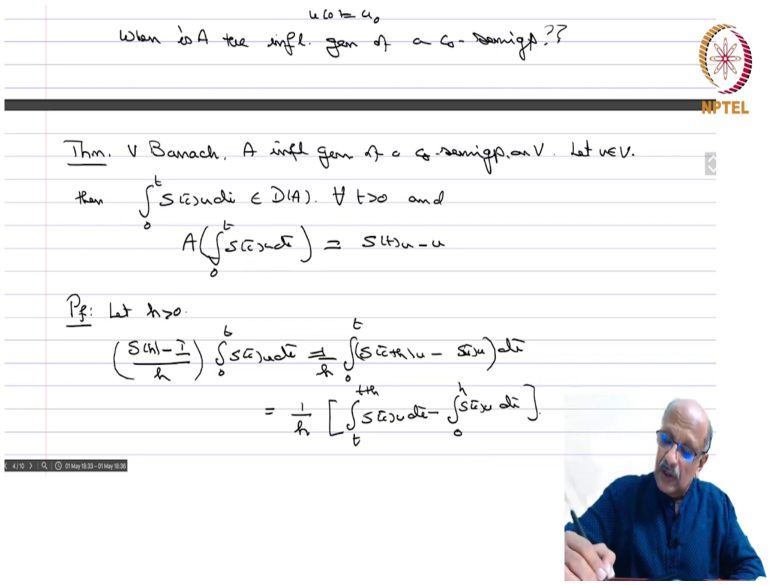

So, we would like to give necessary and sufficient conditions for unbounded operators to be the infinitesimal generators of a C_0 semigroup. So, we will work towards that. So,

Theorem: V Banach A infinitesimal generator of a C_0 semigroup only let $u \in V$. Then we know that t going to $S(t)u$ is continuous and therefore you can $\int_0^t S(\tau)u\, d\tau \in D(A)$. Now, if u is in $D(A)$ then $S(t)u$ is already in $D(A)$ we know that. Now, if u is not in $D(AU)$ is any element of V S t of you will not be in $D(A)$. but the integral of $\int_0^t S(\tau)u\, d\tau \in D(A)$ for all t positive. And so, once it is in $D(A)$. we would like to know what is the action of

$$A\left(\int_0^t S(\tau)u\, d\tau\right) = S(t)u - u.$$

so very beautiful result.

Proof so let $h > 0$ so we just have to check whether $S(t)$ that integral is in this thing. So, by h acting on integral 0 to t s tau of u D tau. Now, what is the property of the integral? How is the integral define it means that every continuous linear functional every continuous linear mapping can pass through the integral sign that is what we have in the definition of the integral itself and built into the definition of integral if you have seen it already.

So, this is equal to integral 0 to t S of tau plus h u minus S tau of u D tau 1 over h. Now, we will make a change of variable tau plus h equals tau. So, this will become h 2 t plus h this integral will become h 2 t plus h then you have S tau of u from t to t plus h minus S tau of u from 0 to t. So, if you simplify this integral you will get integral equal to integral t to t plus h S tau of u d tau minus integral 0 to h S tau of u d tau. So, it is just elementary thing you make a first change variable for the first one make it integral S tau of u and limits will change from h to t plus h. Then you have two different limits and so then you compensate and then you will get this thing.

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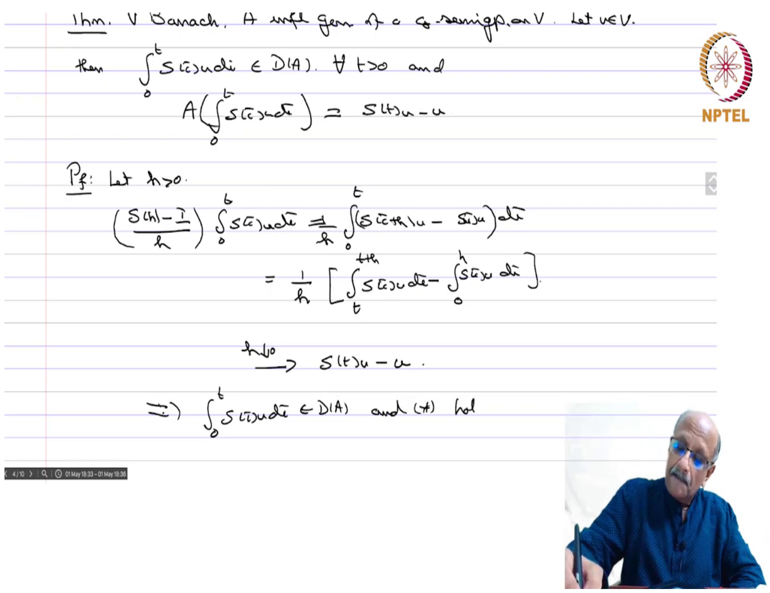

Now, we know the limit of this 1 by h of integral something is nothing but the limit as this goes as h decreases to 0 we know it should go to the lower limit. So, this goes to S t of u minus this will go to the lower limit this is 0 of u which is u. Therefore, A so this implies that integral 0 to t S tau of u d tau belongs to $D(A)$ and star holds.

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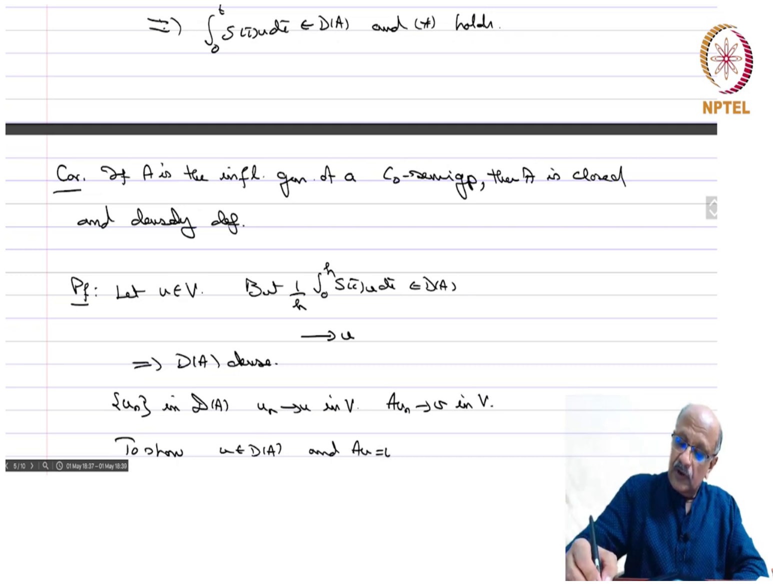

So,

Corollary: if A is the infinitesimal generator of a C_0 semigroup then A is closed and densely defined. So, we have so now already this the importance of closed and densely defined operators comes up because this is a necessary condition is not a sufficient condition for infinitesimal generator to exist to give a C_0 semigroup but we have at least it we have to where do we look for the infinitesimal generators we have to look only amongst closed and densely defined operators.

Proof: so let u$\in V$. So, we want to show the $D(A)$ is dense that means we can approximate u by means of elements of $D(A)$ in this thing. But then we just saw but integral 0 to h S tau of u d tau belongs to $D(A)$. And therefore 1 by h of this also belongs to $D(A)$. And then this converges to u and therefore implies $D(A)$ is dense. So, now we want to show that it is closed so let u n in $D(A)$ such that u n converges to u in v and A u n goes to some v in v. So, to show u belongs to the domain of A and A u equals u so that is the meaning of A B closed.

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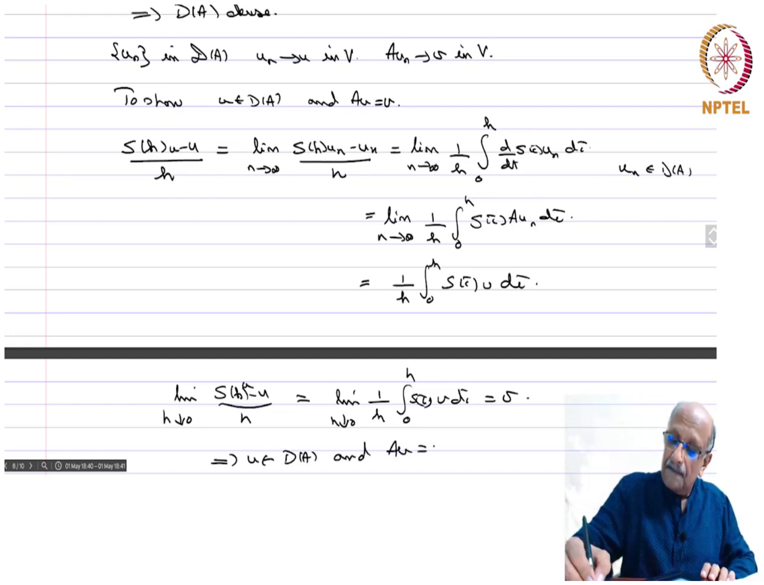

So, we just have to prove this no other way to do S h u minus u by h. So, we have to compute this S h goes to 0 now this is equal to limit as n tends to infinity S h of u n minus u n divided by n h because u n goes to S h is a continuous linear operator and u n goes to u in B. So, this is equal to 1 by h sorry the limit n tending to infinity 1 by h of integral 0 to h d by dt of S tau u n d tau.

Because if you are integrating dy by dt then it is just a difference of the end value. So, S h u n minus S 0 u n is u n. But what is dy by dt f so u n is in $D(A)$ you remember the u n is in $D(A)$ and therefore this equal to limit n tending to infinity 1 by h integral 0 to h of A of sorry s tau of A u n d tau.

But then this limit we know is nothing but 1 by h integral 0 h S tau of A u n goes to B and therefore this equal to d. So, now take the limit. So, limit of S h of u minus u divided by h S h goes to 0 is equal to the limit as h goes to O of 1 by h integral 0 to h s tau of v d tau which is equal to v. And therefore, this implies that u belongs S h of u u belongs to $D(A)$ and A u equals u.

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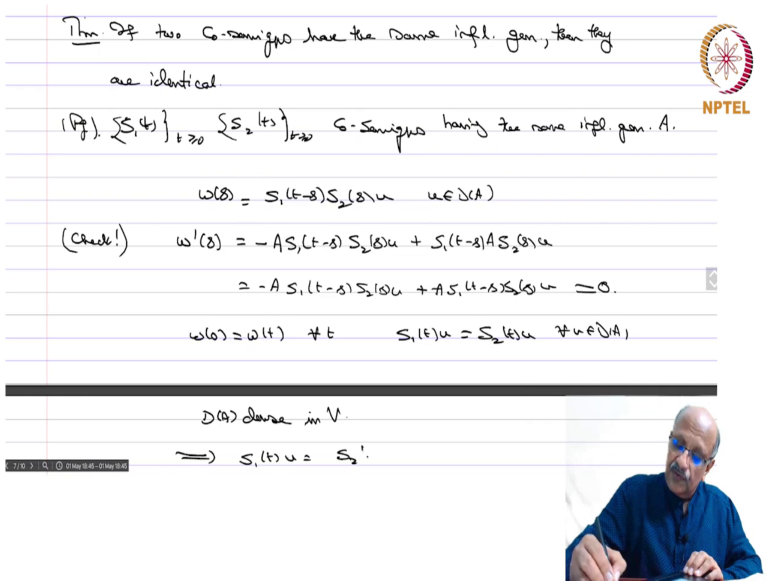

And therefore, it is a closed and densely defined operator. So, we will now presently see what are the conditions are needed for to be satisfied which are necessary and sufficient for unbounded operator to satisfy in order to become the infinitesimal generators of a C semigroup that will be the next thing but before we conclude today so we have one more so if, so

Theorem: If two semi groups have the same infinitesimal generator then they are identical.

So, infinitesimals generator to C_0 semi group is a 1 to 1 correspondence given a C_0 semigroup you have an infinitesimal generator and if you have a unbounded operators which qualifies to be infinitesimal generator for C_0 semigroup then there is only one C_0 semigroup which it can generate it can generate more than one. Therefore, for instance if you have S_1. So,

Proof: $S_1(t)$, $S_2(t)$ are C_0 semigroups having the same infinitesimal generator A.

So, now we set as before w s equals $S_1(t - s)S_2(s)u$, $u \in D(A)$. So, this will be well defined because u is in $D(A)$ $S_2(s)$ will be in $D(A)$. So, $S_1(t - s)$ of that is still define. And then you can differentiate it because it is in $D(A)$. So, this will be equal to if you again we have to check so check again the product rule will apply. So, this will be minus A $S_1(t - s)S_2(s)u + S_1(t - s)$ differentiate A of $S_2(s)$ du.

But A commutes so this equal to -A$S_1(t - s)S_2(s)u + S_2(s) + AS_2(t - s)S_2(s)u = 0$. and therefore that is equal to zero. So, again w is a constant. So, $w(0) = w(t)$ for all t is that is

And then $D(A)$ is dense in V. Because it is a infinitesimal generator and therefore, we have $S_1(t)u = S_1(t)u,\ \forall u \in V$. So, they are one in the same.

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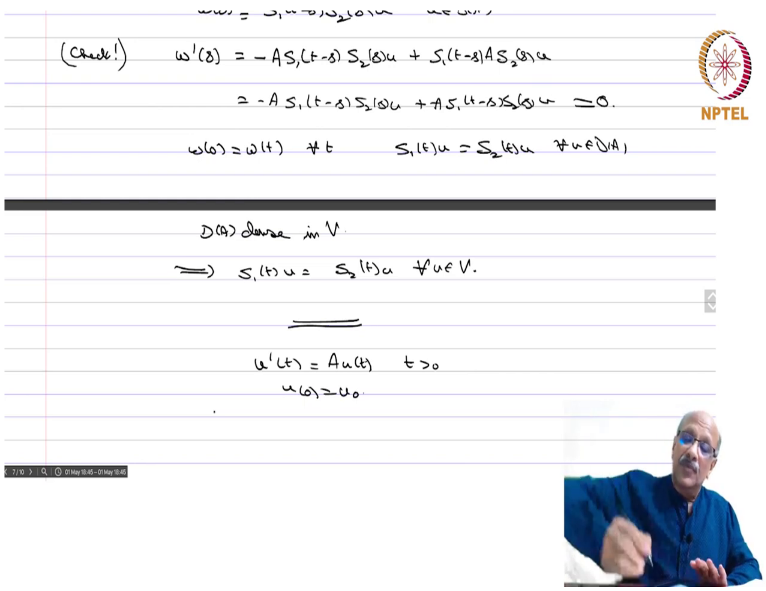

So, this proves that the so we are now seen come to the following state we are interested in solving

$$u'(t) = Au(t),\ \ t > 0$$

$$u(0) = u_0$$

So, if we know that A is infinitesimal generator of a C_0 semigroup $S(t)$ and u_0 is in the domain then we have a classical solution for this namely $u(t) = S(t)Au$ and so we are interested in identifying those unbounded operators which are infinitesimal generators of C_0 semigroups.

And as a starting point we have said the necessary condition for this is that A has to be closed and densely defined. So, we will now explore this thing further. And that will finally lead to the Hille Yosida Sita theorem which completely characterizes those unbounded operators which generates C_0 semi groups.