Sobolev Spaces and Partial Differential Equations
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Lecture 72
Unbounded Operators – Part 2

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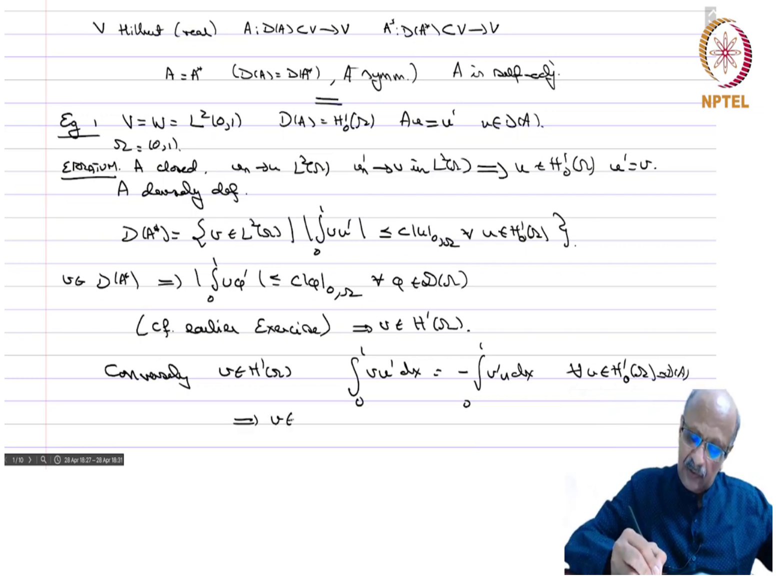

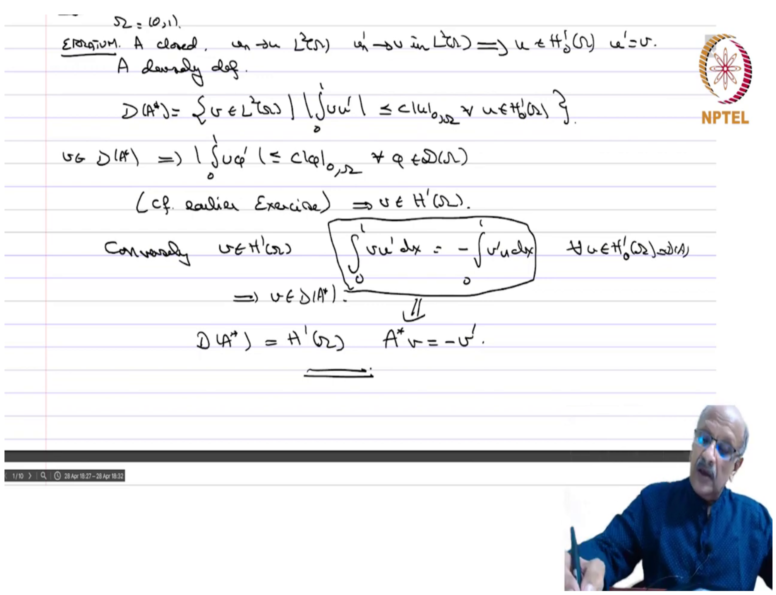

We were discussing the adjoint of a map and in case V is a Hilbert space. Then if $A: D(A) \subset V \rightarrow V,\ A^{*}: D(A^{*}) \subset V \rightarrow V$. And if you have that $A = A^{*}$, that means $D(A) = D(A^{*})$, A is symmetric as I said then you say A is self-adjoint.

So, now we will see a couple of examples for the adjoint. So, first example.

Example. so we go back to the examples which we looked at last time. So, $V = W = L^{2}(0, 1)$

$D(A) = H^{1}_{0}(\Omega),\ \Omega = (0, 1),\ Au = u',\ u \in D(A).$

So, A, so there was one erratum in the previous lecture. So, we have that A is closed. So, then I said if u n goes to u in L2 of omega and u n dash goes to v in L2 omega.

Then this implies we have seen this before $u \in H^{1}_{0}(\Omega)$, yesterday, last time I mistakenly wrote it as $L^{2}(\Omega)$, so please correct it. And you have $u' = v$. So, this is how we prove that A is closed. So, we now go back to this example.

So, A is densely defined and therefore the adjoint can be defined. So,

$$D(A^{*}) = \{v \in L^{2}(\Omega) : |\int_{0}^{1} u'v| \leq c|u|_{0,\Omega}\ \forall\, u \in H^{1}_{0}(\Omega)\}.$$

So if $v \in D(A^{*}) \Rightarrow |\int_{0}^{1} \phi' v| \leq c|\phi|_{0,\Omega}\ \forall\, \phi \in H^{1}_{0}(\Omega)$.

And then we have seen, comparing earlier exercise on Sobolev spaces, that this implies that $v \in H^{1}(\Omega)$.

Conversely if $v \in H^{1}(\Omega)$, then you have $\int_{0}^{1} u'v = -\int_{0}^{1} uv'$

And therefore this implies that $v \in D(A^{*})$. So, we have shown $D(A^{*}) = H^{1}(\Omega)$.

and from this condition here this implies that $A^{*}v = -v'$. So, this is the example.

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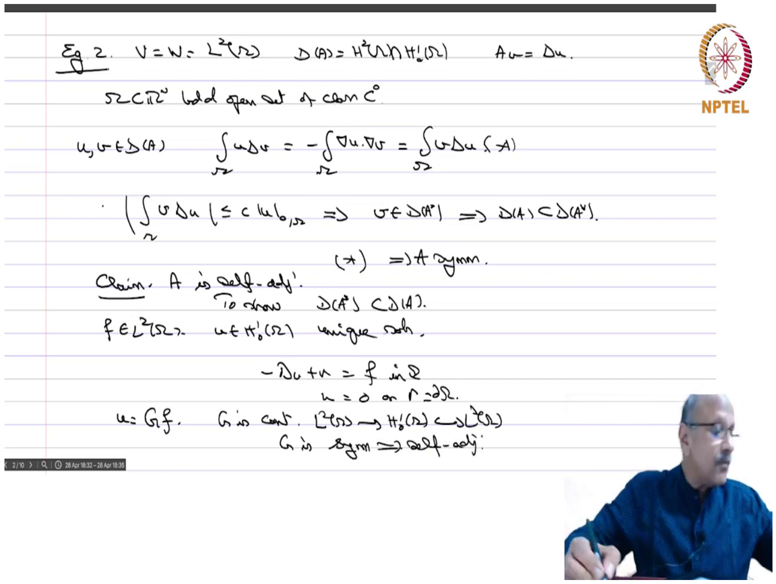

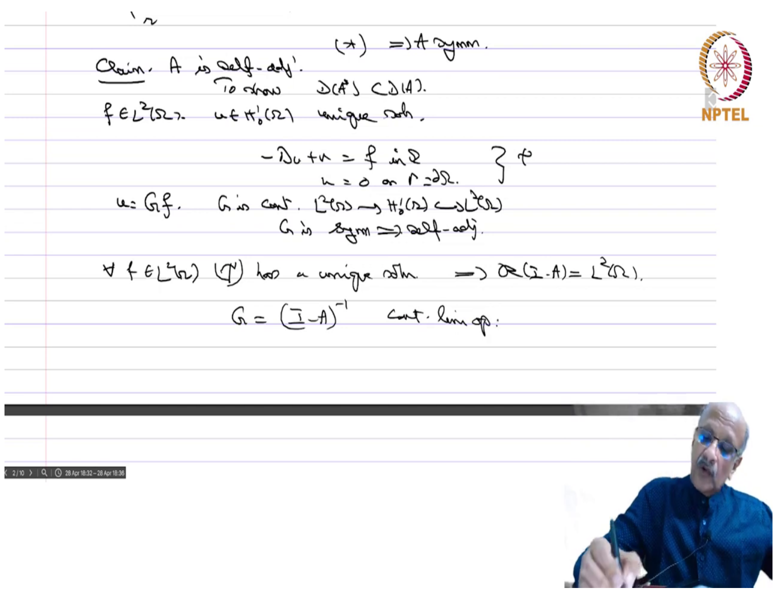

So, now let us go to the other example which we looked at last time so example 2.

Example 2. So, now again $V = W = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H^1{}_0(\Omega)$. So, $\Omega \subset \mathbb{R}^N$ bounded open set of class C^0. And then $Au = \Delta u$. And again this is a densely defined and closed operator and therefore we now want to compute its adjoint.

So, let $u, v \in D(A)$, then you have

$$\int u\Delta v = \int \nabla u . \nabla v = \int \Delta u.\ v \qquad (*)$$

So, $|\int \Delta u.\ v| \leq c|u|_{0,\Omega} \Rightarrow v \in D(A^{*}) \Rightarrow D(A) \subset D(A^{*})$.

And also by this relation, the symmetric relationship in place that A is also symmetric for all these terms, star implies A is symmetric. Therefore, you have that.

so we claim A self-adjoint, therefore to show $D(A^{*}) \subset D(A\ )$. So, that is what we have to show. So, let $f \in L^{2}(\Omega)$ and $u \in H^{1}{}_{0}(\Omega)$, the unique solution of

$$-\ \Delta u + u = f \ in\ \Omega,$$

$$u = 0\ on\ \Gamma = \partial\Omega. \quad (*)$$

So, u denote as usual as G of f, then we know that G is continuous L2 omega into H 1 0 of omega back into L2 of omega. So, it is continuous and of course, G is also self adjoint. We have seen this in the connection of the eigenvalue problem and so on. So, we have, we know that it is symmetric and therefore it is, G is symmetric in place self-adjoint.

Now for every f in L2 of omega dagger has a unique solution. Therefore, it implies that $R(I - A) = L^{2}(\Omega)$, and $G = (I - A)^{-1}$ is a continuous linear operator.

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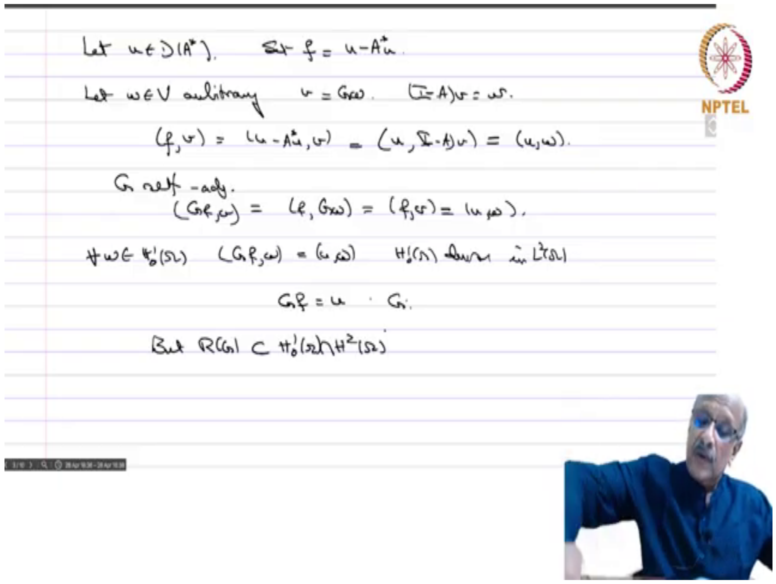

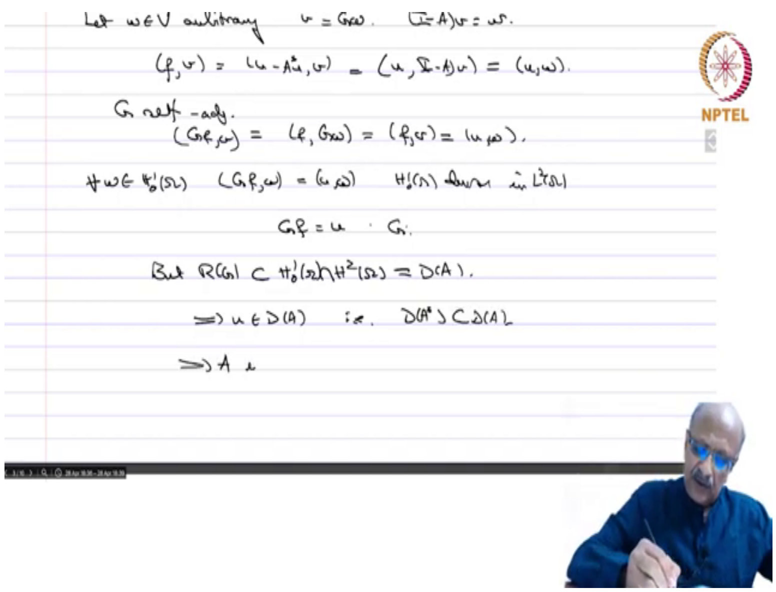

So, now let $u \in D(A^{*})$, and you set $f = u - A^{*}u$, let $w \in V$ *be* arbitrary and $v = G(w)$. So, $(I - A)v = w$. So, now this is the inner product in the L2. So,

$$(f, v) = (u - A^{*}u,\ v) = (u, (I - A)v) = (u, w).$$

Now, G is self-adjoint, therefore $(Gf, u) = (f, Gw) = (f, v) = (u, w)$.

So, for every $w \in H^1_0(\Omega)$, you have $(Gf, w) = (u, w)$, H^1_0 *is dense in* L^2. So, you have $Gf = u$. But $R(G) \subset H^1_0(\Omega) \cap H^2(\Omega) = D(A)$. So $u \in D(A) \Rightarrow D(A^*) \subset D(A)$.

Therefore, this implies that A is self adjoint.

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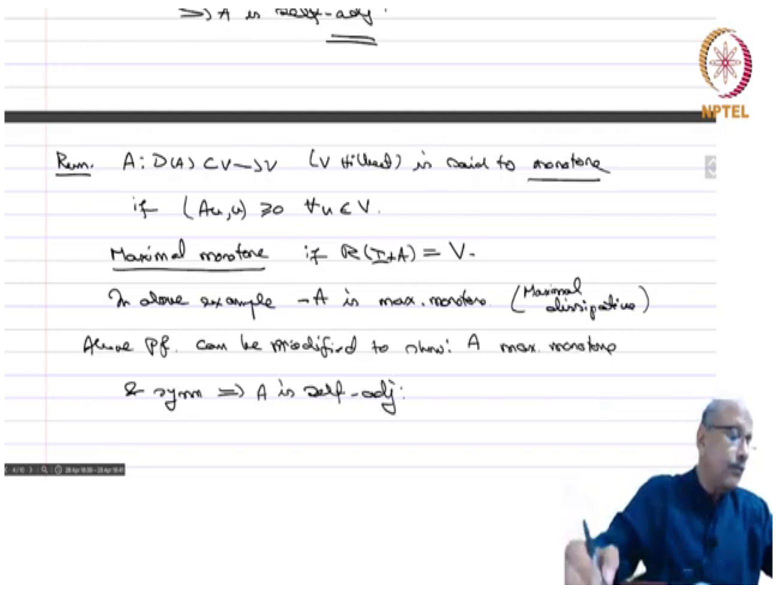

So, remark.

Remark: $A: D(A) \subset V \to V$, V Hilbert, is said to be monotone if $(Au, u) \geq 0$, $\forall u \in V$.

And then it is said to be maximal monotone if $R(I + A) = V$.

Now in the above example -A is maximal monotone (say maximal dissipative). So, the above proof can be modified to show A maximum monotone. And symmetric $\Rightarrow$A is self-adjoint.

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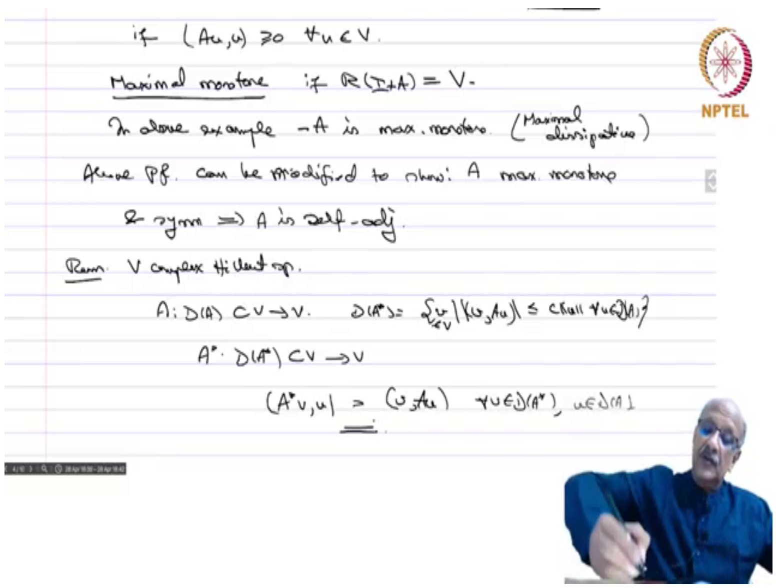

Remark: If V is a complex Hilbert space we can still do this complex Hilbert space. So,

$A: D(A) \subset V \to V,\ D(A^*) = \{v \in V:\ |(v, Au)| \leq c||u||,\ \forall\, u \in D(A)\},\ A^*: D(A^*) \subset V \to V,$

$$(A^* v, u) = (v, Au),\ \forall u \in D(A^*),\ u \in D(A).$$

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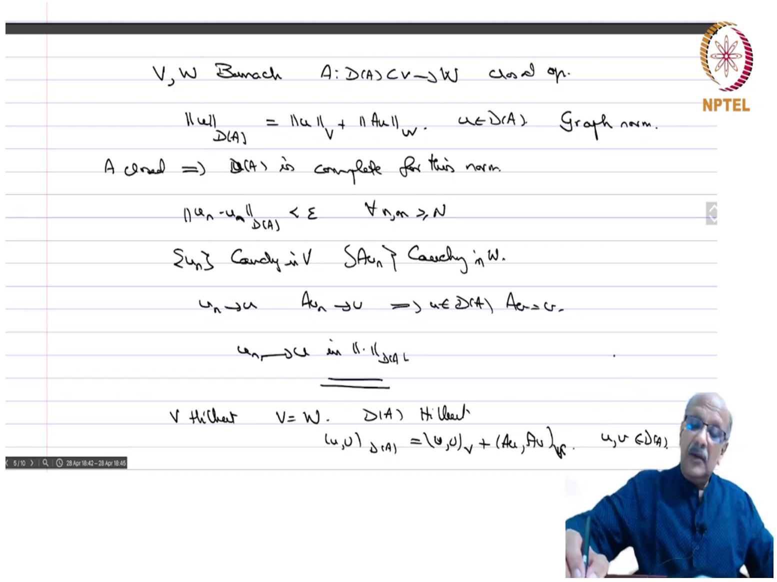

And finally V and W Banach and $A: D(A) \subset V \rightarrow W$ closed operator. Then we define the norm

$$||u||_{D(A)} = ||u||_V + ||Au||_W, \ u \in D(A)\text{- graph norm.}$$

Then A closed implies that D(A) is complete for this norm. So, you take any Cauchy sequence

$$||u_n - u_m||_{D(A)} < \epsilon, \ \forall n, m \ \geq N$$

therefore then $\{u_n\}$ Cauchy in V and $\{Au_n\}$is Cauchy in W. And therefore $u_n \rightarrow u$ and $Au_n \rightarrow v \in V \ \Rightarrow u \in D(A) \ and \ Au = v$ and therefore the Cauchy sequence $u_n \rightarrow u$ in $||.||_{D(A)}$.

And V is Hilbert and V = W then you know that $D(A)$ is also Hilbert with the inner product

$$(u, v)_{D(A)} = (u, v)_V + (Au, Av)_W, \ u, v \in D(A).$$

So, this becomes an inner product which will give the above norm the graph norm and that completes. So, this is a rapid revision of unbounded operators.