Sobolev Spaces and Partial Differential Equations
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Lecture 74

C_0 Semigroups- Part 1

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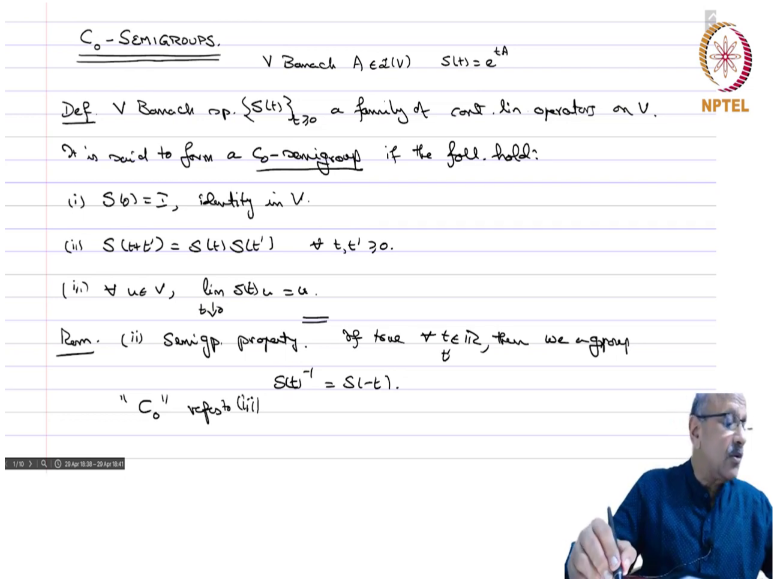

So, we now talk about the core objects of this chapter namely C_0 Semigroups. So, we were looking at V Bamach $A \in L(V)$ bounded linear operators continuous linear operator. And then we were looking at $S(t) = e^{tA}$ which is group of operators so okay. So, now we want to imitate the properties of this in an abstract sense. So, we start with the following definition. So,

Definition: V Bamach space and $\{S(t)\}_{t\geq 0}$ a family of continuous linear operators on V.

It is said to be to form a C_0 semigroup if the following hold

(i) $S(0) = I$;

(ii) $S(t + t') = S(t)\, S(t')$ $for\ all\ t, t' \geq 0$;

(iii) $for\ every\ u \in V, \quad \lim_{t \to 0} S(t)u = u.$

So, if these three properties are satisfied then you have a what is called a C_0 semigroup. So,

Remark: (ii) is called semi group property.

And if it is a group if this is true for all $t \in \mathbb{R}$ if true for all $t \in \mathbb{R}$ then you have $t, t' \in \mathbb{R}$ then we have a group and you have $S(t)$ inverse is $S(-t)$. $S(t)$ will be invertible and its inverse will be $S(-t)$. Now, C_0 refers to property three namely you have some kind of continuity at the limit at t equal to 0.

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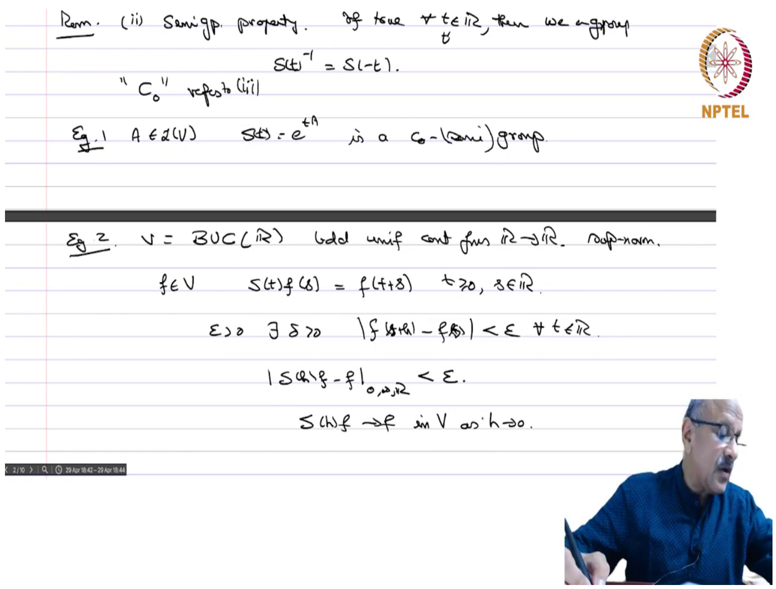

So,

Example 1: So, $A \in L(V)$ bounded continuous linear operator. Then you have $S(t) = e^{tA}$ is a C_0 semigroup I put semi within brackets because it is in fact a group in this case. So, now we will look at a genuine another example.

Example 2: $V = BUC(\mathbb{R})$ so bounded uniformly continuous functions $\mathbb{R}$ to $\mathbb{R}$ with the sup norm.

So, this is a Banach space as you probably know otherwise you can check it and then for $f \in V$. So, we define

$$S(t)(f)(s) = f(t + s), \quad t \geq 0, \quad s \in \mathbb{R}.$$

Then of course $S(0) = I$ that is obvious because $S(0)(f)(s)$ is nothing but f(s) again and then the semigroup properties obvious because if you translate once and then you translate by another number it is a same as translating by $t_1 + t_2$.

And then the continuity comes because you have they are all uniformly continuous functions. Therefore, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(s + h) - f(s)| \leq \varepsilon$$

for all $t \in \mathbb{R}$ or maybe I should put $f(s + h) - f(s)$. And therefore,

$$|S(h)(f) - f|_{0,\infty} \leq \varepsilon.$$

And therefore you have that $S(h)(f) \rightarrow f$ in V the as $h \rightarrow 0$. So, this is an example of a C_0 semigroup.

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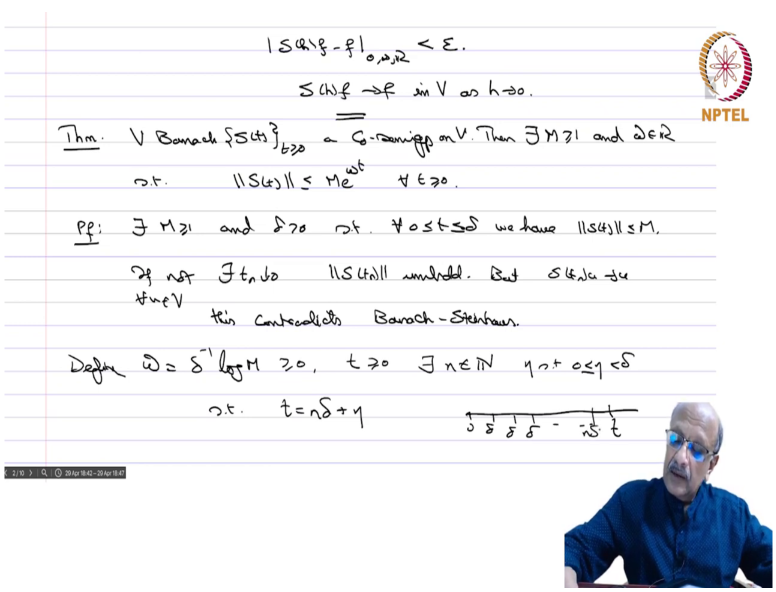

So, now let us quickly start various properties of the semigroup so they are all 1,

Theorem: V Banach and $\{S(t)\}_{t\geq 0}$ a C_0 semigroup on V. Then there exists $M \geq 1$ and ω ∈ ℝ such that

$||S(t)|| \leq Me^{\omega t}, \ \forall, t \geq 0$.

Proof: so there exists an $M \geq 1$ and delta positive such that for all $0 \leq t \leq \delta$, $||S(t)|| \leq M$.

So, if not so $M \geq 1$ without loss of generality, so you have a constant M. And then know if not there exists a $\{t_n\} \to 0$ just that

$||S(t_n)||$ is unbounded. But $S(t_n)u \to u$ for every $u \in V$ that is property three. And therefore by the Banach Stien house so this contradicts Banach Stein house. So, it is pointwise bounded so, it has to be uniformly bounded. So, it goes converges for every u and therefore it is pointwise bounded.

But then $S(t_n)$ is not uniformly bounded and this contradicts. Therefore, there exists an M which you can take bigger than equal to M. So, now you define omega so define $\omega = \frac{1}{\delta} log\, M \geq 0$.. Then given any $t \geq 0$ there exists $n \geq 0$ *and* η *satisfying* $0 \leq \eta < \delta$,

such that $t = n\delta + \eta$.

So, what you are doing is, you are taking you have t here you have zero here so, you go by steps of delta till you come near it. And then t will be finally n delta and then plus an eta so, you go n steps and then you will get here.

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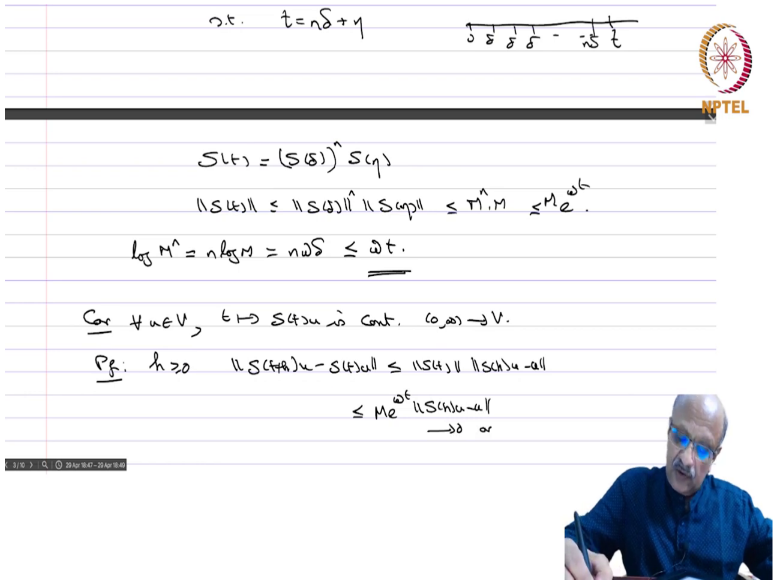

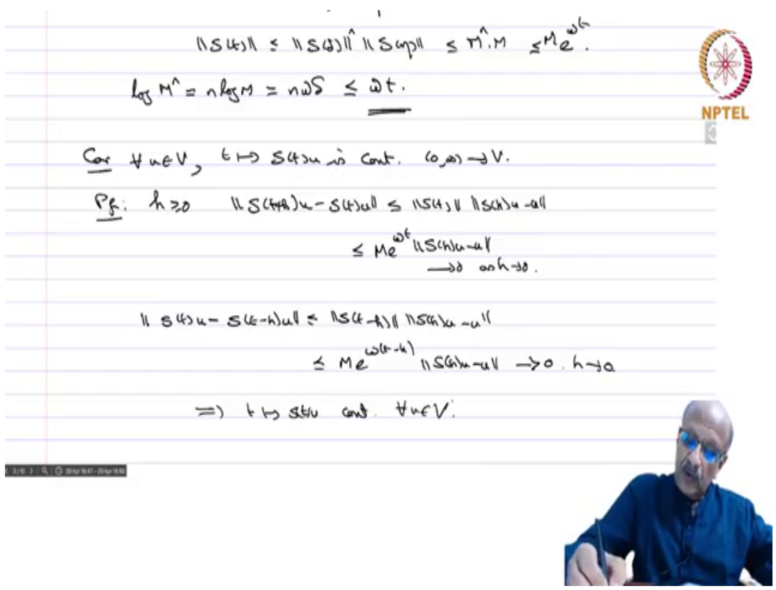

So, now for you have $\{S(t)\}$ by a semigroup property this is nothing but S delta power n times eta. So,

$$||S(t)|| \leq ||S(\delta)||^{n}||S(\eta)|| \leq M^{n}M \leq Me^{\omega t}$$

So, this is less than or equal to $log\ M$ power n is less than or equal to omega t. So, power n is less than e power omega t times M. So, therefore, this proves the theorem.

Corollary: so every $u \in V$ you have $t \rightarrow S(t)u$ is continuous from $[0, \infty)$ into V. So,

Proof: So, let $h \geq 0$,

$$||S(t + h)u - S(t)u|| \leq ||S(t)|| \, ||S(h)u - u|| \leq Me^{\omega t}||S(h)u - u||$$

And that is

and this goes to 0 as h goes to 0. Similarly,

$$||S(t)u - S(t - h)u|| \leq ||S(t - h)|| \, ||S(h)u - u|| \leq Me^{\omega(t-h)}||S(h)u - u||$$

and again that goes to 0 as h goes to 0. So, this proves that in place t going to S t of u continuous for every $u \in V$.

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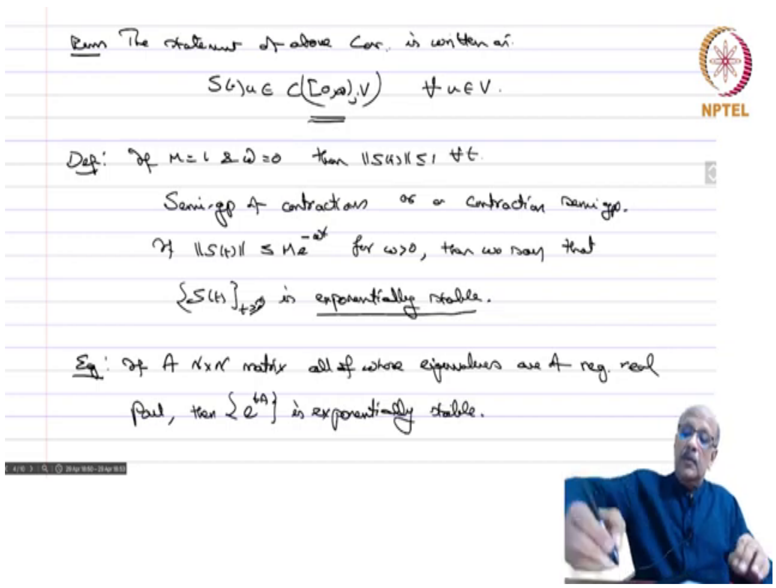

So, we say we symbolically write this remark the statement of the above theorem above

Corollary: is written as

$S(.)u \in C([0, \infty); V)$.

So it is a Banach space continuous function from for every t that means $t \rightarrow S(t)u$ for every u in V. So, this is the statement which we have there. So, definition if $M = 1$ and $\omega = 0$, then $||S(t)|| \leq 1, \ \forall t \geq 0$.

So, this is called a semi group of contractions or a contraction semigroup, if

$||S(t)|| \leq Me^{-\omega t}$, $\omega > 0$.

Then we say that, that is also possible I mean we have showed that it is less than $Me^{\omega t}$ that is a general number which we have got.

But in reality it can happen that can also be less than this. And then we say that $\{S(t)\}$ is exponentially stable. So

Example, if A is a n by n matrix all of whose eigenvalues are of negative real part then e^{tA} is exponentially stable we can prove this. I am just stating it now but we can we will probably see it in the exercises or in the EM assignments.

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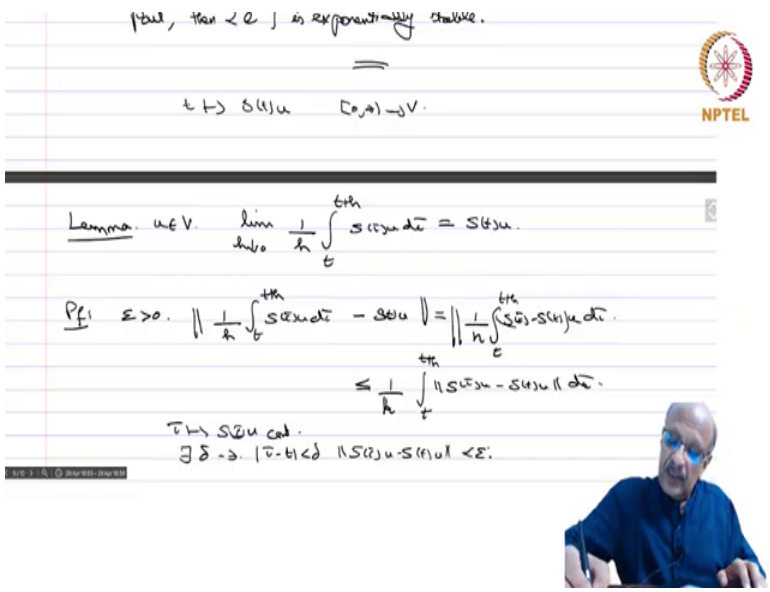

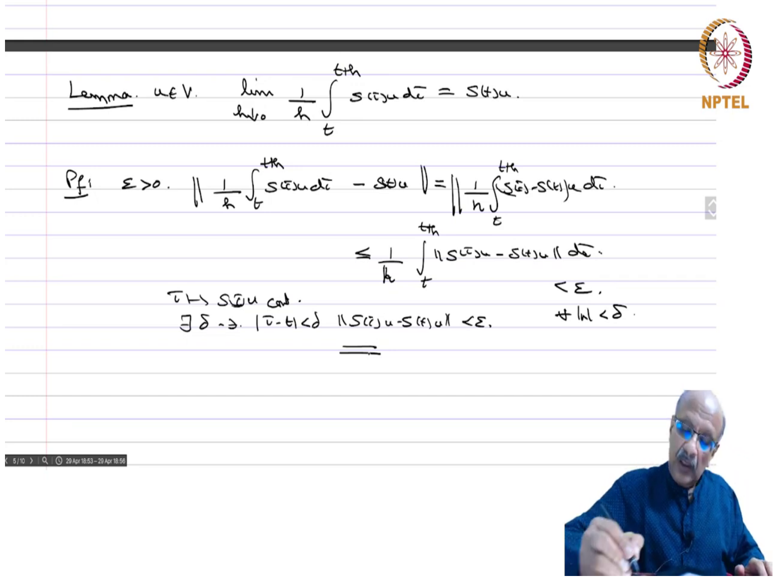

So, since you have the $t \to S(t)u$ is continuous from $[0, \infty)$ into V. Therefore, we can integrate the continuous function with values in a Banach space. So

Lemma: let $u \in V$. Then

$$\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} S(\tau)u \, d\tau$$

is nothing but $S(\tau)u$ the lower end the value at the lower end. So,

Proof. so let $\varepsilon > 0$ be given. So,

$$\left|\left| \frac{1}{h} \int_{t}^{t+h} S(\tau)u \, d\tau - S(t)u \right|\right| = \left|\left| \frac{1}{h} \int_{t}^{t+h} (S(\tau) - S(t)u) d\tau \right|\right|.$$

Now be then this is a constant so you can pull it into the integral. And from the properties of the integral this is

$$\leq \frac{1}{h} \int_{t}^{t+h} ||S(\tau) - S(t)u|| d\tau.$$

rather let us let me not write it that way. So, S tau of u minus S t of u d tau. But you know that t going to S t of u or tau going to S tau of u is continuous.

Therefore, there exists a $\delta > 0$ such that for all

$|h| < \delta$ we have or rather if tau minus t less than delta we have norm of S tau of u minus S t of u is less than epsilon. That is just the definition of continuity and therefore you have that this thing is less than ε and therefore you have for all mod $|h| < \delta$. So, this proves the theorem. So, now we are going to introduce a very important concept.

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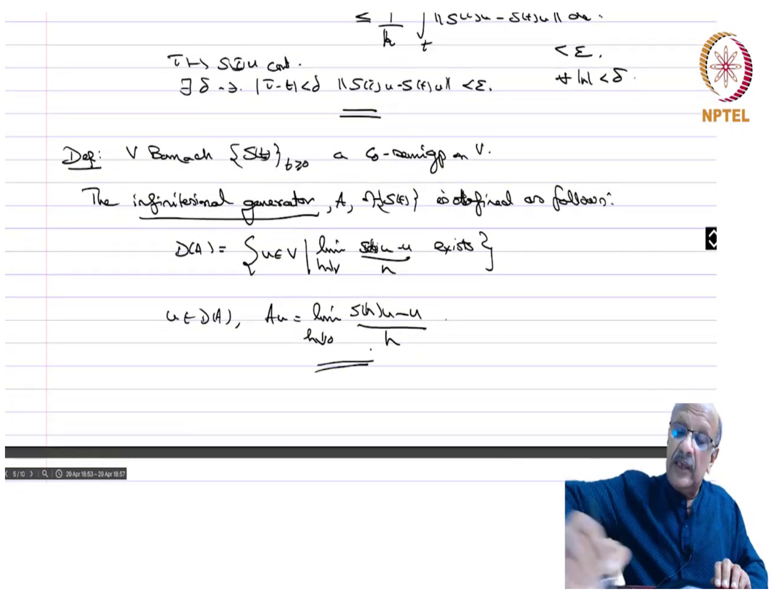

So,

Definition, so V Banach $\{S(t)\}_{t\geq 0}$ a C_0 semigroup, the **Infinitesimal generator** A of $\{S(t)\}_{t\geq 0}$ is defined as follows. So, $D(A)$ the domain of A it is an unbounded operator which we will see. So, this is set of all $u \in V$ such that

$$D(A) = \left\{ u \in V : \lim_{h \to 0} \frac{S(h)u - u}{h} \ exists \right\}$$

and if $u \in D(A)$ we have

$$Au = \lim_{h \to 0} \frac{S(h)u - u}{h} = D^{+} f(s)$$

So, this is the definition of the infinitesimal generator which is which will play a very key role in the theory, so the very important definition.

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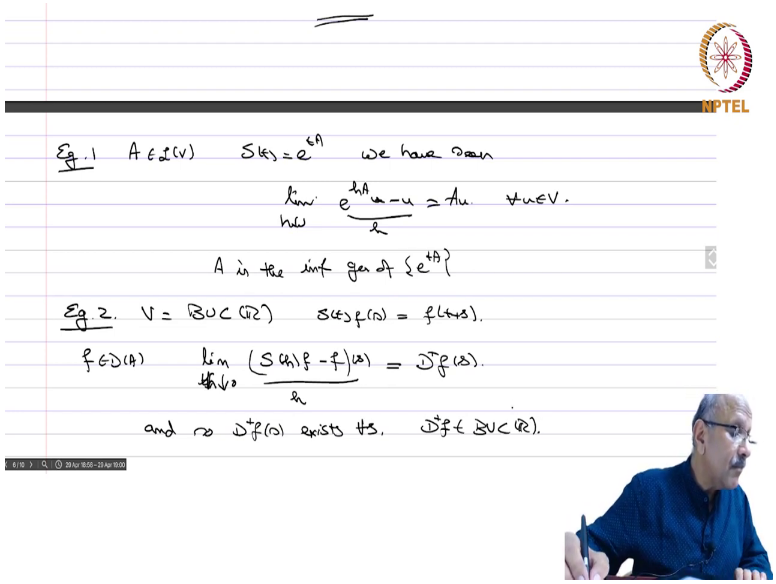

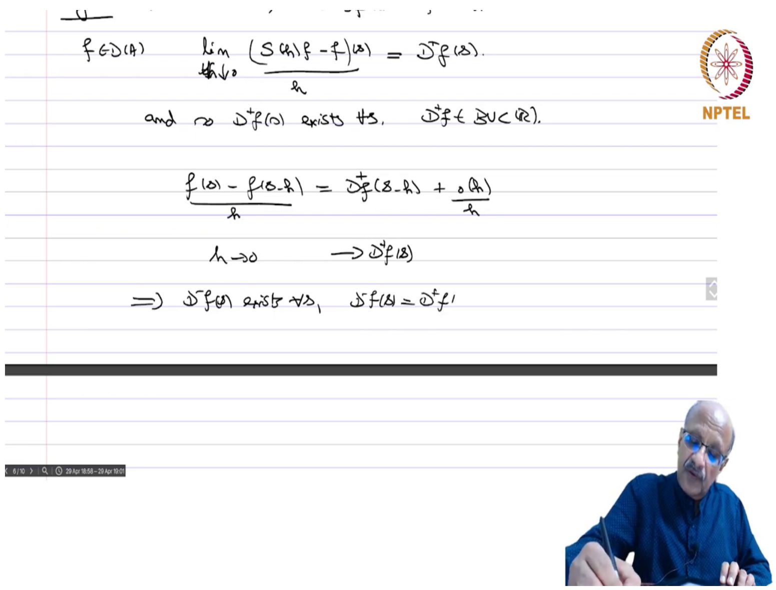

Example 1: So, you have $A \in L(V)$ bounded linear operator and you have

$S(t) = e^{tA}$.

Then we have seen

$$\lim_{h \to 0} \frac{e^{hA}u - hu}{h} = Au, \ \ \forall u$$

we did this competition yesterday. And therefore we have that A is the infinite decimal generator. So, this is true for all u in V of e^{tA}. And in fact it is a bounded linear operator.

So, in this case, it is not just an unbounded A defined everywhere. So, now let us look at another the other

Example. So, let us like V as usual bounded uniformly continuous functions on $\mathbb{R}$ and

S t f at s is f of t plus s the translation semigroup. So, you translate by t is a semigroup.

So, then suppose $f \in D(A)$ that means

$\lim_{h \to 0} \frac{S(h)f - f}{h}$ should exist. And this but we know if the limit exists this is $D^{+}f(s)$ the derivative one sided derivative. And so $D^{+}f(s)$ exists for all s and you have $D^{+}f$ belongs to BUC it is a bounded uniformly continuous function because we know that the limit should.

Now you look at

$$\frac{f(s) - f(s-h)}{h} = D^{+}f(s - h) + \frac{o(h)}{h}$$

by this little o we mean that the numerator goes to 0 this quotient goes to 0 as h goes to 0 so little $\frac{o(h)}{h}$. And so, as h goes to 0 this will converge to $D^{+}f(s)$ because $D^{+}f$ is a continuous in fact bounded uniformly continuous and this part go has to go to 0. So, that means $D^{-}f(s)$ exists for all s and $D^{+}f(s) = D^{-}f(s)$

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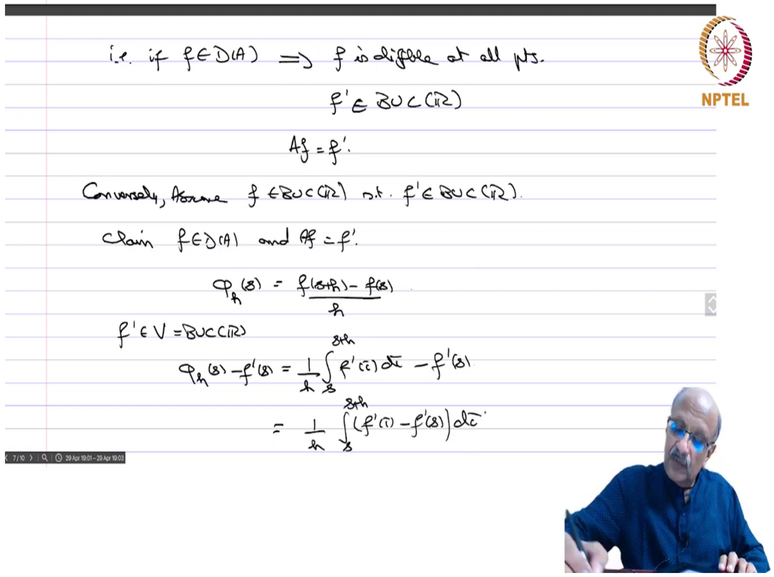

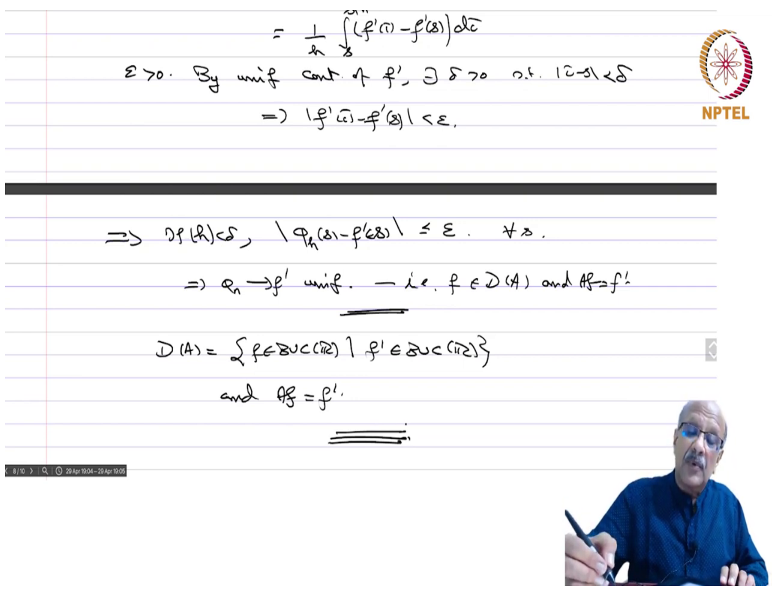

Therefore, that is, if f belongs to the infinite domain of the infinite decimal generator then we have the f is differentiable at all points and the $f^{'} \in BUC(\mathbb{R})$ and you have A of f is nothing but f dash so this is the not it sorry so in fact okay $Af = f^{'}$. So, conversely assume $f \in BUC(\mathbb{R})$ such that $f^{'} \in BUC(\mathbb{R})$. Then we claim f s in D(A) and $Af = f^{'}$.

So, for that we denote

$\varphi_h(s) = \frac{f(s+h)-f(s)}{h}$.

Now, $f' \in V$ which is $BUC(\mathbb{R})$ bounded uniformly continuous functions in $\mathbb{R}$ and therefore

$$\varphi_h(s) - f'(s) = \frac{1}{h}\int_s^{s+h} [f'(\tau) - f'(s)]\, d\tau$$

this fundamental theorem of calculus Because $f'(s)$ is a constant as far as this integration is concerned.

Now, if you if you have $\varepsilon > 0$ by uniform continuity of f' there exists a $\delta > 0$ such that

$|\tau - h| < \delta$,

implies $|f'(\tau) - f'(s)| < \varepsilon$ this is uniform continuity. And therefore this means that if

$|h| < \delta$,

we have

$$|\varphi_h(s) - f'(s)| = |\frac{1}{h}\int_s^{s+h} [f'(\tau) - f'(s)]\, d\tau| \leq \varepsilon$$

you pull it out and then you have $\int_s^{s+h} [f'(\tau) - f'(s)]\, d\tau$ so, the whole thing is less than ε, for all s.

That means $\varphi_h \to f'$ uniformly that means but that is exactly saying that this $f \in D(A)$ and $Af = f'$.

Therefore, you have that $D(A)$ equals set for f in $BUC(\mathbb{R})$ such that f dash belongs to be $BUC(\mathbb{R})$. And $Af = f'$. so this is the infinitesimal generator in this case we will see lot more about infinitesimal generators subsequently and we will investigate the properties of the infinitesimal generator next.