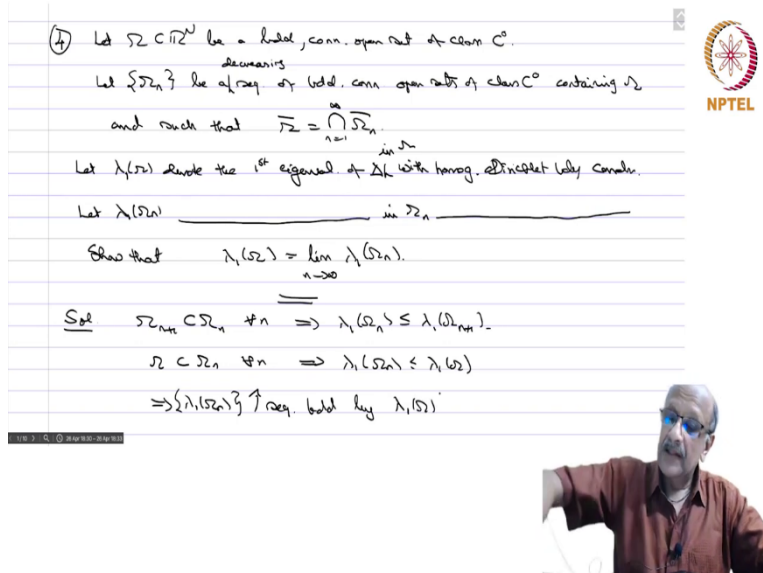
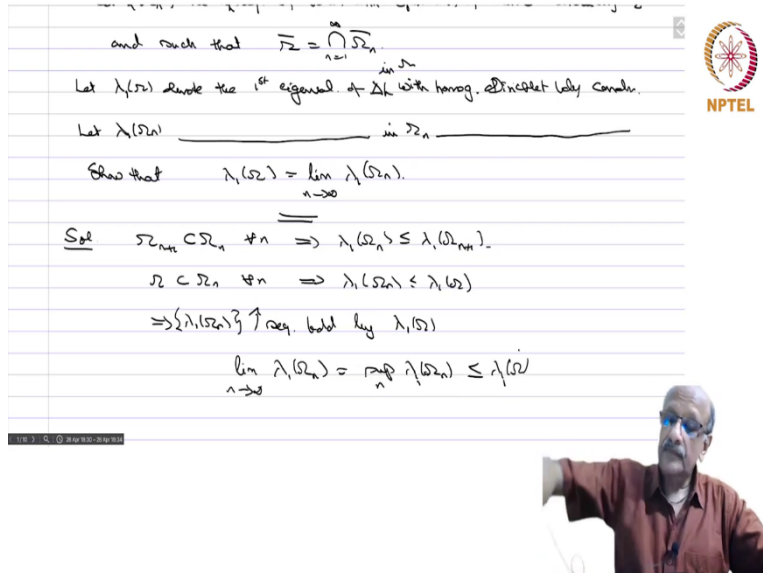


Sobolev Spaces and Partial Differential Equations
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Exercises – Part 13

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(4) Let $\Omega \subset \mathbb{R}^N$ be a bounded, conn. open set of class C^0 .
 Let $\{\Omega_n\}$ be a seq. of bounded, conn. open sets of class C^0 containing Ω
 and such that $\overline{\Omega} = \bigcap_{n=1}^{\infty} \overline{\Omega_n}$.
 Let $\lambda_1(\Omega_n)$ denote the 1st eigenval. of Δ_h with homog. Dirichlet bdy. condn.
 Let $\lambda_1(\Omega_n)$ in Ω_n
 Show that $\lambda_1(\Omega) = \lim_{n \rightarrow \infty} \lambda_1(\Omega_n)$.
Sol $\Omega_{n_m} \subset \Omega_n \neq n \Rightarrow \lambda_1(\Omega_{n_m}) \leq \lambda_1(\Omega_n)$.
 $\Omega \subset \Omega_n \neq n \Rightarrow \lambda_1(\Omega_n) \leq \lambda_1(\Omega)$
 $\Rightarrow \{\lambda_1(\Omega_n)\} \uparrow$ seq. bdd by $\lambda_1(\Omega)$



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 $\Omega \subset \Omega_n \neq n \Rightarrow \lambda_1(\Omega_n) \leq \lambda_1(\Omega)$
 $\Rightarrow \{\lambda_1(\Omega_n)\} \uparrow$ seq. bdd by $\lambda_1(\Omega)$
 $\lim_{n \rightarrow \infty} \lambda_1(\Omega_n) = \sup_n \lambda_1(\Omega_n) \leq \lambda_1(\Omega)$

(4) Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected, open set of class C^0 . Let $\{\Omega_n\}$ be a sequence of bounded, connected, open sets of class C^0 containing Ω and such that $\overline{\Omega} = \bigcap_{i=1}^{\infty} \overline{\Omega_i}$. Let $\lambda_1(\Omega)$ denote the first eigenvalue of Δ with homogeneous Dirichlet boundary conditions. Let $\lambda_1(\Omega_n)$ denote the first eigenvalue, etcetera in

$\lambda_1(\Omega)$, and here it is, in Ω_n with homogeneous Dirichlet boundary conditions. Show that $\lambda_1(\Omega) = \lim_{n \rightarrow \infty} \lambda_1(\Omega_n)$.

proof: so $\Omega_{n+1} \subset \Omega_n, \forall n$, so this implies that $\lambda_1(\Omega_n) \leq \lambda_1(\Omega_{n+1})$. Also, $\Omega \subset \Omega_n, \forall n$, and therefore this implies that $\lambda_1(\Omega_n) \leq \lambda_1(\Omega)$. So, this implies that $\{\lambda_1(\Omega_n)\}$ is a monotonically increasing sequence and bounded by $\lambda_1(\Omega)$. And therefore, we have that $\lim_{n \rightarrow \infty} \lambda_1(\Omega_n)$ exists and

$$\lim_{n \rightarrow \infty} \lambda_1(\Omega_n) = \sup_n \lambda_1(\Omega_n) \leq \lambda_1(\Omega).$$

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Let $u_n \in H_0^1(\Omega_n)$ be an eigenfn. for $\lambda_1(\Omega_n)$ $\int_{\Omega_n} u_n^2 = 1$.

\tilde{u}_n (arbitr. long zero) $\in H_0^1(\Omega_1)$

$$\int_{\Omega_1} |\tilde{u}_n|^2 dx = \int_{\Omega_n} |u_n|^2 dx = \lambda_1(\Omega_n) \leq \lambda_1(\Omega_1).$$

$$\int_{\Omega_1} |\tilde{u}_n|^2 dx = \int_{\Omega_n} |u_n|^2 dx = 1.$$

$\Rightarrow \{\tilde{u}_n\}$ is a bounded seq. in $H_0^1(\Omega_1)$

\exists subseq. $\tilde{u}_{n_k} \rightharpoonup \tilde{g}$ in $H_0^1(\Omega_1)$ weakly

$\tilde{u}_{n_k} \rightarrow \tilde{g}$ in $L^2(\Omega_1)$ (Rellich-K)

$\tilde{u}_{n_k} \rightarrow \tilde{g}$ pointwise a.e.



$\int_{\Omega_1} |\nabla u_n|^2 dx = \int_{\Omega_n} |\nabla u_n|^2 dx = \lambda_1(\Omega_n) \leq \lambda_1(\Omega)$
 $\int_{\Omega_1} \tilde{u}_n^2 dx = \int_{\Omega_n} u_n^2 dx = 1$
 $\Rightarrow \{\tilde{u}_n\}$ is a bdd. seq. in $H^1_0(\Omega_1)$
 \exists subseq $\tilde{u}_{n_k} \rightharpoonup z$ in $H^1_0(\Omega_1)$ weakly
 $\tilde{u}_{n_k} \rightarrow z$ in $L^2(\Omega_1)$ (Rellich-K)
 $\tilde{u}_{n_k} \rightarrow z$ pointwise a.e.
 In particular $\int_{\Omega_1} |z|^2 dx = 1$

So, let $u_n \in H^1_0(\Omega_n)$ be an Eigenfunction for $\lambda_1(\Omega_n)$ and $\int_{\Omega_n} u_n^2 = 1$. So, then $\tilde{u}_n \in H^1_0(\Omega_1)$ (extension by zero), and also you have

$$\int_{\Omega_1} |\nabla \tilde{u}_n|^2 = \int_{\Omega_n} |\nabla u_n|^2 = \lambda_1(\Omega_n) \leq \lambda_1(\Omega).$$

And you have $\int_{\Omega_1} |\tilde{u}_n|^2 = \int_{\Omega_n} u_n^2 = 1$. So, this implies that \tilde{u}_n is a bounded sequence in $H^1_0(\Omega_1)$. Therefore, there exists a subsequence such that $\tilde{u}_{n_k} \rightharpoonup z \in H^1_0(\Omega_1)$ weakly. And therefore, by the Relic theorem, $\tilde{u}_{n_k} \rightarrow z$ in $L^2(\Omega_1)$. And for the further subsequence, which you will choose as the subsequence in question, that $\tilde{u}_{n_k} \rightarrow z$ pointwise almost everywhere. So, in particular, the integral $\int_{\Omega_1} |z|^2 = 1$.

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$$\Rightarrow \{u_n\} \text{ is a weakly seq. in } H_0^1(\Omega_1)$$

$$\exists \text{ subseq. } \tilde{u}_n \rightarrow z \text{ in } H_0^1(\Omega_1) \text{ weakly}$$

$$\tilde{u}_n \rightarrow z \text{ in } L^2(\Omega_1) \text{ (Rellich-K)} \quad \times$$

$$\tilde{u}_n \rightarrow z \text{ a.e.}$$

$$\text{In particular } \int_{\Omega_1} z^2 dx = 1.$$



$$\text{Let } x \in \Omega_1 \setminus \bar{\Omega}_2. \exists N(x) \text{ st. } \forall n \geq N(x) \quad x \in \bar{\Omega}_n.$$

$$\Rightarrow \tilde{u}_n(x) = 0 \quad \forall n \geq N(x).$$

$$\Rightarrow z(x) = 0 \text{ a.e. } x \in \Omega_1 \setminus \bar{\Omega}_2.$$

$$\Rightarrow z \in H_0^1(\Omega_2).$$



$$\Rightarrow \tilde{u}_n(x) = 0 \quad \forall n \geq N(x).$$

$$\Rightarrow z(x) = 0 \text{ a.e. } x \in \Omega_1 \setminus \bar{\Omega}_2.$$

$$\Rightarrow z \in H_0^1(\Omega_2).$$

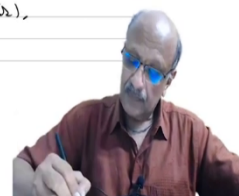
$$\lambda_1(\Omega_2) \leq \int_{\Omega_2} |\nabla z|^2 dx = \int_{\Omega_1} |\nabla z|^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega_1} |\nabla \tilde{u}_n|^2 dx$$

$$= \liminf_{n \rightarrow \infty} \lambda_1(\Omega_{n_k})$$

$$\leq \limsup_{n \rightarrow \infty} \lambda_1(\Omega_{n_k}) \leq \lambda_1(\Omega_2)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \lambda_1(\Omega_{n_k}) = \lambda_1(\Omega_2).$$

$$\text{But } \{\lambda_1(\Omega_{n_k})\} \text{ is g.t. } \Rightarrow \lim_{n \rightarrow \infty} \lambda_1(\Omega_n) = \lambda_1(\Omega_2).$$



So, let $x \in \Omega_1 \setminus \bar{\Omega}_2$, therefore, there exists $N(x)$ such that for all $n \geq N(x)$, $x \in \bar{\Omega}_n$,

$$\tilde{u}_n(x) = 0, \forall n \geq N(x) \Rightarrow z(x) = 0 \text{ a.e. } x \in \Omega_1 \setminus \bar{\Omega}_2.$$

$$\Rightarrow z \in H_0^1(\Omega_2).$$

$$\text{So, } \lambda_1(\Omega) \leq \int_{\Omega} |\nabla z|^2 dx = \int_{\Omega_1} |\nabla z|^2 dx \leq \liminf_{n_k \rightarrow \infty} \int_{\Omega_1} |\nabla \tilde{u}_{n_k}|^2 dx$$

$$= \liminf_{n_k \rightarrow \infty} \lambda_1(\Omega_{n_k}) \leq \limsup_{n_k \rightarrow \infty} \lambda_1(\Omega_{n_k}) \leq \lambda_1(\Omega).$$

$$\Rightarrow \lim_{n_k \rightarrow \infty} \lambda_1(\Omega_{n_k}) = \lambda_1(\Omega).$$

But we already know that $\{\lambda_1(\Omega_n)\}$ is convergent and we have a subsequence converging to λ_1 of Ω and therefore this implies that $\lim_{n \rightarrow \infty} \lambda_1(\Omega_n) = \lambda_1(\Omega)$.

So, with this, we will wind up this chapter, and we will now start next time a study of Semi-Groups of Operators and their Applications to Evolution Equations.