Sobolev Spaces and Partial Differential Equations
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Lecture 07
Support of a distribution

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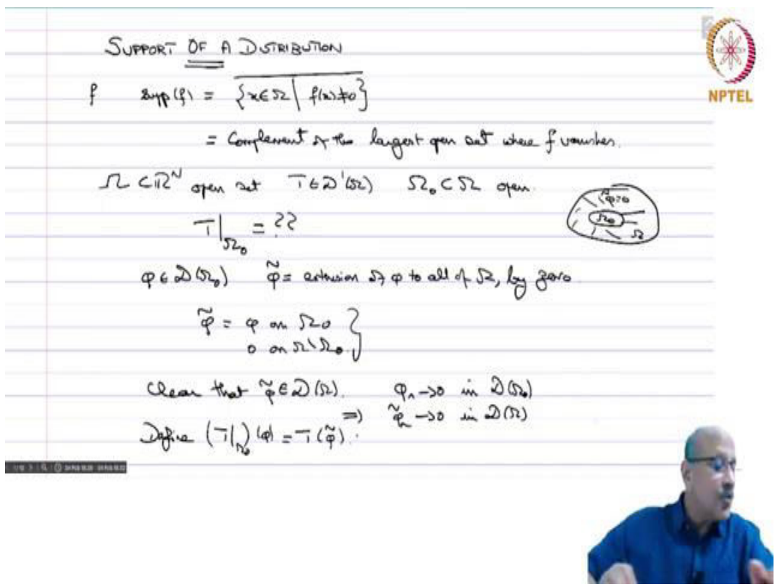

We now talk about the support of a distribution. So, given a function f you know what the support is,

$$\text{supp}(f)=\underline{\{x \in \Omega: f(x) \neq 0\}}.$$

=complement of the largest open set where f vanishes.

Now, we want to extend the notion of support to a distribution that we immediately run into difficulty. Namely, a distribution is defined on an open set, it is a continuous linear functional on the space of C infinity functions with compact supports.

Therefore, it is meaningless to say the value of a distribution at a point that has no meaning. So, how do we then extend this relation? So, we can equivalently say the support is equal to the complement of the largest open set where f vanishes. So, it is easy to see that these two are the same. So, the complement largest open set where the f finishes to complement will be a closed set and you can say that will be precisely the closure of the set of all x, fx is not equal to 0.

Now, the second statement is amenable to us for generalization. So, given $\Omega \subset \mathbb{R}^N$ open set, $T \in \mathcal{D}'(\Omega)$, $\Omega_0 \subset \Omega$ open, we would like to know how to define $T|_{\Omega_0}$?.

So, this is what we want to say.

Let $\phi \in \mathcal{D}(\Omega_0)$ and $\tilde{\phi}$ = extension of ϕ to all of Ω by 0 outside of Ω.

And therefore, that is called the extension. So,

$$\tilde{\phi} = \phi \text{ on } \Omega_0$$

$$= 0 \text{ on } \Omega \setminus \Omega_0.$$

Now, it is clear that $\tilde{\phi} \in \mathcal{D}(\Omega)$, because all that we have done is extended by 0 the support of ϕ is a compact set contained inside Ω_0. So, near the boundary fairly close to the boundary of Ω, ϕ will be 0 and you are further extending by 0.

Now, if $\phi_n \to 0$ in $\mathcal{D}(\Omega_0)$ then $\widetilde{\phi_n} \to 0$ in $\mathcal{D}(\Omega)$ because the support of $\widetilde{\phi_n}$ is the same as the support of ϕ_n which are all contained in fixed compact set K and on that compact set K everything goes to 0 uniformly all the derivatives etcetera. And therefore, this also in places. So, now we define T restricted to omega naught acting on fi is nothing but T acting on phi tilde.

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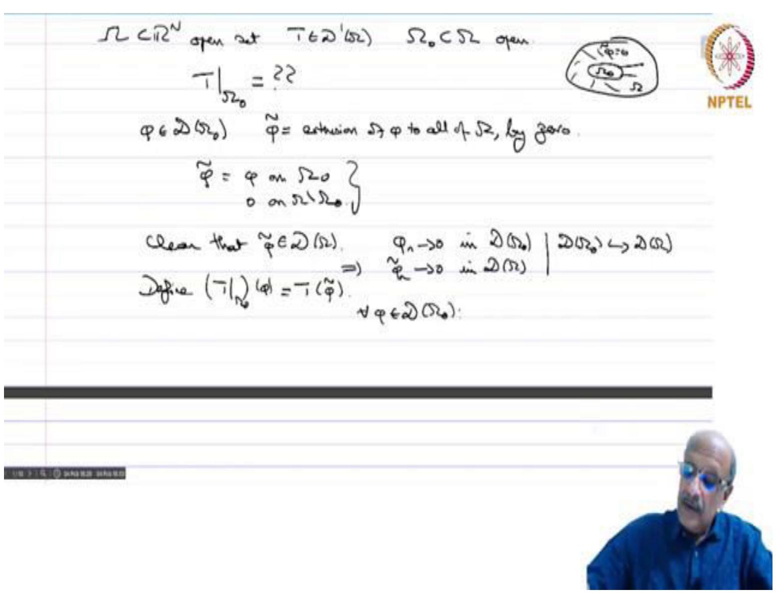

So, this way we define. So, this just says the $\mathcal{D}(\Omega_0)$ is included in $\mathcal{D}(\Omega)$ in a continuous fashion. This hook Tarot means that the inclusion is an enclosed set enclosure which is also an occlusion map is continuous. So, we define this for all $\phi \in \mathcal{D}(\Omega)$. So, in this way we can extend the, I mean define what do we mean by the $T|_{\Omega_0}$.

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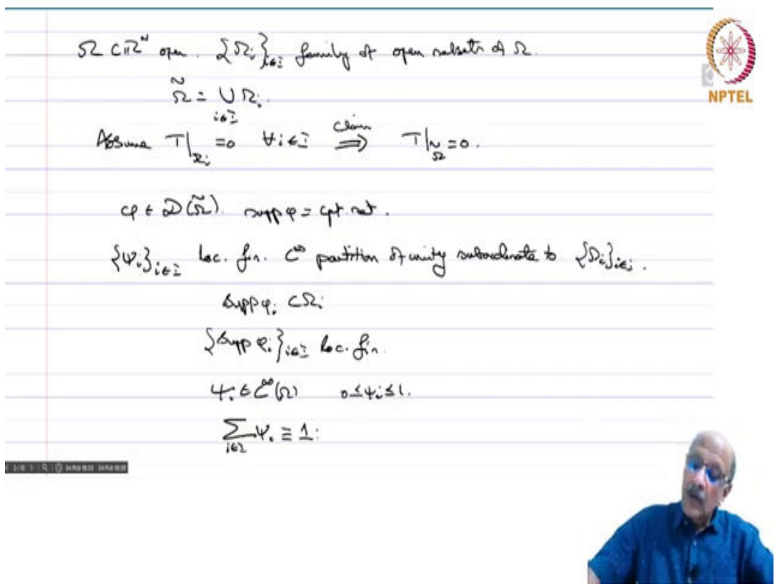

So, now let $\Omega \subset \mathbb{R}^n$ open set, $\{\Omega_i\}_{i}$- family of open subsets of Ω.

$$\widetilde{\Omega} = \bigcup_{i \in I} \Omega_i .$$

So, assume $T|_{\Omega_i} = 0, \forall i \in I$, then we claim: $T|_{\widetilde{\Omega}} = 0$. So, how do we do this?

So, let us take $\phi \in \mathcal{D}(\widetilde{\Omega})$. Now supp($\phi$) equals a compact set. So, $\{\psi_i\}_{i \in I}$ - locally finite C infinity partition of unity subordinate to Ω_i, $i \in I$. So, what does it mean ?

$$\text{supp}(\psi_i) \subset \Omega_i \ ; \ \{\text{supp}(\psi_i)\}_{i \in I} \text{ locally finite ;}$$

$$\psi_i \in C^{\infty}(\Omega) \quad ; \quad 0 \leq \psi_i \leq 1, \forall i \in I.$$

$$\sum_{i \in I} \psi_i = 1.$$

So, this is the locally finite infinity partition of unity which we know always exists.

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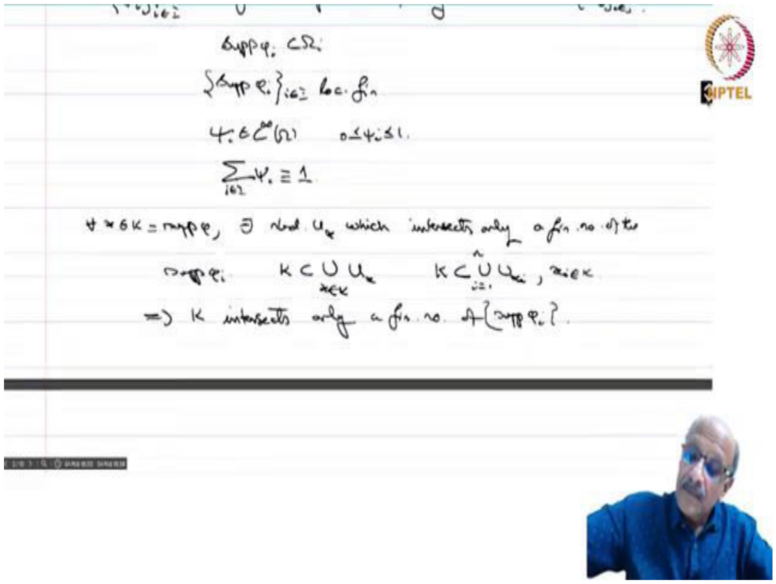

Now $x \in K$= the support of ϕ, there exists a neighborhood U_x which intersects only a finite number of the support of ϕ_i. Now, $K \subset \cup_{x\in K} U_x$, but then K is compact. So, there exists a finite subcover. So, there exists n such that: $K \subset \cup_{i=1}^{n} U_{x_i}$, $x_i \in K$. So, each of these U_{x_i} will intersect only a finite number of support phi i. So, this implies that K intersects only a finite number of supports of ϕ_i.

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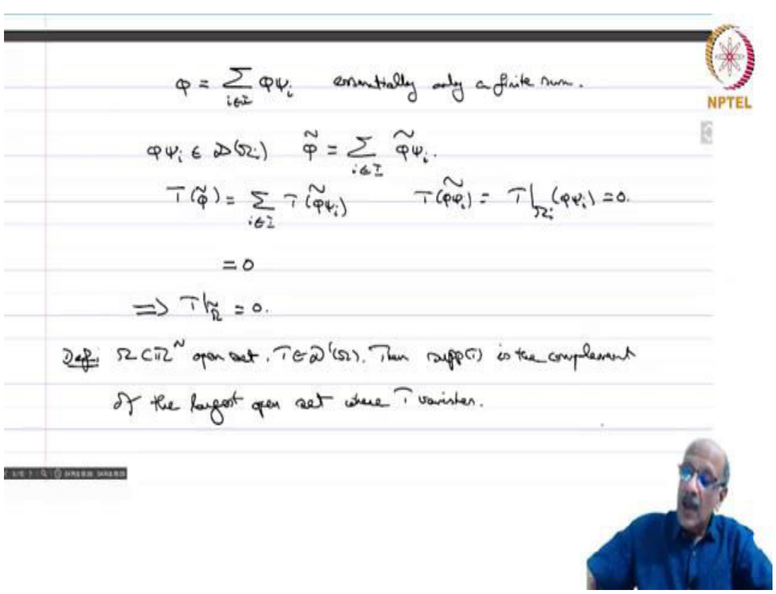

So, therefore we can write

$$\phi = \sum_{i\in I} \phi\psi_i \quad \text{-essentially only a finite sum.}$$

Now $\phi\psi_i \in \mathcal{D}(\Omega_i)$ because psi i C infinity with support in omega i phi i C function with compact support. So, the product will have compact support and it will be contained the support of psi i which is contained in omega i. Also you have

$$\tilde{\phi} = \sum_{i\in I} \widetilde{\phi\psi_i}$$

$$\widetilde{T(\phi)} = \sum_{i\in I} \widetilde{T(\phi\psi_i)} = 0. \qquad [\quad T(\widetilde{\phi\psi_i}) = T|_{\Omega_i}(\phi\psi_i) = 0\ ]$$

So, now we have T of phi tilde i, T of psi tilde is equal to sigma i and I, T of phi psi i tilde and T of phi psi i tilde is nothing but T restricted to omega i of phi psi i and that is given to be 0 because T restrict to omega i is 0. So, this each of these terms, which is only a finite number of them can be non-zero and even those are all 0 and therefore, T of phi tilde equal to 0, so this implies that

$$T|_{\tilde{\Omega}} = 0$$

So, if T vanishes on a certain number of open sets, then T vanishes on the Union.

So, it makes sense to say there is a largest open set on which T vanishes. And consequently, we can make the following definition:

Definition: $\Omega \subset \mathbb{R}^n$ open set, $T \in \mathcal{D}'(\Omega)$, then supp(T) is the complement of the largest open set where T vanishes.

So, this makes perfect sense now, and therefore, and therefore, the support is again a closed set.

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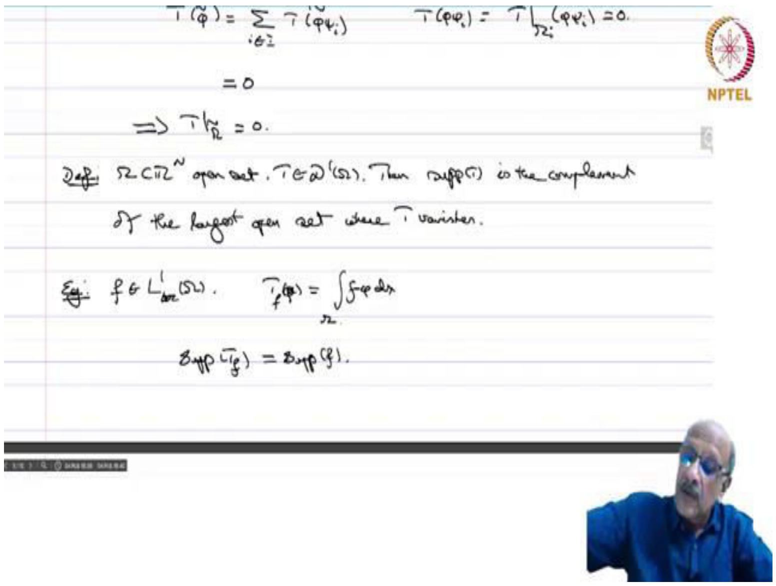

So, let us take example:

Example: □ ∈ □^{1}□□□(□). then □□(□) = ∫ □□□□.

T of fx is nothing but the entirety of phi sorry is integral f phi dx over omega and clearly now, this will vanish wherever f vanishes, this will also vanish on every open set where f is 0 Tf will also be 0 similarly, and therefore, the support you can easily check supp(□□)=supp(f)in the classical sense. So, this is for the function nothing changes.

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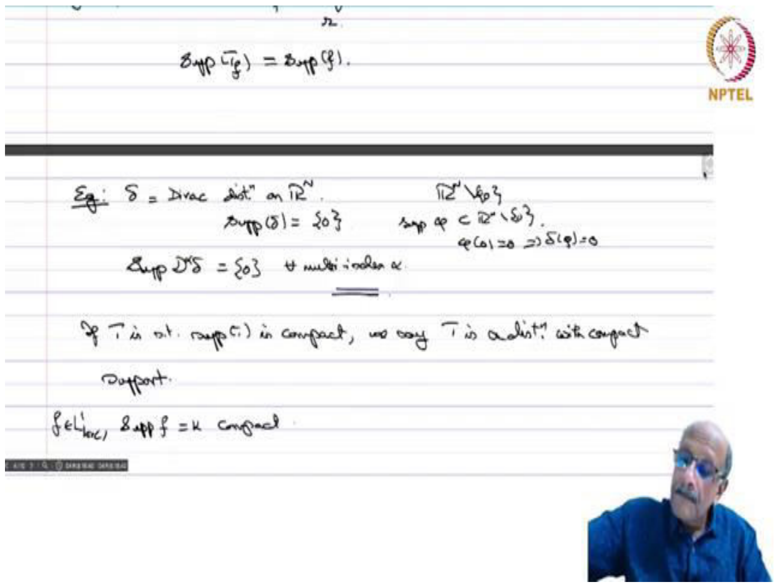

Example: Now, let us look at one more example, δ = Dirac distribution on $\mathbb{R}^N$, let us say, and then you have supp(δ)={0}, because if you take RN minus 0 and you take phi, which is set to support a phi that is contained in RN minus 0 though then phi of 0 is 0, this implies that delta phi is 0. So, RN minus 0, but if you have RN then you can have functions which do not vanish at 0 having compact support and being C infinity and therefore, T will not vanish. So, the largest open set where the delta vanishes is RN minus 0.

So, it is compliment, is singleton 0 and therefore supportive of delta is Singleton 0, and in fact support you can check this also D alpha delta is also Singleton 0 for every multi-index alpha. So, now, so if T is such that supp(T) is compact, we say T is a distribution with compact support that is natural.

So, let us take a closer look at distributions with compact support. So, let us assume that the $f \in L^1_{loc}(\Omega)$ and supp(f)=K- compact.

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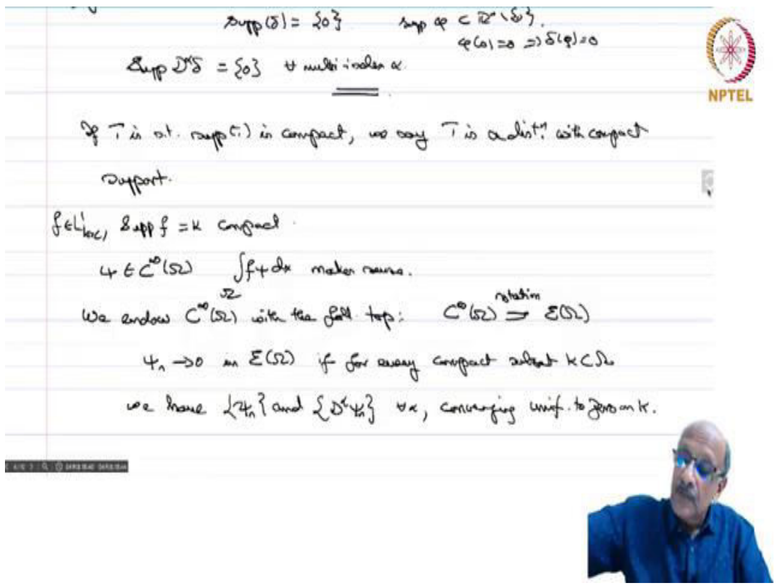

Now, if you take $\psi \in C^\infty(\Omega)$, then $\int_\Omega f\psi\,dx$ makes sense because the integral is essentially only over a compact set, psi bounded on a compact set and f is locally integrable therefore, f is integrable on that compact set so, this is well defined and it makes sense. So, now if we endow $C^\infty(\Omega)$ with the following topology: (we will henceforth call $C^\infty(\Omega) = \mathcal{E}(\Omega)$.

$\psi_n \to 0$ in $\mathcal{E}(\Omega)$ if for every compact subset $K \subset \Omega$, we have $\{\psi_n\}$ and $\{D^\alpha \psi_n\}, \forall\, \alpha$, converge uniformly to 0 on K.

So, this is similar to the $\mathcal{D}(\Omega)$ convergence, but only now you see because you cannot assume that the supports are all in some fixed compact set, that does not happen because the functions themselves do not have compact support, yes C infinity functions and therefore, on every compact set, we say psi n and all the derived sequences go to 0.

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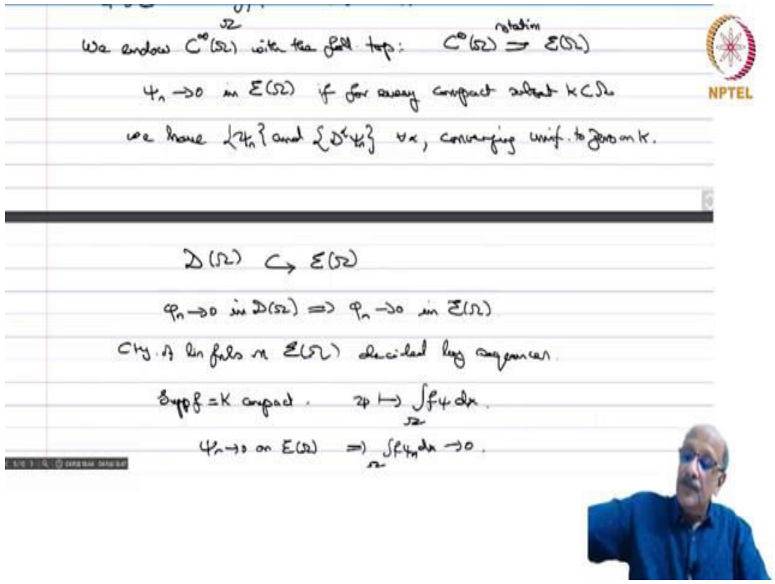

So, if you think then it is very easy to see, the $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$, it is a subset and it is also continuous because phi n goes to 0 in D of omega, that means all the supports are in the fixed compact set and they are psi n and all the D wave sequences converge to 0 uniformly any other compact set will either be intersect with this or they will be disjoint with the psi n and phi n will all be 0 and therefore, this implies that phi n goes to 0 in E of omega as well. Just thinking about it in detail is a very trivial thing.

And therefore, we say the D omega is continuously included in E of omega and this topology on E of omega is again fresh a space which means it is induced by some metric and therefore, continuity of linear functional So, continuity of linear functional on E of omega is also decided by sequences.

So, now if you took, so if supp(f)=K- compact, then you look at $\phi \mapsto \int_{\Omega} f\phi\,dx$, then this is because if $\psi_n \to 0$ in $\mathcal{E}(\Omega)$, then automatically it implies that all these good is either uniformly f is integrable and therefore, you have on the on this compact set on any compact set in particular the support of f which is k and therefore, this implies that $\int_{\Omega} f\psi_n\,dx \to 0$ because it does so, yeah all of them converge to 0 uniformly on the compact set and therefore, you have this immediately.

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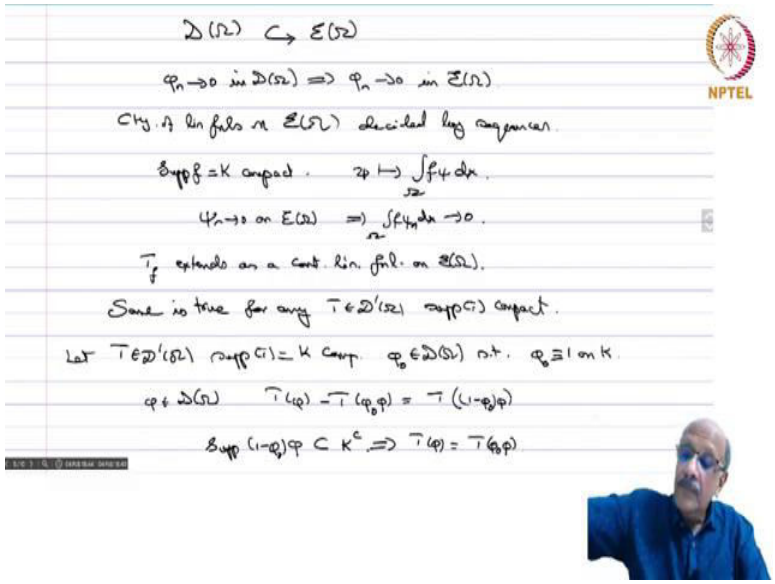

So, now, so, if so, T_f extends as a continuous linear functional on $\mathcal{E}(\Omega)$. Now, this is true same is true for any $T \in \mathcal{D}'(\Omega)$, supp(T)- compact. So, let us see how this happens. So, let $T \in \mathcal{D}'(\Omega)$, supp(T)- compact. So, you can find $\phi_0 \in \mathcal{D}(\Omega)$ such that $\phi_0 \equiv 1$ on K.

So, we saw this on the very first day. Now, if you take

$$\phi \in \mathcal{D}(\Omega)\ ;\ T(\phi) - T(\phi_0\phi) = T((1-\phi_0)\phi)$$

$$\text{supp}((1-\phi_0)\phi) \subset K^c \Rightarrow T(\phi) = T(\phi_0\phi).$$

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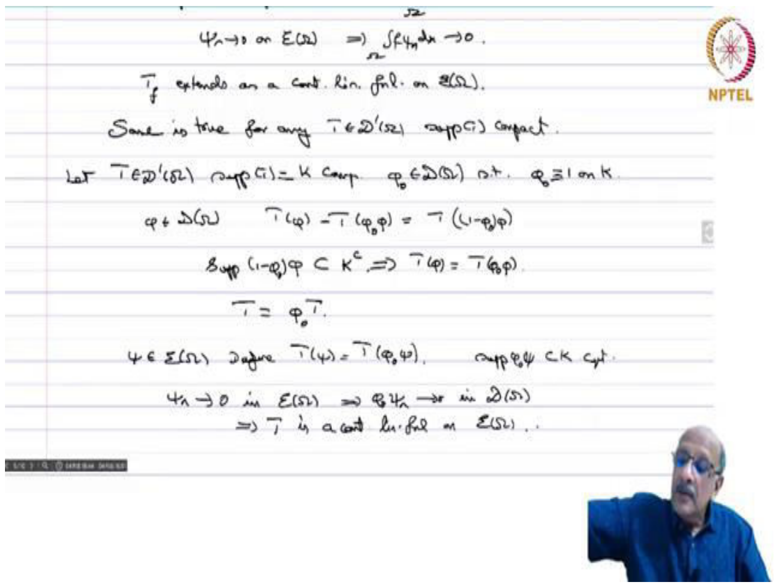

So, in fact, if you have the support of T is compact and take any function $\phi_0 \in$ □(□), which is 1 on K, then we have □ $=$ □$_0$□ in the notation which we introduced for multiplication by a C infinity function.

So, now you take any $\psi \in$ □(□) and define □(□) $=$ □(□$_0$□). This makes sense because supp(□$_0$□) $\subset$ □-compact.

And therefore, this is C infinity function with compact support and therefore, this defines distribution and of course, if you have that if $\psi_{□} \rightarrow 0$ in □(□) $\Rightarrow \phi_0$□$_{□} \rightarrow 0$ in □(□) because now all the supports on the fixed compact set K phi naught is C infinity function and therefore, by Leibniz formula, so, it is all these derivatives are bound in k and therefore, you have that phi naught psi n go to 0 uniformly because psi n goes to 0 uniformly with all its derivatives on any compact set k in particular will do it on this support of T and therefore, this goes to 0 and therefore, you have that this implies that T is a continuous linear functional on □(□).

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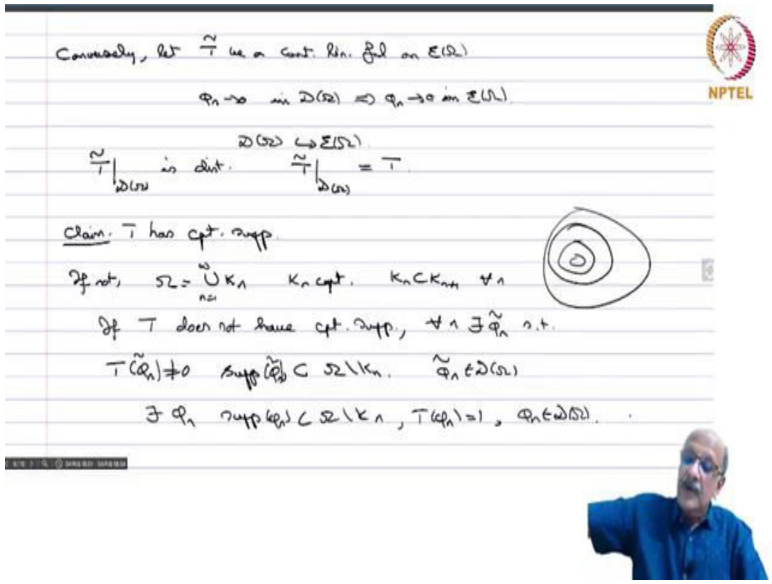

Now, what about the converse?

So, conversely let $\tilde{T}$ be a continuous linear functional on $\mathcal{E}(\Omega)$. So, we already saw that the $\phi_n \to 0$ in $\mathcal{D}(\Omega)$, this implies that $\phi_n \to 0$ in $\mathcal{E}(\Omega)$ and therefore, that D of omega is continuously included in E of omega. So, $\tilde{T}|_{\mathcal{D}(\Omega)}$ is distribution and $\tilde{T}|_{\mathcal{D}(\Omega)} = T$.

claim: T has compact support.

So, if not we can write $\Omega = \cup_{n=1}^{\infty} K_n$, K_n- compact, $K_n \subset K_{n+1}, \forall n$.

So, you have omega here and I can take an increasing sequence of compact sets which cover ultimately omega we have already seen this when we proved that if integral f phi is 0 for every phi in D omega then f has to be 0 for the locally integrable function at the time during the in the second step of the proof.

We already used this fact to construct omega as the increasing union of relatively compact open sets so you can make them also take the closures of the sets so then it becomes the increasing union of compact sets. So, then you can do this now, if T does not have compact support then it cannot vanish on the compliment of any Kn.

So, for every n that exists $\tilde{\phi}_n$ such that,

$$T(\tilde{\phi}_n) \neq 0, \ supp(\tilde{\phi}_n) \subset \Omega \setminus K_n, \ \tilde{\phi}_n \in D(\Omega).$$

so, you normalize you divide by this number, so, you get there exists

$$\phi_n \in D(\Omega), supp(\tilde{\phi}_n) \subset \Omega \setminus K_n, T(\phi_n) = 1.$$

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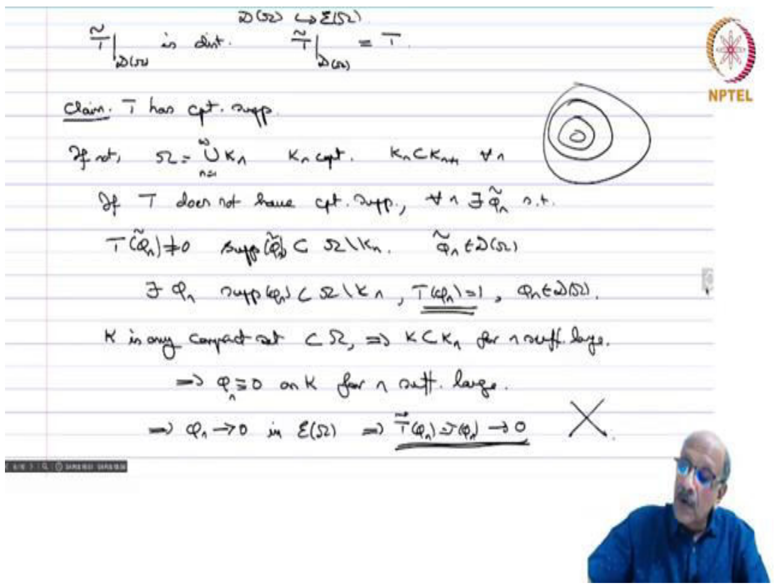

So, we can have such a thing now, we will get the contradiction. So, if K is any compact set contained in omega, then this implies that $K \subset K_n$ for n sufficiently large. Now, the support of finding this contains omega minus Kn So, this means that

$$\phi_n \equiv 0 \text{ on K for n sufficiently large.}$$

$$\Rightarrow \phi_n \to 0 \text{ in } E(\Omega) \Rightarrow T(\phi_n) = \tilde{T}(\phi_n) \to 0.$$

So, you have a contradiction and therefore, T has compact support.

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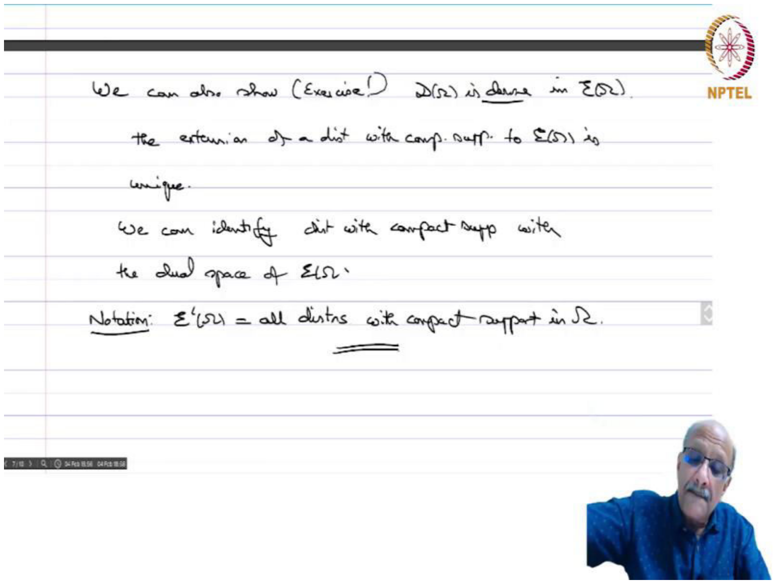

So, we can also show (exercise !) $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}(\Omega)$ it is very easy to show that. And therefore, the extension of a distribution with compact support to $\mathcal{E}(\Omega)$ is unique because you have a dense subset yet extending it to the whole thing. So, it is going to be unique therefore, we can identify the distribution with compact support with the dual space of E omega. So, we have the following notation:

Notation: $\mathcal{E}'(\Omega)$ = all distributions with compact support in Ω. So, we will look at some other properties of distributions with compact support subsequently.