Sobolev Spaces and Partial Differential Equations
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Exercises – Part 13

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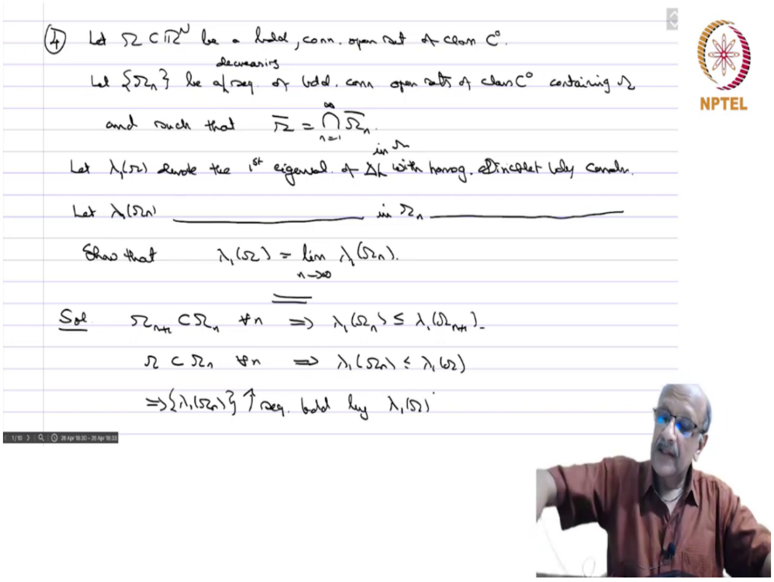

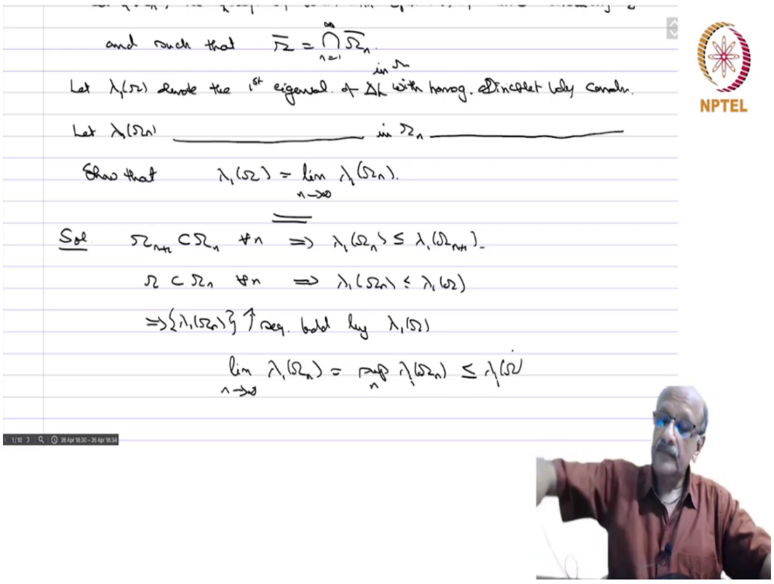

(4) Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected, open set of class C^0. Let $\{\Omega_n\}$, be a sequence of bounded, connected, open sets of class C^0 containing Ω and such that $\overline{\Omega} = \cap_{i=1}^{\infty} \overline{\Omega_i}$. Let $\lambda_1(\Omega)$ denote the first eigenvalue of Δ with homogeneous Dirichlet boundary conditions. Let $\lambda_1(\Omega_n)$ denote the first eigenvalue, etcetera in

$\lambda_1(\Omega)$, and here it is, in Ω_n with homogeneous Dirichlet boundary conditions. Show that $\lambda_1(\Omega\ ) = \lim_{n\to\infty} \lambda_1(\Omega_n)$.

proof: so $\Omega_{n+1} \subset \Omega_n$, $\forall\, n$, so this implies that $\lambda_1(\Omega_n) \leq \lambda_1(\Omega_{n+1})$. Also, $\Omega\ \subset \Omega_n$, $\forall\, n$, and therefore this implies that $\lambda_1(\Omega_n) \leq \lambda_1(\Omega\ )$. So, this implies that $\{\lambda_1(\Omega_n)\}$ is a monotonically increasing sequence and bounded by $\lambda_1(\Omega\ )$. And therefore, we have that $\lim_{n\to\infty} \lambda_1(\Omega_n)$ exists and $\lim_{n\to\infty} \lambda_1(\Omega_n) = \sup \lambda_1(\Omega_n) \leq \lambda_1(\Omega\ )$.

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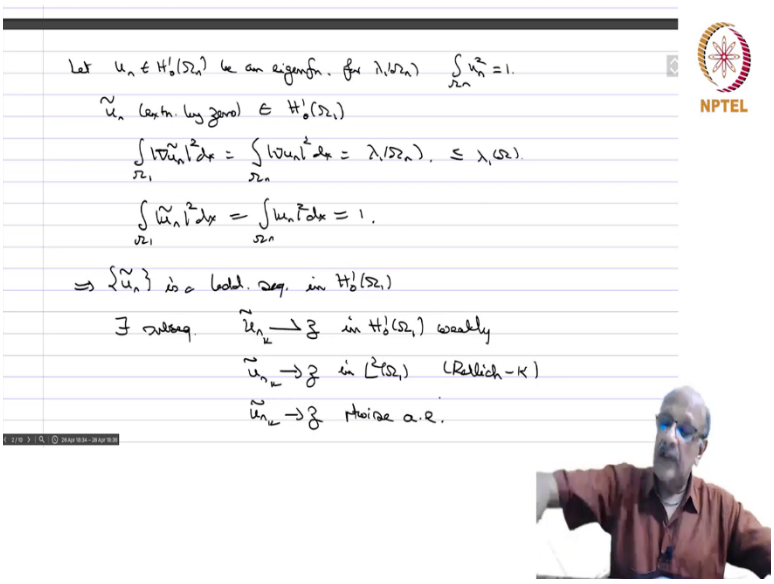

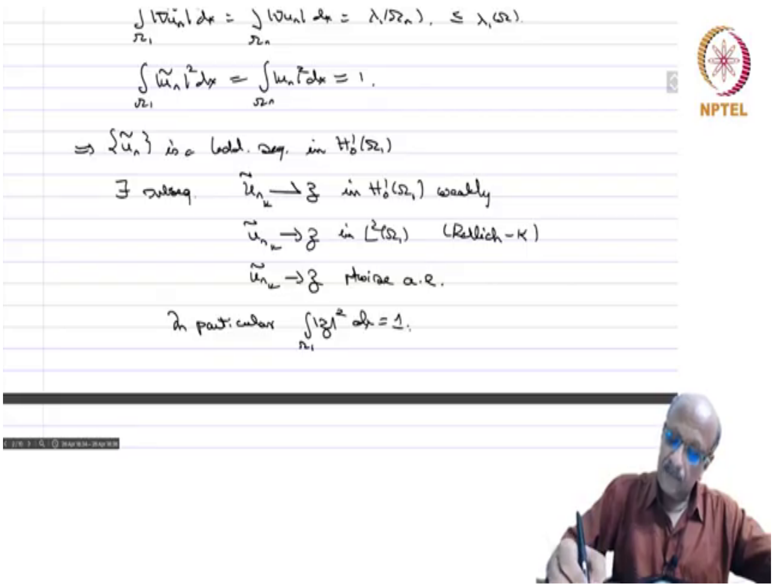

So, let $u_n \in H^1_0(\Omega_n)$ be an Eigenfunction for $\lambda_1(\Omega_n)$ and $\int_{\Omega_n} u_n^2 = 1$. So, then un $\tilde{u}_n \in H^1_0(\Omega_1)$ (extension by zero), and also you have

$$\int_{\Omega_1} |\nabla \tilde{u}_n|^2 = \int_{\Omega_n} |\nabla u_n|^2 = \lambda_1(\Omega_n) \leq \lambda_1(\Omega\ ).$$

And you have $\int_{\Omega_1} |\tilde{u}_n|^2 = \int_{\Omega_n} u_n^2 = 1$. So, this implies that $\tilde{u}_n$ is a bounded sequence in $H^1_0(\Omega_1)$. Therefore, there exists a subsequence such that $\tilde{u}_{n_k} \to z \in H^1_0(\Omega_1)$ weakly. And therefore, by the Relic theorem, $\tilde{u}_{n_k} \to z$ in $L^2(\Omega_1)$. And for the further subsequence, which you will choose as the subsequence in question, that $\tilde{u}_{n_k} \to z$ pointwise almost everywhere. So, in particular, the integral $\int_{\Omega_1} |z|^2 = 1$.

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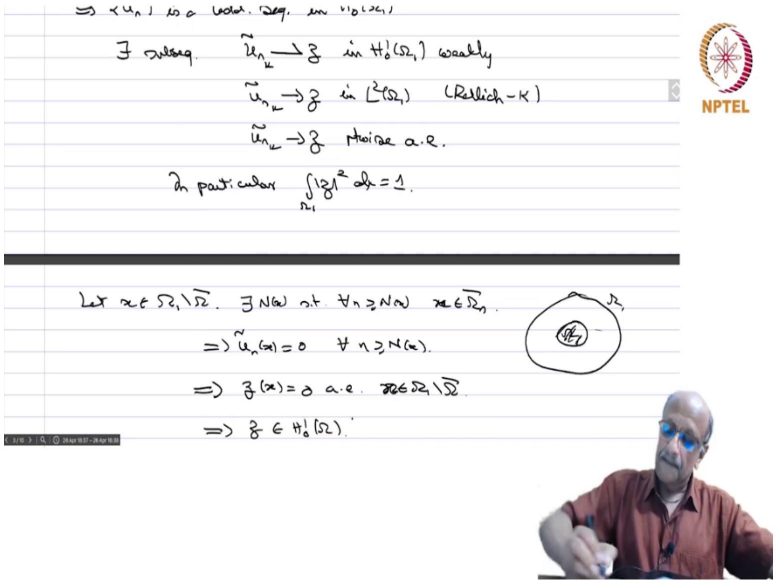

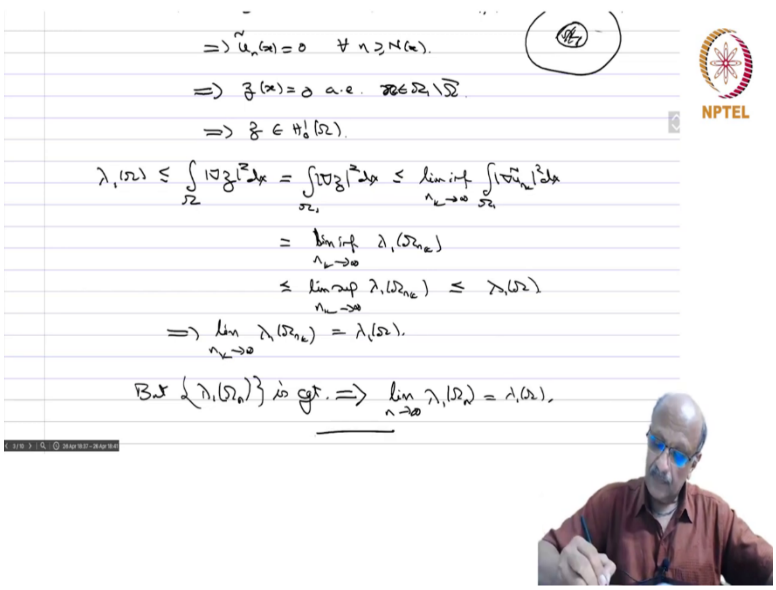

So, let $x \in \Omega_1 \backslash \overline{\Omega}$, therefore, there exists $N(x)$ such that for all $n \geq N(x)$, $x \in \overline{\Omega_n}$,

$$\widetilde{u}_n(x) = 0, \ \forall n \geq N(x) \Rightarrow z(x) = 0 \ a.e. \ x \in \Omega_1 \backslash \overline{\Omega}.$$

$$\Rightarrow z \in H^1_0(\Omega).$$

So, $\lambda_1(\Omega) \leq \int_\Omega |\nabla z|^2 dx = \int_{\Omega_1} |\nabla z|^2 dx \leq \liminf_{n_k \to \infty} \int_{\Omega_1} |\nabla \widetilde{u}_{n_k}|^2 dx$

$$= \liminf_{n_k \to \infty} \lambda_1(\Omega_{n_k}) \leq \limsup_{n_k \to \infty} \lambda_1(\Omega_{n_k}) \leq \lambda_1(\Omega).$$

$$\Rightarrow \lim_{n_k \to \infty} \lambda_1(\Omega_{n_k}) = \lambda_1(\Omega).$$

But we already know that $\{\lambda_1(\Omega_n)\}$ is convergent and we have a subsequence converging to lambda 1 of omega and therefore this implies that $\lim_{n \to \infty} \lambda_1(\Omega_n) = \lambda_1(\Omega)$.

So, with this, we will wind up this chapter, and we will now start next time a study of Semi-Groups of Operators and their Applications to Evolution Equations.