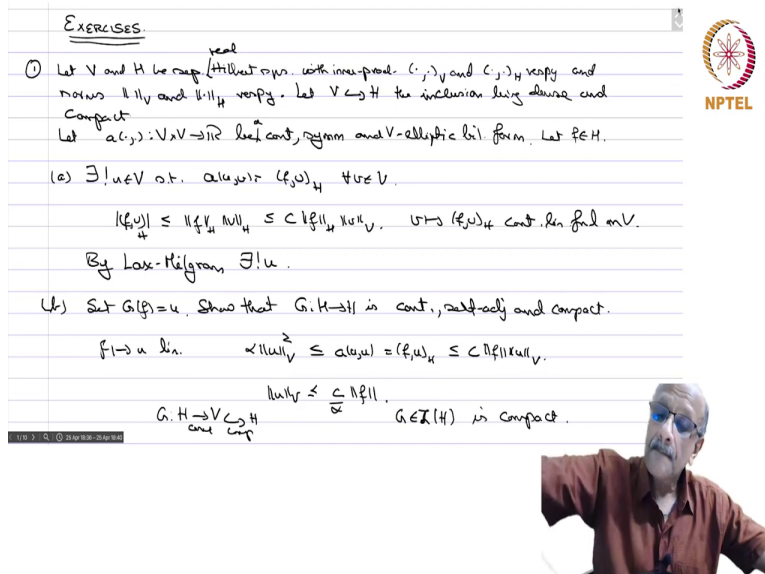


Sobolev Spaces and Partial Differential Equations
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Exercise – Part 12

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EXERCISES. ^{read}

① Let V and H be separable Hilbert spaces with inner product $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_H$ respectively and norms $\|\cdot\|_V$ and $\|\cdot\|_H$ respectively. Let $V \hookrightarrow H$ the inclusion being dense and compact. Let $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ be a continuous, symmetric and V -elliptic bilinear form. Let $f \in H$.

(a) $\exists! u \in V$ s.t. $a(u, v) = (f, v)_H \quad \forall v \in V$.

$|f(v)| \leq \|f\|_H \|v\|_H \leq c \|f\|_H \|v\|_V$, $\Rightarrow (f, v)_H$ is a continuous linear form on V .

By Lax-Milgram $\exists! u$.

(b) Set $G(f) = u$. Show that $G : H \rightarrow V$ is continuous, linear and compact.

$f \mapsto u$ lin. $\|u\|_V \leq \|a\|_V = \|f\|_H \leq c \|f\|_H$.

$\|u\|_V \leq c \|f\|_H$ or $G \in \mathcal{K}(H)$ is compact.

$G : H \rightarrow V \hookrightarrow H$ and $G : H \rightarrow H$

So, let us do some more exercises:

(1), let V and H be separable Hilbert spaces with a inner product $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_H$ respectively and norms $\|\cdot\|_V$ and $\|\cdot\|_H$ respectively. Let us look $V \rightarrow H$, the inclusion being dense and compact. Let $a : V \times V \rightarrow \mathbb{R}$ be continuous, symmetric and V -elliptic by linear form. Let $f \in H$.

(a): there exists a unique $u \in V$ such that $a(u, v) = (f, v)_H$, for every $v \in H$.

So, the immediate consequence of the Lax-Milgram lemma. a is asymmetric V elliptic continuous by linear form. So, if you look at

$$|(f, v)_H| \leq \|f\|_H \|v\|_H \leq c \|f\|_H \|v\|_V, \text{ therefore } v \mapsto (f, v)_H$$

is a continuous linear functional on V . Therefore, by Lax-Milgram there exists a unique u .

(b): set $G(f) = u$. Show that $G: H \rightarrow H$ is continuous self adjoint and compact. We are of course, talking of real Hilbert spaces that are understood to be dealing with real functions.

So, this we are trying to imitate what we did for the Laplacian. So, this is the abstract framework.

So, $f \rightarrow u$ is certainly linear, $\alpha \|u\|_V^2 \leq a(u, u) = (f, u)_H \leq c \|f\| \|u\|_V$

$$\Rightarrow \|u\|_V \leq \frac{c}{\alpha} \|f\|.$$

And therefore, you have $G: H \rightarrow V$ is continuous and this inclusion is continuous and compact.

Therefore, $G \in L(H)$ is compact. It is self adjoint as we have seen many times.

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Q. Set $Gf = u$. Show that $G: H \rightarrow H$ is cont., self-adj. and compact.

Ans. $(Gf, Gg) = 0$ in L^2 .

$f \mapsto u$ lin. $\alpha \|u\|_V^2 \leq a(u, u) = (f, u)_H \leq c \|f\| \|u\|_V$.

$\|u\|_V \leq \frac{c}{\alpha} \|f\|$.

$G: H \rightarrow V \xrightarrow{\text{incl}} H$ $G \in \mathcal{K}(H)$ is compact.

$(Gf, Gg)_H = a(Gg, Gf) = a(Gf, Gg) = (f, Gg)_H$.

G is self-adj.

$(Gf, f) = a(Gf, Gf) \geq \alpha \|Gf\|^2$ $Gf = 0 \Rightarrow (f, u) = 0 \forall u \in V$.

$(Gf, f) > 0$ if $f \neq 0$. $\forall u \in H \Rightarrow (f, f) = 0 \Rightarrow f = 0$.



G is self-adjoint.

$$(Gf, f) = a(Gf, Gf) \geq \alpha \|Gf\|^2, \quad Gf = 0 \Rightarrow (f, v) = 0 \quad \forall v \in V.$$

$$(Gf, f) > 0 \text{ if } f \neq 0, \quad V \text{ dense in } H \Rightarrow (f, f) = 0 \Rightarrow f = 0.$$

(c) Image of G dense in V .



$a(\cdot, \cdot)$ symm, elliptic and bil form $\langle u, v \rangle = a(u, v)$

defines an inner-product and the norm $\|u\|_a = \sqrt{a(u, u)}$ is equiv to usual norm.

Assume $v \in V$ s.t. $a(u, v) = 0 \quad \forall u \in \mathcal{R}(G), \quad u = Gf.$

$$\Rightarrow (f, v) = 0 \quad \forall f \in H \Rightarrow (v, v) = 0 \Rightarrow v = 0.$$

H-B thm. $\mathcal{R}(G)$ is dense in V .

So, we have $(Gf, g) = a(Gg, Gf) = a(Gf, Gg) = (f, Gg)_H$ and therefore, G is self adjoint. So, also $(Gf, f) = a(Gf, Gf) \geq \alpha \|Gf\|^2$. So, if $Gf = 0 \Rightarrow (f, v) = 0, \forall v \in V$, but then V is dense in H implies $(f, f) = 0 \Rightarrow f = 0$. So, therefore, you have $(Gf, f) > 0$ if $f \neq 0$.

(c): $Im(G)$ is dense in V .

So, a is a symmetric elliptic continuous bilinear form.

Therefore, $(u, v) = a(u, v)$ defines in the inner product and the norm, $\|u\|_a = \sqrt{a(u, u)}$ is equivalent to usual.


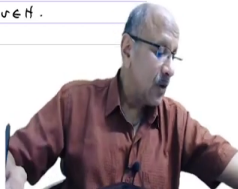
So, assume that assume $v \in V$ such that $a(u, v) = 0, \forall u \in Range(G)$, but $u = G(f)$. So,

and that means $(f, v) = 0, \forall v \in V$ but $V \subset H$ and therefore, this means $(v, v)_H = 0 \Rightarrow v = 0$


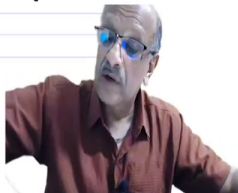
So, therefore by the Hahn Banach theorem the range of G is dense in V .

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(d) \exists an o.n. basis of H , $\{u_n\}$ and $\{\lambda_n\}$ seq. of pos. nos.
 s.t. $a(u_n, v) = \lambda_n(u_n, v) \quad \forall v \in H$.
 $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \rightarrow \infty$.
 $G: H \rightarrow H$ comp. self-adj. $\exists \{\mu_n\}$ eigenvalues $\mu_n \rightarrow 0$.
 dim of eigenspace of μ_n is finite.
 $\mu_n \neq 0 \quad Gu_n = \mu_n u_n \Rightarrow Gu_n = 0 \Rightarrow u_n = 0$.
 $(Gu_n, u_n) > 0 \quad u_n \neq 0 \Rightarrow \mu_n > 0$.
 $\lambda_n = \mu_n \quad \lambda_n \rightarrow \infty \quad Gu_n = \mu_n u_n$
 $\Rightarrow u_n = \frac{1}{\lambda_n} Gu_n$
 $a(u_n, v) = \frac{1}{\lambda_n} (Gu_n, v) = \lambda_n(u_n, v) \quad \forall v \in H$.

$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \rightarrow \infty$.
 $G: H \rightarrow H$ comp. self-adj. $\exists \{\mu_n\}$ eigenvalues $\mu_n \rightarrow 0$.
 dim of eigenspace of μ_n is finite.
 $\mu_n \neq 0 \quad Gu_n = \mu_n u_n \Rightarrow Gu_n = 0 \Rightarrow u_n = 0$.
 $(Gu_n, u_n) > 0 \quad u_n \neq 0 \Rightarrow \mu_n > 0$.
 $\lambda_n = \mu_n \quad \lambda_n \rightarrow \infty \quad Gu_n = \mu_n u_n$
 $\Rightarrow u_n = \frac{1}{\lambda_n} Gu_n$
 $a(u_n, v) = \frac{1}{\lambda_n} (Gu_n, v) = \lambda_n(u_n, v) \quad \forall v \in H$.
 Try to show Rayleigh quotient characterization of eigenvals.
 $\forall v \neq 0 \quad R(v) = \frac{a(v, v)}{\|v\|^2}$.

(d): there exists an orthonormal basis of H , $\{u_n\}$ and $\{\lambda_n\}$ sequence of positive numbers such that $a(u_n, v) = \lambda_n(u_n, v)$, $\forall v \in H$. So, this is the eigenvalue problem and you can write therefore, $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \rightarrow \infty$.

So, $G: H \rightarrow H$ is compact self adjoint bounded linear operator So, there exists Eigenvalues $\{\mu_n\}$ such that $\mu_n \rightarrow 0$. So, the sequence of Eigenvalues and dimension of Eigenspace of μ_n is finite. Now, $\mu_n \neq 0$ because $Gu_n = \mu_n u_n \Rightarrow Gu_n = 0 \Rightarrow u_n = 0$ this implies G of u_n equal to 0 and

that implies that u_n equal to 0 because $G f f$ equal to 0 we know implies f equal to 0. And therefore, μ_n is not so, μ_n cannot be 0 and so, we have and also $(Gu_n, u_n) > 0$ if $u_n \neq 0 \Rightarrow u_n > 0$.

Therefore, you put $\lambda_n = \frac{1}{\mu_n}$, then $\lambda_n \rightarrow \infty$ and $Gu_n = \mu_n u_n$

That means $u_n = G(\lambda_n u_n)$. That means $a(u_n, v) = (\lambda_n u_n, v) = \lambda_n (u_n, v)$, $\forall v \in H$.

So, this proves it.

Now you can go ahead and try to pause the show so, try to show Rayleigh quotient characterization of eigenvalues. So, you can write

$$R(v) = \frac{a(v,v)}{\|v\|_H^2}, \quad v \in V, \quad v \neq 0.$$

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② Eigenvalue Pb. (Neumann case). Ω bdd open set, $\partial\Omega \in \Gamma$ conn.

$$-\Delta u = \lambda u \text{ in } \Omega$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma.$$

$$-\Delta u + u = (\lambda + 1)u = \Lambda u.$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} uv \quad \begin{matrix} V = H^1(\Omega) \\ H = L^2(\Omega) \end{matrix}$$

All hypotheses of problem 1 satisfied. $\Rightarrow \exists$ o.n. basis $\{w_n\} \subset L^2(\Omega)$

and $\{\lambda_n\}$ pos. seq. $a(w_n, v) = \lambda_n(w_n, v) \quad \forall v \in H^1(\Omega)$

$$\Rightarrow \int_{\Omega} \nabla w_n \cdot \nabla v + \int_{\Omega} w_n v = \lambda_n \int_{\Omega} w_n v \quad \forall v \in H^1(\Omega).$$

$$\lambda_n = \lambda_{n-1} + 1.$$

$$\int_{\Omega} \nabla w_n \cdot \nabla w_n dx \geq 0 \Rightarrow \lambda_n \geq 0 \quad \forall n$$



So, this is the way you can study various eigenvalue problems based on this abstract framework.

So, we will give you one example now.

(2) (eigenvalue problem, Neumann case).

So, we have done the Dirichlet's case in the lectures. So, we are now looking at Neumann from.

So, let $\Omega \subset \mathbb{R}^N$ bounded open set and $\partial\Omega = \Gamma$ and consider

$$-\Delta u = \lambda u \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma.$$

So, we will also say connected this case. So, now you rewrite this equation as:

$$-\Delta u + u = (\lambda + 1)u = \Lambda u.$$

So, we will take $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} uv$, $V = H^1(\Omega)$, $H = L^2(\Omega)$.

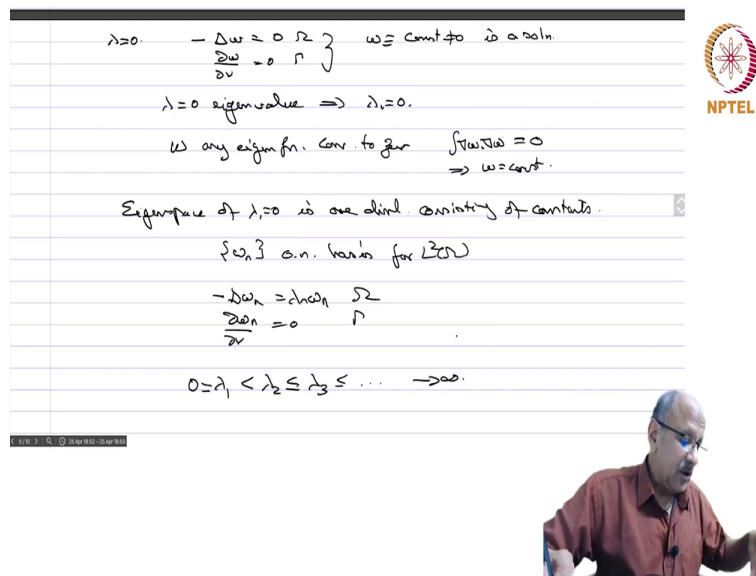
So, then on the hypothesis of problem 1 satisfied. So, this implies that there exists an orthonormal basis $\{w_n\}$ of $L^2(\Omega)$ and $\{\lambda_n\}$ positive sequence such that $a(w_n, v) = \lambda_n(w_n, v)$, for all $v \in H^1(\Omega)$.

So, this will imply that

$$\int_{\Omega} \nabla w_n \cdot \nabla v = (\Lambda_n - 1) \int_{\Omega} w_n v, \quad \forall v \in H^1(\Omega).$$

So, you set $\lambda_n = \Lambda_n - 1$ and therefore, since $\int_{\Omega} \nabla w_n \cdot \nabla w_n \geq 0 \Rightarrow \lambda_n \geq 0, \quad \forall n.$

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Handwritten notes on a digital screen:

$$\lambda = 0 \quad \left. \begin{array}{l} -\Delta w = 0 \text{ in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \end{array} \right\} w \equiv \text{const. is a soln.}$$

$\lambda = 0$ eigenvalue $\Rightarrow \lambda_1 = 0$.

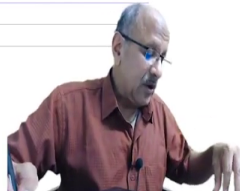
w any eigenfn. corr. to λ_1 $\int_{\Omega} w \Delta w = 0 \Rightarrow w = \text{const.}$

Eigenspace of $\lambda_1 = 0$ is one diml. consisting of constants.

$\{w_n\}$ o.n. basis for $L^2(\Omega)$

$$\left. \begin{array}{l} -\Delta w_n = \lambda_n w_n \text{ in } \Omega \\ \frac{\partial w_n}{\partial \nu} = 0 \text{ on } \Gamma \end{array} \right\}$$

$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \rightarrow \infty$.



So, what is the take, so assume what about $\lambda = 0$. So, you have

$$-\Delta u = \lambda u \text{ in } \Omega; \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma.$$

and $w \equiv \text{const.} \neq 0$ is a solution obviously and therefore, $\lambda = 0$ is an eigenvalue and this implies that $\lambda_1 = 0$. So, this is the first Eigen value will be 0 and then what is the Eigenspace?

Suppose you have an Eigen function for $\lambda = 0$. So, minus Laplacian w what happens if w any Eigenfunction corresponding to 0 we have

$$\int_{\Omega} \nabla w \cdot \nabla w = 0 \Rightarrow w = 0.$$

So, eigenspace of $\lambda_1 = 0$ is 1 dimensional consisting of constants. So, we have $\{w_n\}$

orthonormal basis for $L^2(\Omega)$ and then $-\Delta w_n = \lambda_n w_n$ in Ω ; $\frac{\partial w_n}{\partial \nu} = 0$ on Γ .

and you have $0 = \lambda_1 < \lambda_2 \leq \dots \rightarrow \infty$.

So, this is the thing and you can also do the Rayleigh quotient characterization.

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③ (a) Find the best constant in Poincaré's inequality.

$\Omega \subset \mathbb{R}^N$ bounded open set $|u|_{0,\Omega} \leq C|u|_{1,\Omega} \quad \forall u \in H_0^1(\Omega)$

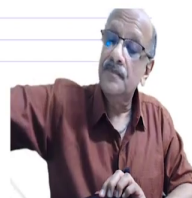
Best const. $C \Rightarrow \frac{1}{C} = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$

$\lambda_1^D = 1^{\text{st}}$ eigenval of Δ with $u=0$ on Γ (Dirichlet)

$\lambda_1^D = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} = \min_u \frac{|u|_{1,\Omega}^2}{|u|_{0,\Omega}^2}$

\Rightarrow Best constant $C = \frac{1}{\sqrt{\lambda_1^D}}$

Equality achieved for $u = w_1$



(3) (a), find the best constant in Poincaré inequality. $\Omega \subset \mathbb{R}^N$ bounded open set. So, you have

$$|u|_{0,\Omega} \leq C|u|_{1,\Omega}, \quad \forall u \in H_0^1(\Omega).$$

So, what is the best possible constant? So, best constant $C \Rightarrow \frac{1}{C} = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$.

But we know that if λ_1^D -first Eigenvalue of Δ with $u = 0$ on Γ , that is the Dirichlet boundary

condition. Then, you know that $\lambda_1^D = \min_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} = \min_{u \in H_0^1(\Omega), u \neq 0} \frac{|u|_{1,\Omega}^2}{|u|_{0,\Omega}^2}.$

$$\Rightarrow \text{Best constant } C = \frac{1}{\sqrt{\lambda_1^D}}.$$

equality achieved for $u = w_1$.

So, in any domain the first Eigenvalue of the Dirichlet Laplacian gives you the best constant for the thing.

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(b) Find the best constant in Poincaré-Wirtinger inequality.

$\Omega \subset \mathbb{R}^N$ bounded, $u \in H^1(\Omega)$, $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u$



$$|u - \bar{u}|_{0,\Omega} \leq C |u|_{1,\Omega}.$$

$$\Rightarrow |u - \bar{u}|_{0,\Omega} \leq C |u - \bar{u}|_{1,\Omega}.$$

$u \in H^1$, $u - \bar{u}$ is such that $\int_{\Omega} (u - \bar{u}) = 0$

P-W $\Leftrightarrow |u|_{0,\Omega} \leq C |u|_{1,\Omega}$ $\forall u \in H^1(\Omega), \int_{\Omega} u = 0$.

Best const. $C \Rightarrow \frac{1}{C} = \inf_{\substack{u \in H^1 \\ \int_{\Omega} u = 0}} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$

P-W $\Leftrightarrow |u|_{0,\Omega} \leq C |u|_{1,\Omega}$ $\forall u \in H^1(\Omega), \int_{\Omega} u = 0$.


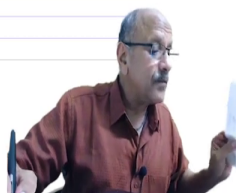
Best const. $C \Rightarrow \frac{1}{C} = \inf_{\substack{u \in H^1 \\ \int_{\Omega} u = 0}} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$

$0 = \lambda_1 < \lambda_2 \leq \dots$ $-\Delta u = \lambda^N u$ Ω
 $\frac{\partial u}{\partial \nu} = 0$ Γ

(Check!) $\lambda_2^N = \min_{\substack{u \perp V_1 \\ \int_{\Omega} u^2 = 1}} \int_{\Omega} |\nabla u|^2$ $V_1 = \text{1st const fn.}$
 $= \text{eigenfn of } \lambda_1^N = 0$

$\Rightarrow \lambda_2^N = \min_{\substack{u \perp V_1 \\ \int_{\Omega} u^2 = 1}} \int_{\Omega} |\nabla u|^2$

Best const. P-W is $C = \frac{1}{\sqrt{\lambda_2^N}}$.

(b) find the best constant in Poincaré Wirtinger inequality. So, $\Omega \subset \mathbb{R}^N$ bounded, $u \in H^1_0(\Omega)$,

$\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u \, dx$ and then $|u - \bar{u}|_{0,\Omega} \leq C |u|_{1,\Omega}$. Now, if you have this, this can also be written as

$|u - \bar{u}|_{0,\Omega} \leq C |u - \bar{u}|_{1,\Omega}$. And $|u| \in H^1$, then $u - \bar{u}$ is such that $\int_{\Omega} (u - \bar{u}) = 0$. So,

Poincare Wirtinger is equivalent to saying $|u|_{0,\Omega} \leq C |u|_{1,\Omega}$, $\forall u \in H^1(\Omega)$, $\int_{\Omega} u = 0$.

So, this is the same. So, the best constant is C implies $\frac{1}{C} = \inf_{\int u=0, u \neq 0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$.

But then if you look at the ((23:08)) problem, you have $0 = \lambda_1^N < \lambda_2^N \leq \dots$. So, these are the Neumann Eigenvalue. So,

$$-\Delta u = \lambda^N u \text{ in } \Omega; \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma.$$

So, these are the Neumann Eigenvalues which he saw and as I told you, you can do the so, then λ_2^N by the variational characterization you should be so, you have to check this, so check will be

$$\lambda_2^N = \min_{u \perp V_1} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2}, \quad V_1 = \text{is the space of constant functions} = \text{eigenspace}$$

of $\lambda_1^N = 0$.

So, this is the variation characterization which we saw and therefore, this implies that

$$\lambda_2^N = \min_{\int u=0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}.$$

So, the best constant in Poincare-Wirtinger is $C = \frac{1}{\sqrt{\lambda_2^N}}$. So, this way you have that.