Sobolev Spaces and Partial Differential Equations
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Lecture 71
Unbounded Operators – Part 1

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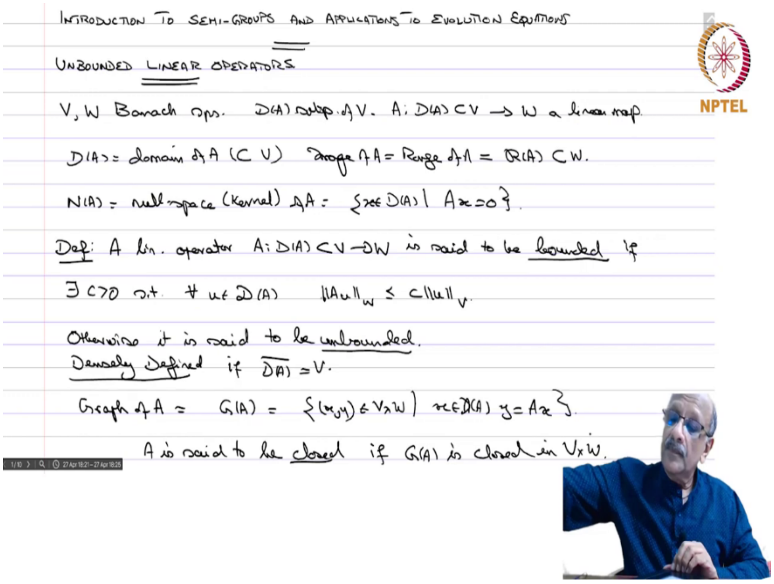

Today, we will start a new chapter, Introduction to Semi-Groups and Applications to Evolutionary Equations. So, the theory of semi-groups plays an important role in the study of evolution equations and this is an abstract framework in which we can study many evolutionary equations. So, one of the essential ingredients of this theory is the notion of an unbounded linear operator.

So, first, we will talk about unbounded linear operators. So, we start with the, so we take V and W Banach spaces and $D(A)$ is a subspace of V and $A: D(A) \subset V \rightarrow W$ a linear map. So, a linear operator will henceforth define be of this kind, namely, it is defined on a subspace of V, not necessarily on the whole of V and it is a linear map.

So, then $D(A)$ is called the domain of A, contained in V. Then

$$Image(A) = Range(A) = R(A) \subset W.$$

Then $N(A) =$ the null space (kernel) of A. So, $N(A) = \{x \in D(A): Ax = 0\}$, so this is what is called the null space of A.

And, so now we have the following definition:

Definition: A linear operator $A: D(A) \subset V \to W$ is said to be bounded if there exists a $C > 0$ such that for every $u \in D(A)$, you have $||Au||_W \leq C||u||_V$.

So, this is the usual boundedness which we have seen in case of linear operators.

Otherwise, if this is not true it is said to be unbounded. So, it is said to be densely defined, if $\overline{D(A)} = V$; that means, the domain is dense, so that is just this thing. And it is set to be the graph of A equal to $G(A) = \{(y, x) \in V \times W: x \in D(A),\ y = Ax\}$.

So, and then A is said to be closed, if $G(A)$ is closed in $V \times W$.

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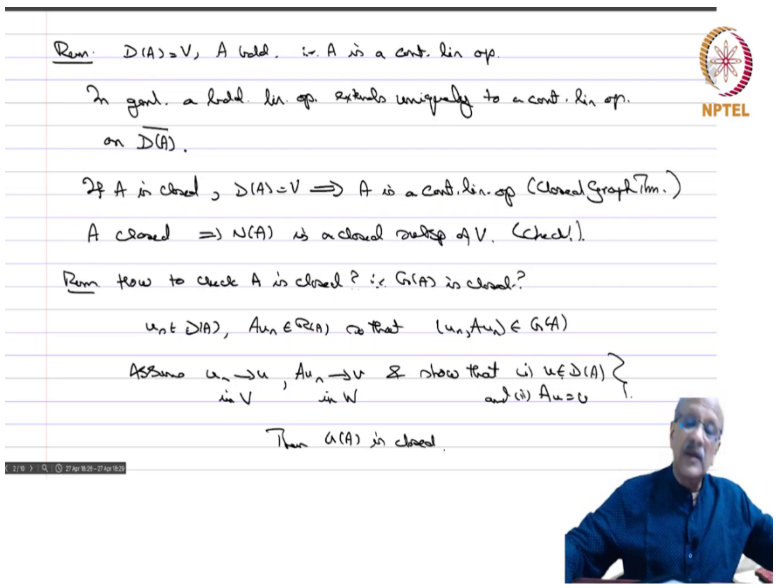

Remark: so if $D(A) = V$ and A is bounded, then that is, A is a continuous linear operator as we have usually known in function analysis. So, in general, a bounded linear operator extends uniquely to a continuous linear operator on $\overline{D(A)}$.

So, if A is closed, that means the graph of A is closed and $D(A) = V$, then this implies that A is a continuous linear operator. This is nothing but the closed graph theorem, one of the famous theorems in Banach space theory.

Now A, closed $\Rightarrow N(A)$ is a closed subspace of V. (Check), very easy to check, so normally, if you have a continuously linear operator then of course null space is a closed subspace, but if it is a general linear operator, nothing can be said about the null space, and if A is a closed operator, that means the graph is closed, then you say that it is a, then the null space will be closed.

So, then remark.

Remark: how to check A is closed? So, what is the method which we usually employ? So, we want to show $G(A)$ is closed. So, you take $u_n \in D(A)$ and $Au_n \in R(A)$, of course, so that $(u_n, Au_n) \in G(A)$. So, assume $u_n \to u \; in \; V$, $Au_n \to v \; in \; W$, and show that, (i) $u \in D(A)$ and (ii) $Au = v$. Then $G(A)$ is closed.

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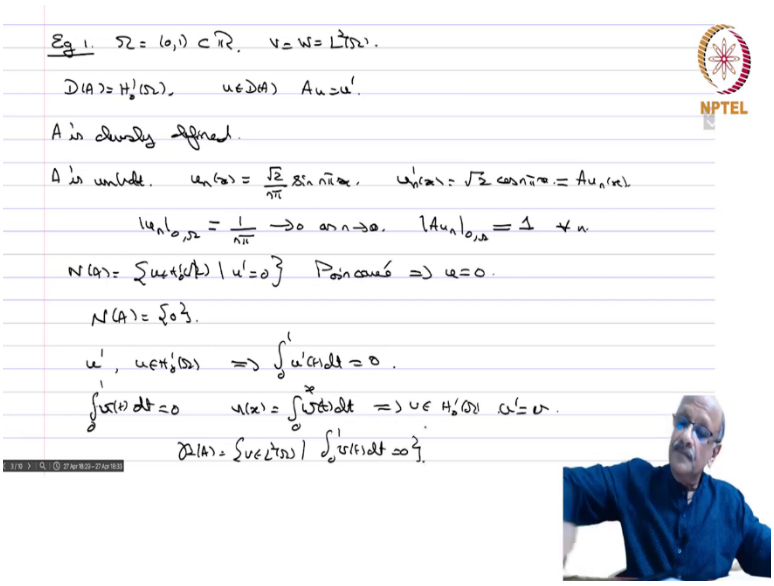

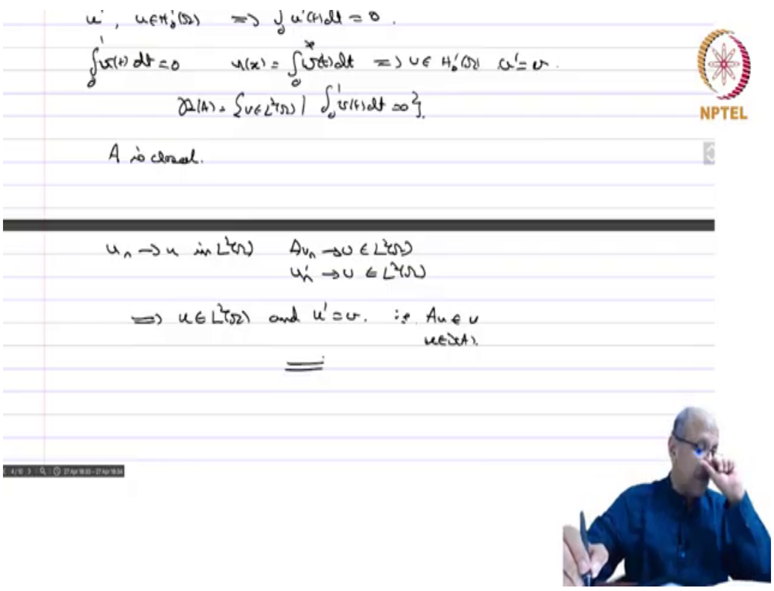

So let us, so most, almost all the operators which we will come across in the course of the next few lectures will be, of course, closed, and densely defined. So anyway, let us start with some examples. So example one:

Example: so let $\Omega = (0,1) \subset \mathbb{R}$, and $V = W = L^2(\Omega)$. So, then you define $D(A) = H^1_0(\Omega)$ and for all $u \in D(A)$, $Au = u'$, namely the distribution derivative. So, then it is well defined.

It is not defined on all of L^2, it is defined this. So, then A is densely defined. Because you know that $D(\Omega)$ is dense in $H^1_0(\Omega)$ and $D(\Omega)$ is dense in $L^2(\Omega)$ also. So, $H^1_0(\Omega)$ is dense in $L^2(\Omega)$, so A is densely defined. Then it is unbounded.

so let us take $u_n(x) = \frac{\sqrt{2}}{n\pi} \sin n\pi x$. So, then $u_n'(x) = \sqrt{2} \cos n\pi x = Au_n(x)$.

And now you have,

$$|u_n|_{0,\Omega} = \frac{1}{n\pi} \to 0 \ as \ n \to \infty \ and \ |Au_n|_{0,\Omega} = 1 \ \forall \, n.$$

So, you cannot have a constant C s.t. $|Au_n|_{0,\Omega} \leq C\,|u_n|_{0,\Omega}$ because this stays at 1 and this goes to 0, therefore this implies that it is unbounded.

Now, $N(A) = \{u \in H^1_0(\Omega) : u' = 0\}$, and therefore by Poincare's inequality $u = 0$. So, $N(A) = \{0\}$. Now what about the range of A?

So, if you have u', $u \in H^1_0(\Omega)$, this implies $\int_0^1 u'(t)dt = 0$. Conversely, if

$\int_0^1 v(t)dt = 0$, then you define $u(x) = \int_0^x v(t)dt, \Rightarrow v \in H^1_0(\Omega) \ and \ u' = v$.

And therefore, $R(A) = \{v \in L^2(\Omega) : \int_0^1 v(t)dt = 0\}$.

Finally, we want to show that A is closed, so let $u_n \to u \, in \, L^2(\Omega)$ and $Au_n \to v \, in \, L^2(\Omega)$. that means $u_n' \to v \, in \, L^2(\Omega)$, but this is one of the very first things we did when we proved Sobolev spaces especially to show that it is complete that means $u \in L^2(\Omega)$ and $u' = v$.

And that is $Au = v \, and \, u \in D(A)$. And therefore, you have that this is a closed operator as well, so this is one example.

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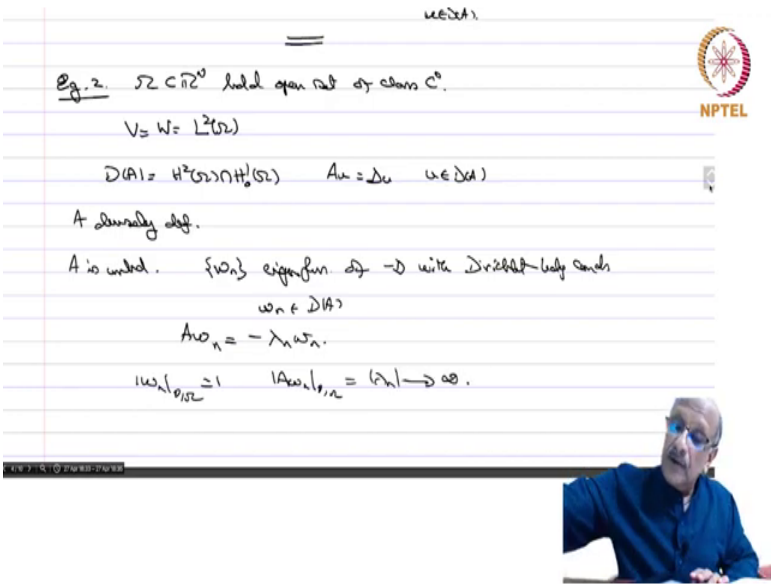

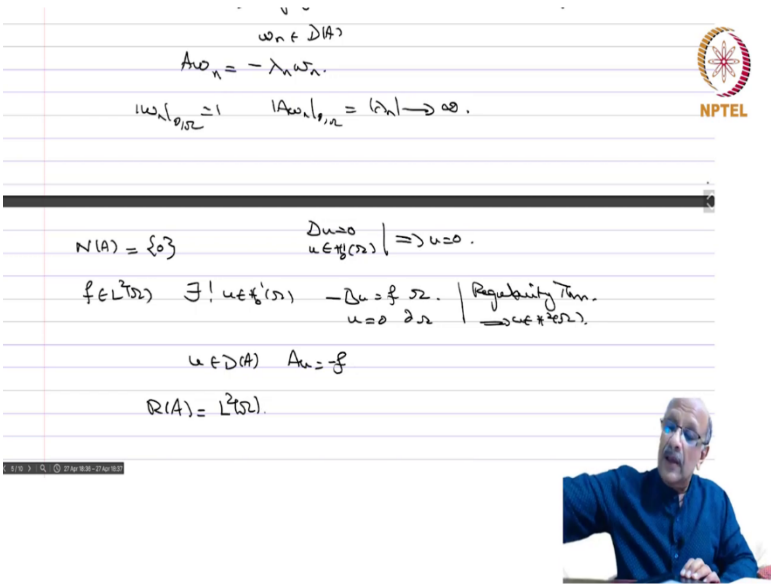

So, second example.

(2) so again, $\Omega \subset \mathbb{R}^N$ bounded open set of class C^0. And then again, we take $V = W = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, $Au = \Delta u$, $u \in D(A)$.

So again, A is densely defined. Now it is unbounded, A is unbounded, so let $\{w_n\}$ be Eigenfunctions of $-\Delta$ with Dirichlet boundary conditions.

That means $w_n \in D(A)$ automatically. And $Aw_n = -\lambda_n w_n$. Now $|w_n|_{0,\Omega} = 1$, we know normalized Eigenfunctions, whereas $|Aw_n|_{0,\Omega} = |\lambda_n| \to \infty$ and therefore this is a clearly unbounded operator.

What about $N(A)$? $N(A)$ is singleton $\{0\}$ because you have $-\Delta u = 0$ and $u \in H^1_0(\Omega)$, then Poincare's inequality tells you that it implies that u =0. Now, if $f \in L^2(\Omega)$, there exists unique $u \in H^1_0(\Omega)$, such that $-\Delta u = f$ and $u = 0$ on $\partial\Omega$.

And regularity theorem, $u \in H^2(\Omega)$. So, $u \in D(A)$, $Au = f$. And therefore, $R(A) = L^2(\Omega)$.

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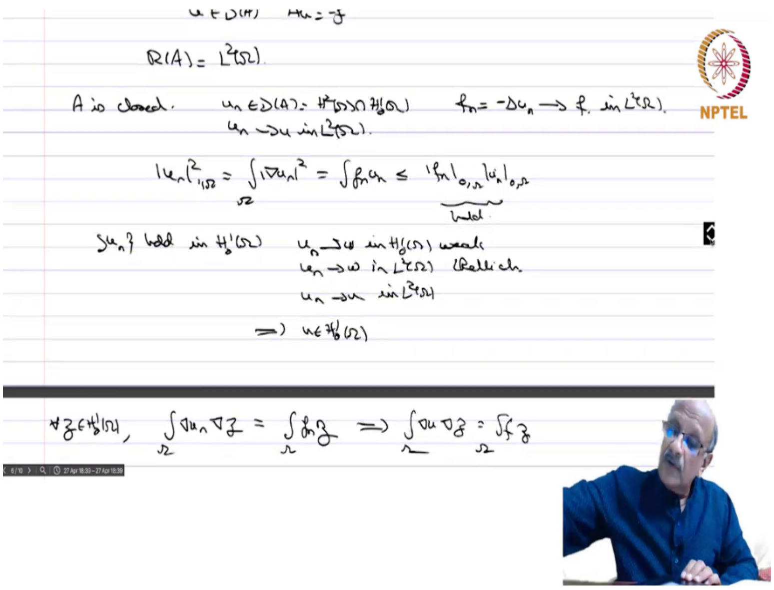

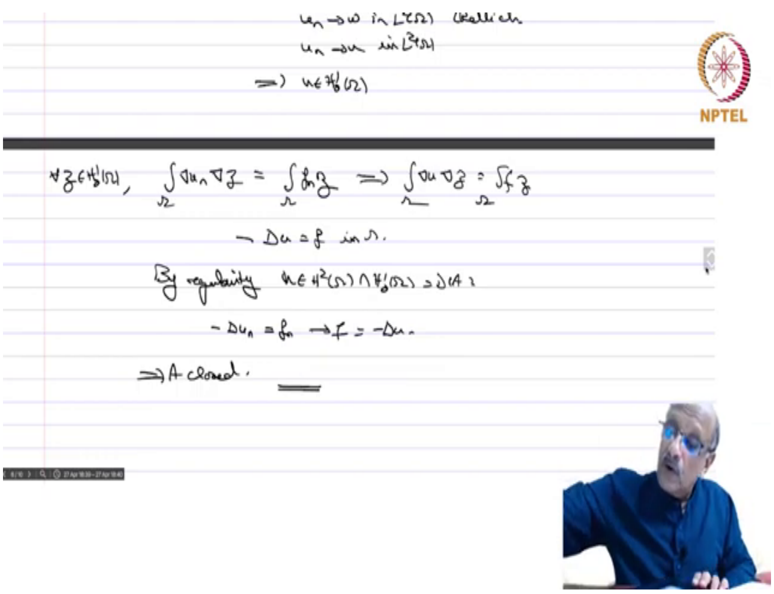

So, now A is again closed, so we have to show this. So, $u_n \in D(A) = H^2(\Omega) \cap H^1_0(\Omega)$,

$u_n \to u \; in \; L^2(\Omega)$. So, let $f_n = - \Delta u_n \to f \; in \; L^2(\Omega)$.

But $|\nabla u_n|^2_{1,\Omega} = \int_\Omega |\nabla u_n|^2 = \int_\Omega f_n u_n \leq |f_n|_{0,\Omega} |u_n|_{0,\Omega}$ $-$and this is bonded.

So u_n is bounded in H^1_0, so $u_n \to u$ in weakly H^1_0 weak to some W and therefore

$u_n \to w$ in $L^2(\Omega)$ by the Rellich theorem. But $u_n \to u$ already in $L^2(\Omega)$. So, this implies that $H^1_0(\Omega)$. Now for all $z \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u_n . \nabla z = \int_\Omega f_n z \Rightarrow \int_\Omega \nabla u \, . \nabla z = \int_\Omega f \; z$$

So, you have $- \Delta u = f \; in \; \Omega$. And by regularity, you have $u \in D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. And therefore, $- \Delta u_n = f_n \to f = - \Delta u$ and this implies that A is closed.

So, these are two, in fact, very important examples, which we will return to a little later.

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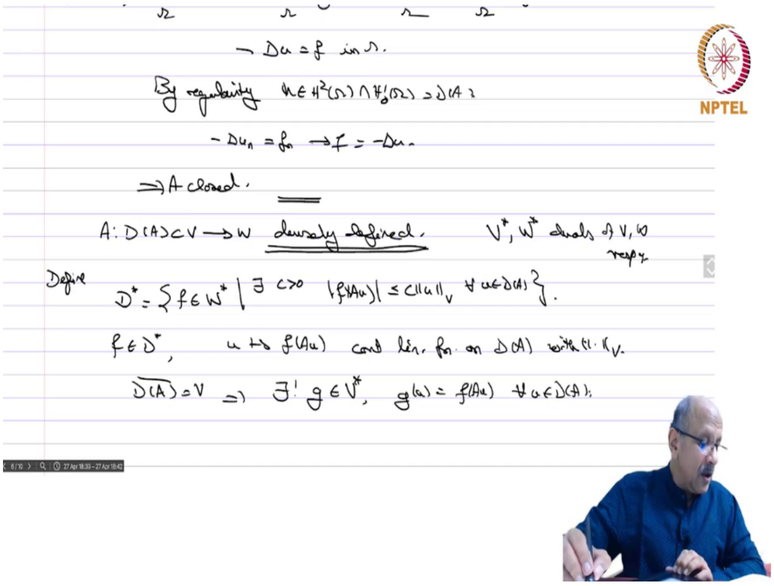

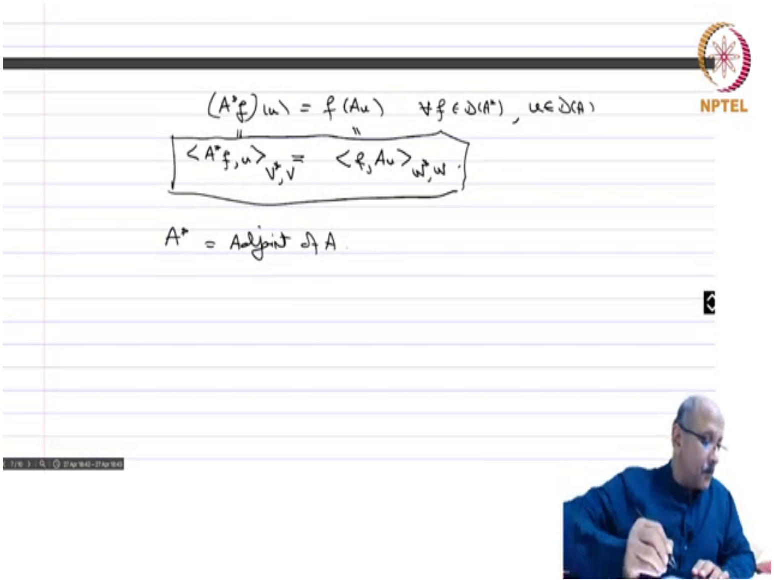

So, now we want to, I want to give a definition which is very important. So, let us take $A: D(A) \subset V \rightarrow W$, which is densely defined. So, this is a very important hypothesis at this moment. So, let V^*, W^* are the duals of V, W respectively. So, now I define,

$$D^* = \{f \in W^* : \exists\, C > 0 \,|f(Au)| \leq C||u||_V, \forall\, u \in D(A)\}.$$

Now, if $f \in D^*$, then $u \rightarrow f(Au)$ is a continuous linear functional on $D(A)$ with norm $||.||_V$.

But $D(A)$ is dense in V implies there exists a unique $g \in V^*$ such that $g(u) = f(Au), \forall u \in D(A)$.

So, then you have $(A^{*}f)(u) = f(Au)\,,\ \forall f \in D(A^{*}),\ \forall u \in D(A)\,.$

So, at this juncture we will use the bracket notation, so

$$< A^{*}f, u>_{V^{*},V} =< f, Au>_{W^{*},W}\,,\ \forall u \in D(A)\,.$$

A^{*} - adjoint of A .

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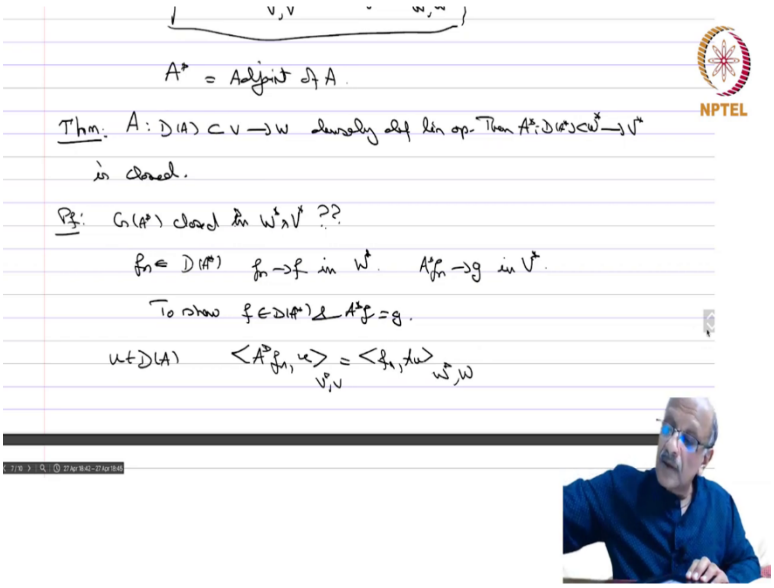

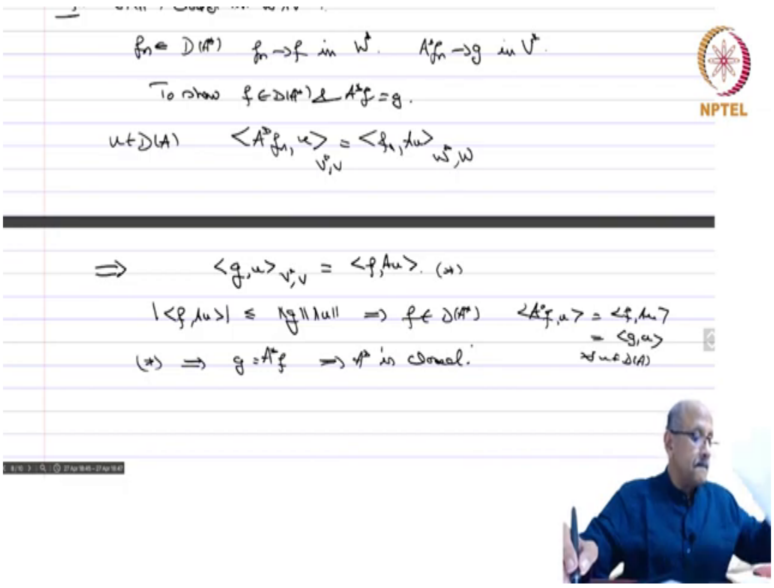

So, the theorem:

Theorem: So, $A: D(A) \subset V \to W$ is a densely defined linear operator. Then $A^*: D(A^*) \subset V^* \to W^*$ is closed.

proof: So to show $G(A^*)$ star is closed, so let us take $W^* \times V^*$.

So, let us take $f_n \in D(A^*)$, $f_n \to f$ in W^*, $Af_n \to g$ in V^*.

So, what do we have to show? So we have to show that $f \in D(A^*)$, $A^* f = g$.

So, if $u \in D(A)$, $< A^* f_n, u>_{V^*,V} =< f_n, Au>_{W^*,W}$.

But this implies, $< g, u>_{V^*,V} =< f, Au >$ ------ (*)

So, $| < f, Au > | \leq ||g|| \, ||Au|| \Rightarrow f \in D(A^*)$

And (*) implies $g^* = A^* f \Rightarrow A^*$ is closed.

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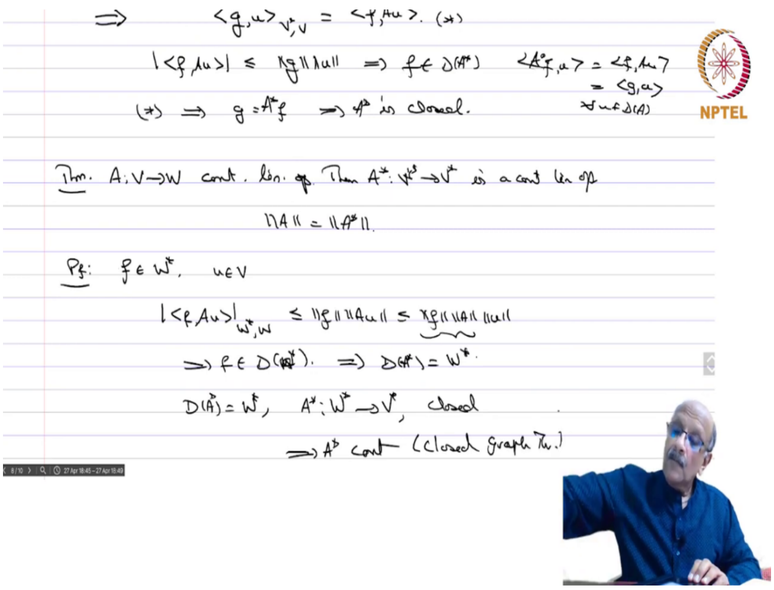

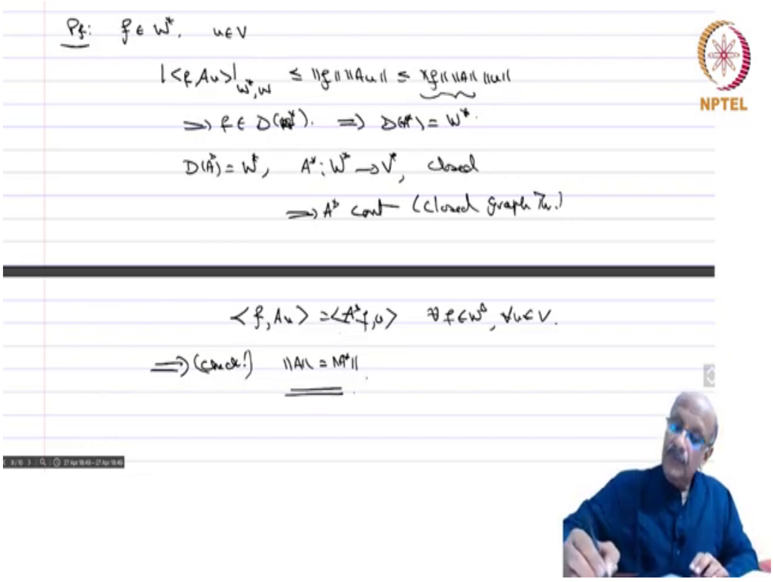

Theorem: Let $A: V \to W$ is a continuous linear operator. Then $A^{*}: W^{*} \to V^{*}$ is continuous and $||A|| = ||A^{*}||$.

proof: So, let $f \in W^{*}$, and $u \in V$ arbitrary. Then

$$| < f, Au > |_{W^{*},W} = ||f|| \, ||Au|| \leq ||f|| \, ||A|| \, ||u||$$

$$\Rightarrow f \in D(A^{*}) \Rightarrow D(A^{*}) = W^{*}$$

and therefore $A^{*}: W^{*} \to V^{*}$ is closed and by by the closed graph theorem A^{*} is continuous.

And, so you have $< f, Au > = < A^{*}u, f > \ , \forall f \in W^{*}, \forall u \in V$.

$$\Rightarrow (Check\,!) \ ||A|| = ||A^{*}||.$$

So, you just use this, take the modulus on either side and then take the inequalities, you will get $||A||$ less than $||A^{*}||$ and $||A^{*}||$ less than equal to $||A||$.

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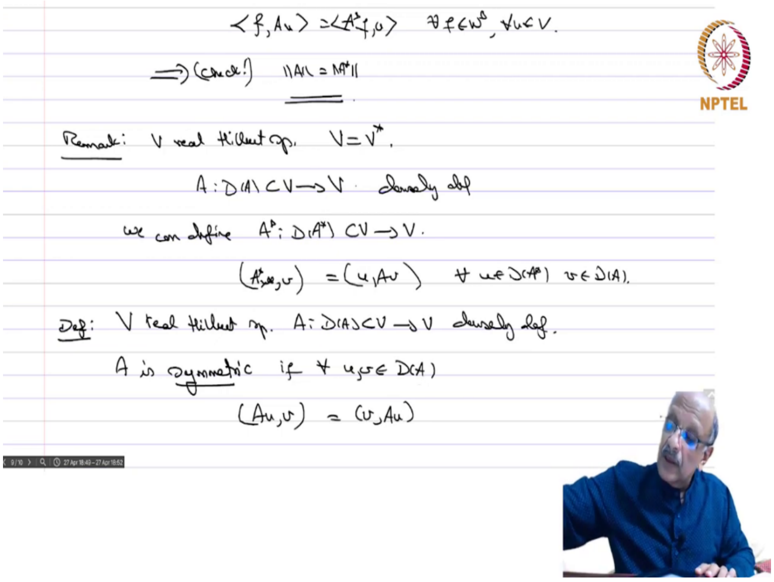

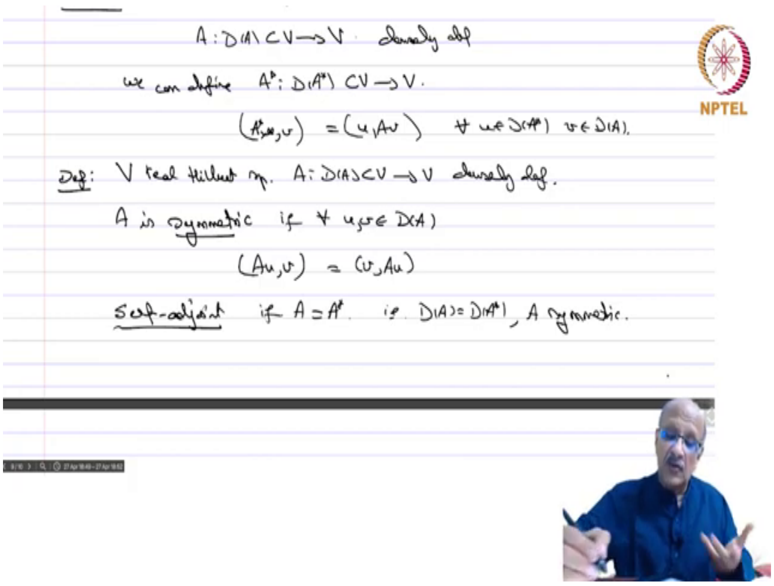

Remark. if V is a real Hilbert space, we can identify $V = V^*$. We have already gone through this thing generally. So, if you take $A: D(A) \subset V \to V$ which is densely defined, we can define $A^*: D(A) \subset V \to V$ by $(A^*u, v) = (u, Av)$, $\forall\, u \in D(A^*)$, $v \in D(A)$.

Definition: V is a real Hilbert space and $A: D(A) \subset V \to V$ is densely defined. Then A is said to be symmetric, if for every $u, v \in D(A)$, you have

$$(Au, v) = (v, Au).$$

Also A is said to be self-adjoint, if $A^* = A$, that is, $D(A^*) = D(A)$ and A is symmetric.

So, if you have a continuous linear operator then, or a bounded linear operator, then there is no distinction between which is densely defined, so it will automatically be a continuous linear operator. And therefore, the symmetry and self-adjointness are one and the same.

But if you have a unbounded operator, which is densely defined then if you, symmetry does not imply so symmetry, so A unbounded remark symmetric only implies that $D(A) \subset D(A^*)$ and that $A^*|_{D(A)} = A$. So, to show that it is self-adjoint that means you have to show the $D(A) = D(A^*)$ and $A = A^*$. So, symmetry is not enough alone. So we will see examples of the adjoint next time.