Sobolev Spaces and Partial Differential Equations
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Lecture 73
The Exponential Map

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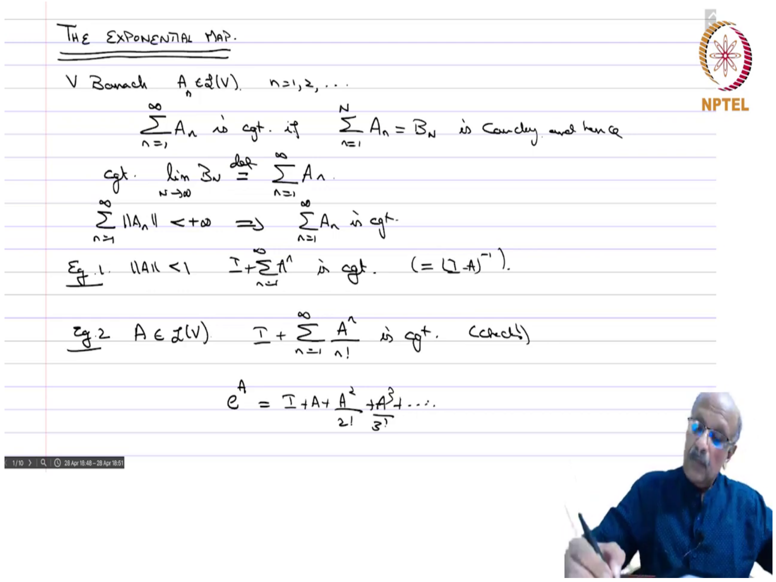

We will now discuss the exponential map. So, let us take V a Banach space and $A_n \in L(V),\ n = 1, 2, 3...$

$\sum_{n=1}^{\infty} A_n$ is said to be is convergent if $\sum_{n=1}^{N} A_n = B_N$ is Cauchy and hence convergent.

So, $\lim_{n \to \infty} B_N = \sum_{n=1}^{\infty} A_n$. Now, if $\sum_{n=1}^{\infty} ||A_n|| < \infty,\ \sum_{n=1}^{\infty} A_n$ is convergent.

example 1. you have $||A|| < 1,\ I + \sum_{n=1}^{\infty} A^n$ is convergent. And in fact it is equal to the sum is equal to $(I - A)^{-1}$, This is called the Neumann series.

example 2: The second example is a exponential series this is for any $A \in L(V)$, you have that

$I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$ is convergent: check ! all you have to do is to show that sigma norm An is convergent and then we call the limit so $e^A = I + A + \frac{A^2}{2!} + ...$

So, this is called the exponential of a bounded linear operator.

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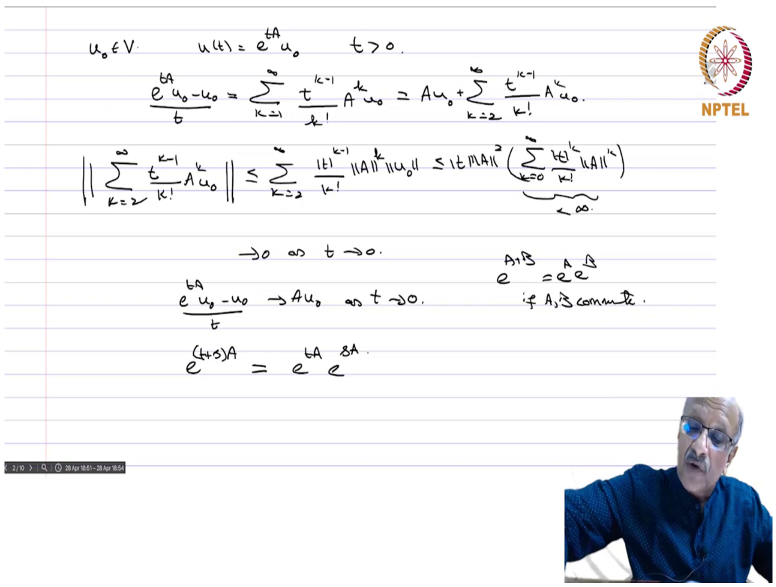

So, now, we will let $u_0 \in V$, $u(t) = e^{tA} u_0$, $t > 0$. So, let us look at the limit of this

$$\frac{e^{tA} u_0 - u_0}{t} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} A^k u_0 = Au_0 + \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} A^k u_0$$

So, now if you look at the term which is remaining there that is

$$|| \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} A^k u_0 || \leq \sum_{k=2}^{\infty} \frac{|t|^{k-1}}{k!} ||A||^k ||u_0|| \leq |t| \, ||A||^2 \sum_{k=0}^{\infty} \frac{|t|^{k-1}}{k!} ||A||^k \rightarrow 0 \; as \; t \rightarrow 0.$$

And therefore, $\frac{e^{tA} u_0 - u_0}{t} \rightarrow Au_0 \; as \; t \rightarrow 0$.

Now $e^{(t+s)A} = e^{tA} . e^{sA}$, It is easy to check.

In fact e power A plus B equals e power A e power B if A B commute this is important otherwise it is not true. So, t n they say always commute and therefore there is no problem. So this commutes and therefore you have this. So, this you will just check this exactly as in the exponential series of a real variable so there is nothing really different. And therefore you will have this u the same proof.

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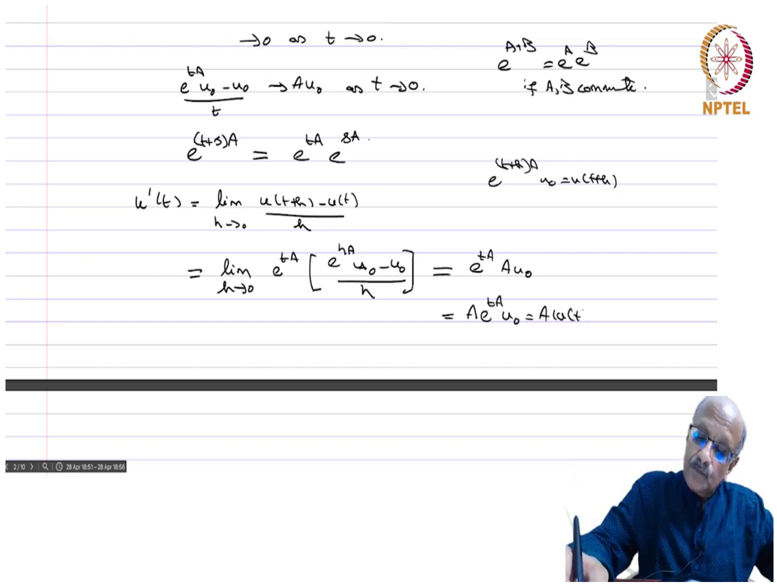

So now, let us take $u'(t) = \lim_{h \to 0} \frac{u(t+h)-u(t)}{h} = \lim_{h \to 0} e^{tA}[\frac{e^{hA}u_0 - u_0}{h}] = e^{tA}Au_o = Ae^{tA}u_0 = Au(t)$.

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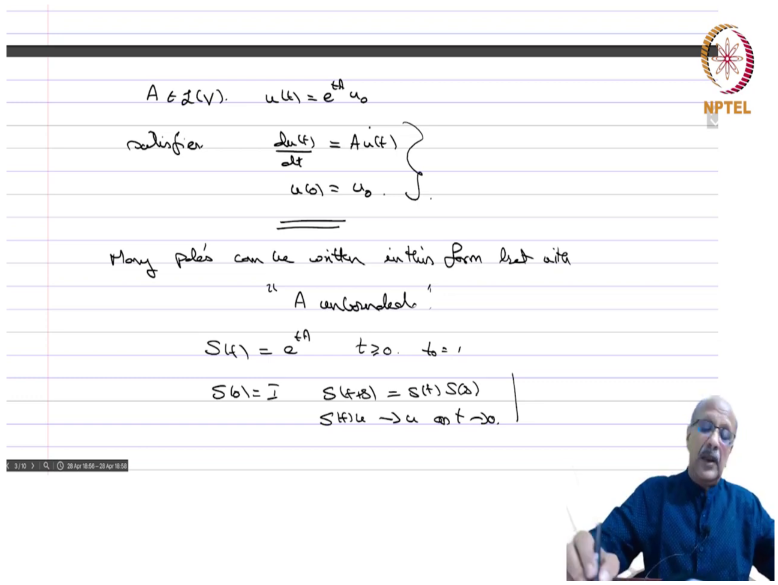

Therefore, if $A \in L(V)$, $u(t) = e^{tA}u_0$ satisfies

$$\frac{du(t)}{dt} = Au(t),\ u(0) = u_0.$$

So, it is a solution to this initial value problem. So, given an initial value problem in a Banach space of du by dt equals A u t where A is a bounded linear operator. Then we can immediately write down the solution and this solution will be unique and this is nothing but equals e power t A times u naught.

Now many PDE's can be written in this form but with A unbounded we will be able to write the heat equation the wave equation then many such pde's evolution equations in this form du by dt equals A u and u 0 equals u at 0 equals u naught. But then A will be unbounded. So, we want to investigate how to handle this situation. And that is where we introduce the generalization of this.

So, if you write $S(t) = e^{tA}$, $t \geq 0$, $S(0) = I$, $S(t+s) = S(t)S(s)$, $S(t)u \rightarrow u\ as\ t \rightarrow 0$.

So, these properties we will generalize to what is called a semigroup C 0 so it says collection of bounded linear operators with some properties and that will be connected to an unbounded operator and we will look at the solution of such differential equations with respect to this unbounded operator. So that is what we plan to do next?