


Sobolev Spaces and Partial Differential Equations
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Lecture 72
Unbounded Operators – Part 2

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V Hilbert (real) $A: D(A) \subset V \rightarrow V$ $A^*: D(A^*) \subset V \rightarrow V$
 $A = A^*$ ($D(A) = D(A^*)$, A symmetric) A is self-adjoint.


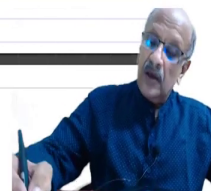
Ex 1 $V = W = L^2(\Omega, \mathbb{R})$ $D(A) = H_0^1(\Omega)$ $Au = u'$ $u \in D(A)$.
 $\Omega = (0, 1)$.
Exercise A closed, $u_n \rightarrow u$ in $L^2(\Omega)$ $u'_n \rightarrow v$ in $L^2(\Omega) \Rightarrow u \in H_0^1(\Omega)$ $u' = v$.
 A densely def.

$D(A^*) = \{ u \in L^2(\Omega) \mid \exists u' \text{ s.t. } \int_0^1 uu' \leq C \|u\|_{L^2} \forall u \in H_0^1(\Omega) \}$.
 $u \in D(A^*) \Rightarrow \int_0^1 u \varphi' \leq C \|\varphi\|_{L^2} \forall \varphi \in D(A)$.
 (cf. earlier Exercise) $\Rightarrow u \in H^1(\Omega)$.
 Conversely $u \in H^1(\Omega)$ $\int_0^1 u u' dx = - \int_0^1 u' u dx \quad \forall u \in H_0^1(\Omega) \cap D(A)$
 $\Rightarrow u \in D(A^*)$




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 $\Rightarrow u \in D(A^*)$.
 $D(A^*) = H^1(\Omega)$ $A^* u = -u'$.

We were discussing the adjoint of a map and in case V is a Hilbert space. Then if $A: D(A) \subset V \rightarrow V$, $A^*: D(A^*) \subset V \rightarrow V$. And if you have that $A = A^*$, that means $D(A) = D(A^*)$, A is symmetric as I said then you say A is self-adjoint.

So, now we will see a couple of examples for the adjoint. So, first example.

Example. so we go back to the examples which we looked at last time. So, $V = W = L^2(0, 1)$

$$D(A) = H_0^1(\Omega), \Omega = (0, 1), Au = u', u \in D(A).$$

So, A , so there was one erratum in the previous lecture. So, we have that A is closed. So, then I said if u_n goes to u in L^2 of Ω and u_n' goes to v in L^2 Ω .

Then this implies we have seen this before $u \in H_0^1(\Omega)$, yesterday, last time I mistakenly wrote it as $L^2(\Omega)$, so please correct it. And you have $u' = v$. So, this is how we prove that A is closed. So, we now go back to this example.

So, A is densely defined and therefore the adjoint can be defined. So,

$$D(A^*) = \{v \in L^2(\Omega): |\int_0^1 u'v| \leq c|u|_{0,\Omega} \forall u \in H_0^1(\Omega)\}.$$

$$\text{So if } v \in D(A^*) \Rightarrow |\int_0^1 \phi'v| \leq c|\phi|_{0,\Omega} \forall \phi \in H_0^1(\Omega).$$

And then we have seen, comparing earlier exercise on Sobolev spaces, that this implies that $v \in H^1(\Omega)$.

$$\text{Conversely if } v \in H^1(\Omega), \text{ then you have } \int_0^1 u'v = - \int_0^1 uv'$$

And therefore this implies that $v \in D(A^*)$. So, we have shown $D(A^*) = H^1(\Omega)$.

and from this condition here this implies that $A^* v = -v'$. So, this is the example.

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$\text{Ex 2. } V = W = L^2(\Omega) \quad D(A) = H^1_0(\Omega) \cap H^1(\Omega) \quad Au = \Delta u.$

$\Omega \subset \mathbb{R}^n$ bounded open set of class C^0

$u, v \in D(A) \quad \int_{\Omega} u \Delta v = - \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} \Delta u \cdot v$

$\left| \int_{\Omega} \Delta u \cdot v \right| \leq C \|u\|_{0,\Omega} \|v\|_{0,\Omega} \Rightarrow v \in D(A^*) \Rightarrow D(A) \subset D(A^*)$



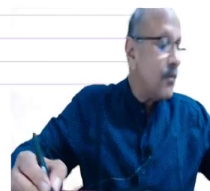
$(*) \Rightarrow A^*$ symmetric.

Claim. A is self-adjoint.
 To show $D(A^*) \subset D(A)$.

$f \in L^2(\Omega), u \in H^1_0(\Omega)$ unique soln,

$-\Delta u + u = f \text{ in } \Omega$
 $u = 0 \text{ on } \Gamma = \partial\Omega.$

$u = Gf, \quad G \text{ is cont. } L^2(\Omega) \rightarrow H^1_0(\Omega) \subset L^2(\Omega)$
 $G \text{ is symm} \Rightarrow \text{self-adj.}$

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$u = Gf, \quad G \text{ is cont. } L^2(\Omega) \rightarrow H^1_0(\Omega) \subset L^2(\Omega)$
 $G \text{ is symm} \Rightarrow \text{self-adj.}$

$\forall f \in L^2(\Omega) \quad (I - A)^{-1}$ has a unique soln $\Rightarrow \mathcal{R}(I - A) = L^2(\Omega)$.

$G = (I - A)^{-1}$ cont. lin op.


So, now let us go to the other example which we looked at last time so example 2.

Example 2. So, now again $V = W = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. So, $\Omega \subset \mathbb{R}^N$ bounded open set of class C^0 . And then $Au = \Delta u$. And again this is a densely defined and closed operator and therefore we now want to compute its adjoint.

So, let $u, v \in D(A)$, then you have

$$\int u \Delta v = \int \nabla u \cdot \nabla v = \int \Delta u \cdot v \quad (*)$$

So, $|\int \Delta u \cdot v| \leq c|u|_{0,\Omega} \Rightarrow v \in D(A^*) \Rightarrow D(A) \subset D(A^*)$.

And also by this relation, the symmetric relationship in place that A is also symmetric for all these terms, star implies A is symmetric. Therefore, you have that.

so we claim A self-adjoint, therefore to show $D(A^*) \subset D(A)$. So, that is what we have to show. So, let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$, the unique solution of

$$-\Delta u + u = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \Gamma = \partial\Omega. \quad (*)$$

So, u denote as usual as G of f, then we know that G is continuous $L^2(\Omega)$ into $H_0^1(\Omega)$ of Ω back into $L^2(\Omega)$. So, it is continuous and of course, G is also self adjoint. We have seen this in the connection of the eigenvalue problem and so on. So, we have, we know that it is symmetric and therefore it is, G is symmetric in place self-adjoint.

Now for every f in $L^2(\Omega)$ dagger has a unique solution. Therefore, it implies that $R(I - A) = L^2(\Omega)$, and $G = (I - A)^{-1}$ is a continuous linear operator.

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Let $u \in D(A^*)$. Set $f = u - A^*u$.

Let $w \in V$ arbitrary $v = Gw$. $(I - A)v = w$.

$$(f, v) = (u - A^*u, v) = (u, (I - A)v) = (u, w).$$


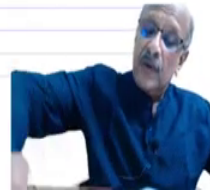
G self-adj.

$$(Gf, w) = (f, Gw) = (f, v) = (u, w).$$

$\forall w \in H_0^1(\Omega)$ $(Gf, w) = (u, w)$ $H_0^1(\Omega)$ dense in $L^2(\Omega)$

$$Gf = u \quad \cdot \quad Gv.$$

But $R(G) \subset H_0^1(\Omega) \cap H^2(\Omega)$

Let $w \in V$ arbitrary $v = Gw$. $(I - A)v = w$.

$$(f, v) = (u - A^*u, v) = (u, (I - A)v) = (u, w).$$

G self-adj.


$$(Gf, w) = (f, Gw) = (f, v) = (u, w).$$

$\forall w \in H_0^1(\Omega)$ $(Gf, w) = (u, w)$ $H_0^1(\Omega)$ dense in $L^2(\Omega)$

$$Gf = u \quad \cdot \quad Gv.$$

But $R(G) \subset H_0^1(\Omega) \cap H^2(\Omega) = D(A)$.

$$\Rightarrow u \in D(A) \quad \text{ i.e. } \quad D(A^*) \subset D(A)$$

$$\Rightarrow A \text{ is}$$



So, now let $u \in D(A^*)$, and you set $f = u - A^*u$, let $w \in V$ be arbitrary and $v = G(w)$. So, $(I - A)v = w$. So, now this is the inner product in the L^2 . So,

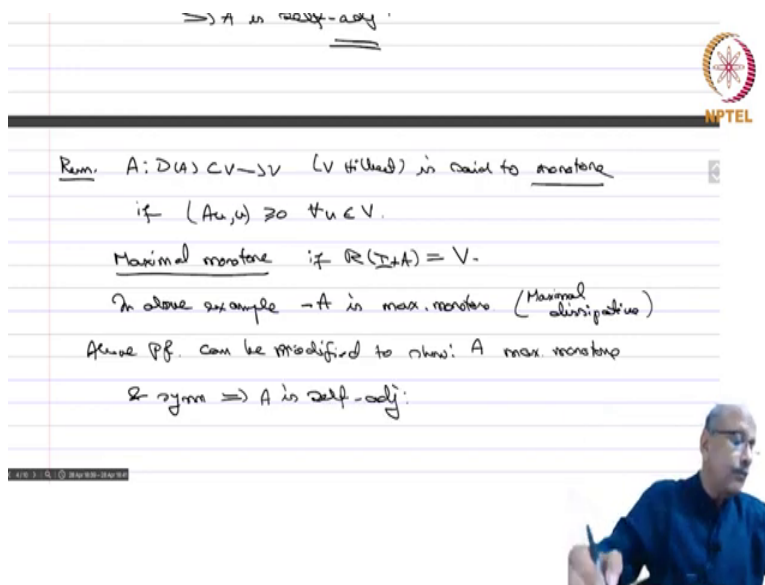
$$(f, v) = (u - A^*u, v) = (u, (I - A)v) = (u, w).$$

Now, G is self-adjoint, therefore $(Gf, u) = (f, Gw) = (f, v) = (u, w)$.

So, for every $w \in H_0^1(\Omega)$, you have $(Gf, w) = (u, w)$, H_0^1 is dense in L^2 . So, you have $Gf = u$. But $R(G) \subset H_0^1(\Omega) \cap H^2(\Omega) = D(A)$. So $u \in D(A) \Rightarrow D(A^*) \subset D(A)$.

Therefore, this implies that A is self adjoint.

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$\Rightarrow A$ is max-adj.

Rem. $A: D(A) \subset V \rightarrow V$ (V Hilbert) is said to be monotone if $(Au, u) \geq 0, \forall u \in V$.

Maximal monotone if $R(I+A) = V$.

In above example $-A$ is max. monotone (Maximal dissipative)

Above pf. can be modified to show: A max. monotone

& sym $\Rightarrow A$ is self-adj.

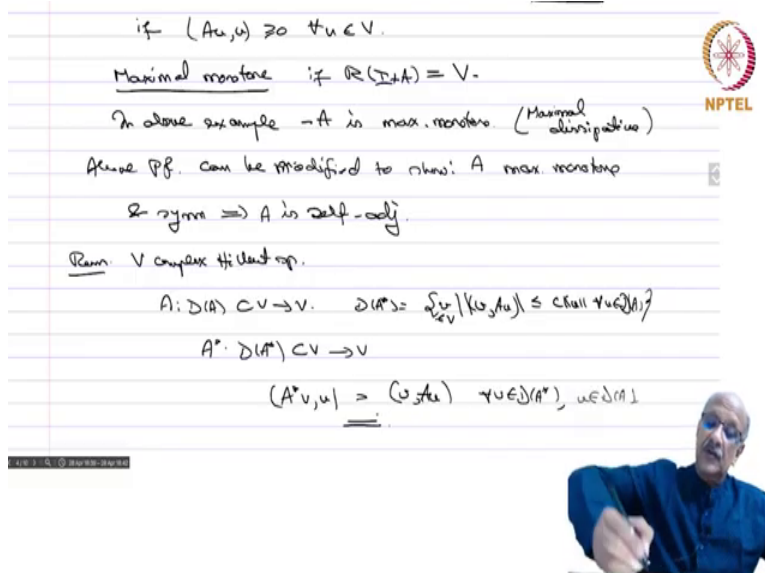
So, remark.

Remark: $A: D(A) \subset V \rightarrow V$, V Hilbert, is said to be monotone if $(Au, u) \geq 0, \forall u \in V$.

And then it is said to be maximal monotone if $R(I + A) = V$.

Now in the above example $-A$ is maximal monotone (say maximal dissipative). So, the above proof can be modified to show A maximum monotone. And symmetric $\Rightarrow A$ is self-adjoint.

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if $(Au, u) \geq 0 \quad \forall u \in V.$

Maximal monotone if $R(I+A) = V.$

In above example $-A$ is max. monotone (Maximal dissipative)

Above pf. can be modified to show: A max. monotone

A sym $\Rightarrow A$ is self-adj.

Defn. V complex Hilbert sp.

$A: D(A) \subset V \rightarrow V. \quad D(A^*) = \{v \in V : |(v, Au)| \leq c||u||, \forall u \in D(A)\}$

$A^*: D(A^*) \subset V \rightarrow V$

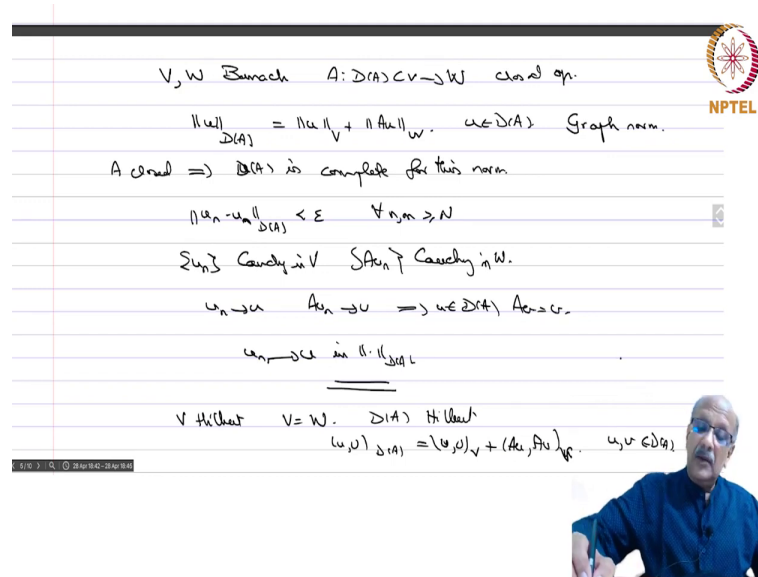
$(A^*v, u) = (v, Au) \quad \forall u \in D(A^*), u \in D(A)$

Remark: If V is a complex Hilbert space we can still do this complex Hilbert space. So,

$$A: D(A) \subset V \rightarrow V, D(A^*) = \{v \in V : |(v, Au)| \leq c||u||, \forall u \in D(A)\}, A^*: D(A^*) \subset V \rightarrow V,$$

$$(A^*v, u) = (v, Au), \forall u \in D(A^*), u \in D(A).$$

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V, W Banach $A: D(A) \subset V \rightarrow W$ closed op.
 $\|u\|_{D(A)} = \|u\|_V + \|Au\|_W, u \in D(A)$ Graph norm.
 A closed $\Rightarrow D(A)$ is complete for this norm.
 $\|u_n - u_m\|_{D(A)} < \epsilon \quad \forall n, m \geq N$
 $\{u_n\}$ Cauchy in V $\{Au_n\}$ Cauchy in W .
 $u_n \rightarrow u \quad Au_n \rightarrow v \Rightarrow u \in D(A), Au = v$.
 $u_n \rightarrow u$ in $\|\cdot\|_{D(A)}$.
 V Hilbert $V = W$. $D(A)$ Hilbert.
 $(u, v)_{D(A)} = (u, v)_V + (Au, Av)_W, u, v \in D(A)$.

And finally V and W Banach and $A: D(A) \subset V \rightarrow W$ closed operator. Then we define the norm

$$\|u\|_{D(A)} = \|u\|_V + \|Au\|_W, u \in D(A) \text{ - graph norm.}$$

Then A closed implies that $D(A)$ is complete for this norm. So, you take any Cauchy sequence

$$\|u_n - u_m\|_{D(A)} < \epsilon, \forall n, m \geq N$$

therefore then $\{u_n\}$ Cauchy in V and $\{Au_n\}$ is Cauchy in W . And therefore $u_n \rightarrow u$ and $Au_n \rightarrow v \in W \Rightarrow u \in D(A)$ and $Au = v$ and therefore the Cauchy sequence $u_n \rightarrow u$ in $\|\cdot\|_{D(A)}$.

And V is Hilbert and $V = W$ then you know that $D(A)$ is also Hilbert with the inner product

$$(u, v)_{D(A)} = (u, v)_V + (Au, Av)_W, u, v \in D(A).$$

So, this becomes an inner product which will give the above norm the graph norm and that completes. So, this is a rapid revision of unbounded operators.