Sobolev Spaces and Partial Differential Equations
Professor. S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Exercise – Part 12

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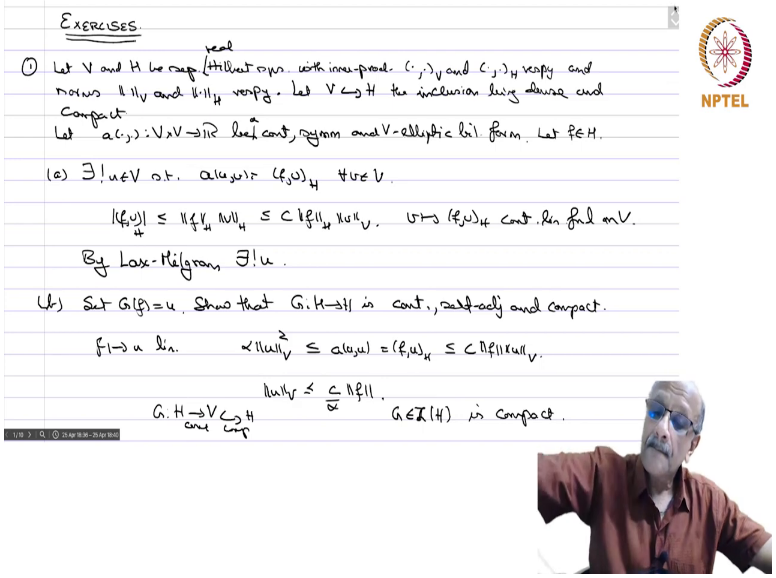

So, let us do some more exercises:

(1), let V and H be separable Hilbert spaces with a inner product $(.,.)_V$ and $(.,.)_H$ respectively and norms $||.||_V$ and $||.||_H$ respectively. Let us look $V \to H$, the inclusion being dense and compact. Let $a: V \times V \to \mathbb{R}$ be continuous, symmetric and V-elliptic by linear form. Let $f \in H$.

(a): there exists a unique $u \in V$ such that $a(u, v) = (f, v)_H$, for every $v \in H$.

So, the immediate consequence of the Lax-Milgram lemma. a is asymmetric V elliptic continuous by linear form. So, if you look at

$$|(f, v)_H| \leq ||f||_H ||v||_H \leq c||f||_H ||v||_V, \; therefore \; v \to (f, v)_H$$

is a continuous linear functional on V. Therefore, by Lax-Milgram there exists a unique u.

(b): set $G(f) = u$. Show that $G: H \to H$ is continuous self adjoint and compact. We are of course, talking of real Hilbert spaces that are understood to be dealing with real functions.

So, this we are trying to imitate what we did for the Laplacian. So, this is the abstract framework. So, $f \to u$ is certainly linear, $\alpha||u||^2_V \leq a(u,u) = (f,u)_H \leq c||f||||v||_V$

$$\Rightarrow ||u||_V \leq \frac{c}{\alpha}||f||.$$

And therefore, you have $G \to H \to V$ is continuous and this inclusion is continuous and compact. Therefore, $G \in L(H)$ is compact. It is self adjoint as we have seen many times.

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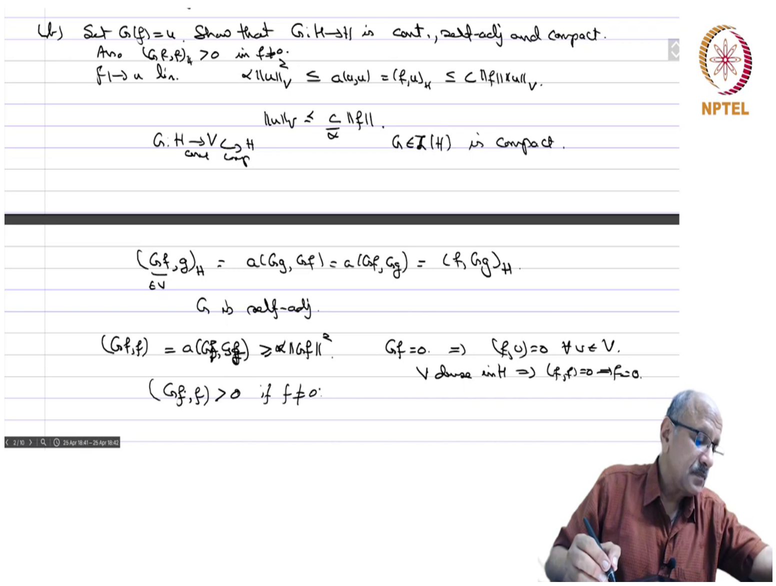

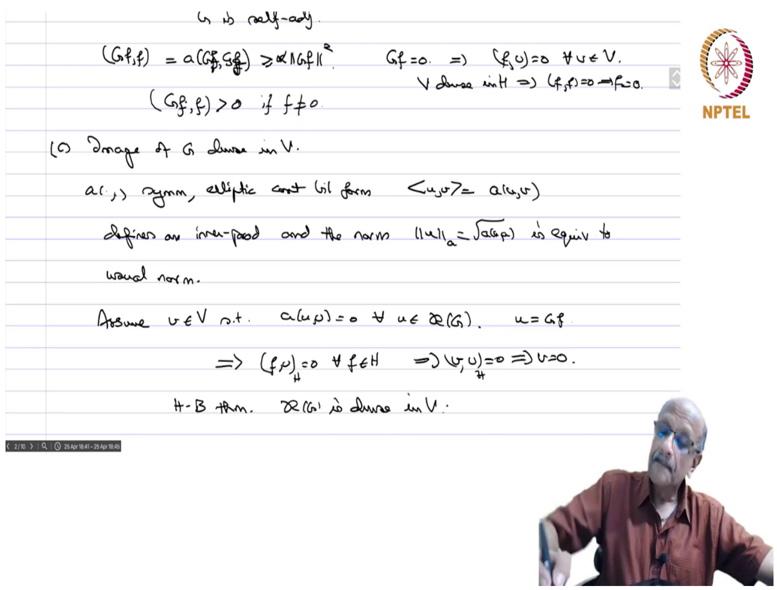

So, we have $(Gf, g) = a(Gg, Gf) = a(Gf, Gg) = (f, Gg)_H$ and therefore, G is self adjoint. So, also $(Gf, f) = a(Gf, Gf) \geq \alpha||Gf||^2$. So, if $Gf = 0 \Rightarrow (f, v) = 0, \forall v \in V$, but then V is dense in H implies $(f, f) = 0 \Rightarrow f = 0$. So, therefore, you have $(Gf, f) > 0 \, if \, f \neq 0$.

(c): $Im(G)$ is dense in V.

So, a is a symmetric elliptic continuous bilinear form.

Therefore, $(u, v) = a(u, v)$ defines in the inner product and the norm, $||u||_a = \sqrt{a(u, u)}$ is equivalent to usual.

So, assume that assume $v \in V$ such that $a(u, v) = 0, \forall u \in Range(G)$, but $u = G(f)$. So,

and that means $(f, v) = 0, \forall v \in V$ but $V \subset H$ and therefore, this means $(v, v)_H = 0 \Rightarrow v = 0$

So, therefore by the Hahn Banach theorem the range of G is dense in V.

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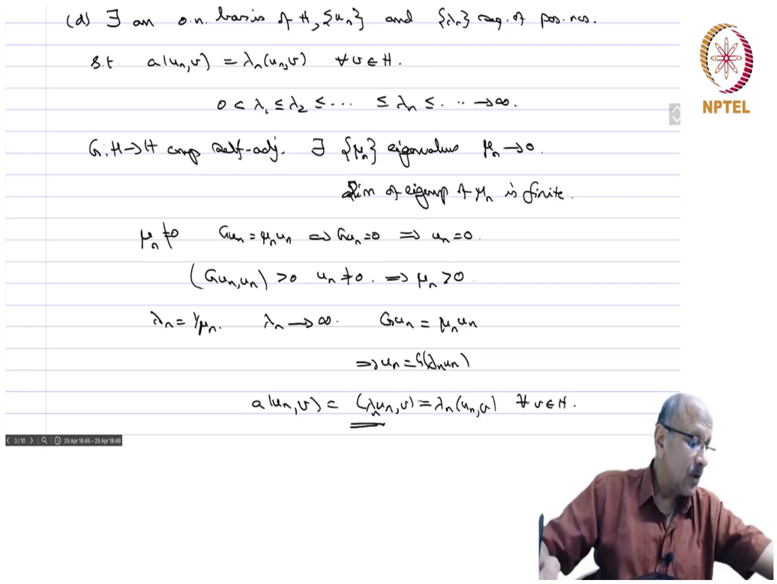

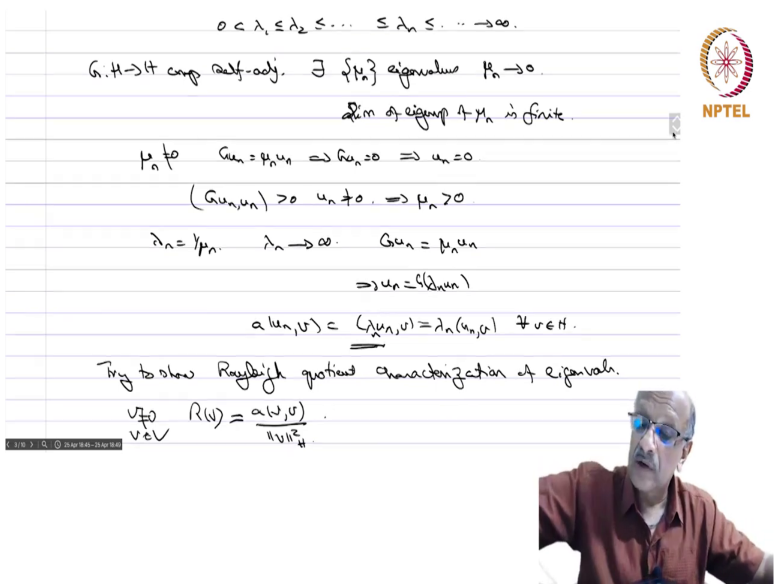

(d): there exists an orthonormal basis of H, $\{u_n\}$ and $\{\lambda_n\}$ sequence of positive numbers such that $a(u_n, v) = \lambda_n(u_n, v), \forall v \in H$. So, this is the eigenvalue problem and you can write therefore, $0 < \lambda_1 \le \lambda_2 \le .... \le \lambda_n .... \to \infty$.

So, $G: H \to H$ is compact self adjoint bounded linear operator So, there exists Eigenvalues $\{\mu_n\}$ such that $\mu_n \to 0$. So, the sequence of Eigenvalues and dimension of Eigenspace of μ_n is finite. Now, $\mu_n \neq 0$ because $Gu_n = \mu_n u_n \Rightarrow Gu_n = 0 \Rightarrow u_n = 0$ this implies G of un equal to 0 and that implies that u n equal to 0 because G f f equal to 0 we know implies f equal to 0. And therefore, mu n is not so, mu n cannot be 0 and so, we have and also $(Gu_n, u_n) > 0\ if\ u_n \neq 0 \Rightarrow u_n > 0$.

Therefore, you put $\lambda_n = \frac{1}{\mu_n}$, $then\ \lambda_n \rightarrow \infty\ and\ Gu_n = \mu_n u_n$

That means $u_n = G(\lambda_n u_n)$. That means $a(u_n, v) = (\lambda_n u_n, v) = \lambda_n (u_n, v)\ , \forall\, v \in H$.

So, this proves it.

Now you can go ahead and try to pause the show so, try to show Rayleigh quotient characterization of eigenvalues. So, you can write

$$R(v) = \frac{a(v,v)}{||v||_H^2}\ ,\ v \in V,\ v \neq 0\,.$$

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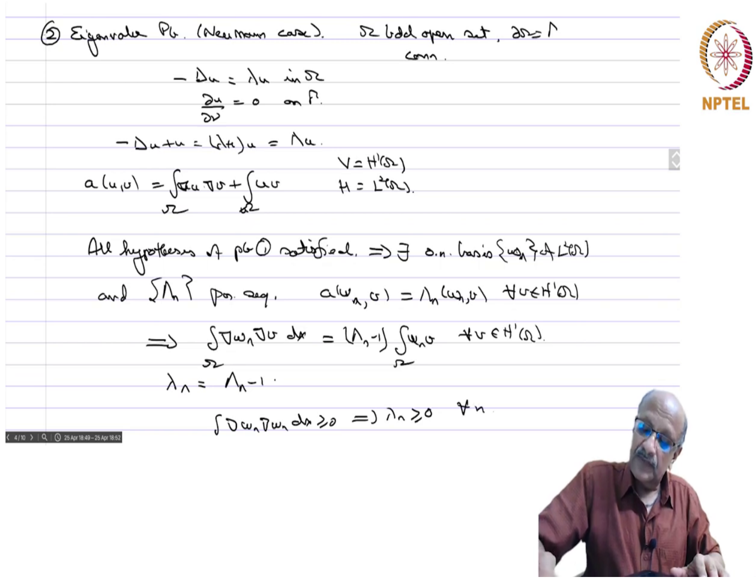

So, this is the way you can study various eigenvalue problems based on this abstract framework. So, we will give you one example now.

(2) (eigenvalue problem, Neumann case).

So, we have done the Dirichlet's case in the lectures. So, we are now looking at Neumann from.

So, let $\Omega \subset \mathbb{R}^N$ bounded open set and $\partial\Omega = \Gamma$ and consider

$$-\Delta u = \lambda u \; in \; \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \; on \; \Gamma.$$

So, we will also say connected this case. So, now you rewrite this equation as:

$$-\Delta u + u = (\lambda + 1)u = \Lambda u.$$

So, we will take $a(u, v) = \int_\Omega \nabla u . \nabla v + \int_\Omega uv$, $V = H^1(\Omega)$, $H = L^2(\Omega)$.

So, then on the hypothesis of problem 1 satisfied. So, this implies that there exists an orthonormal basis $\{w_n\}$ of $L^2(\Omega)$ and $\{\Lambda_n\}$ positive sequence such that $a(w_n, v) = \Lambda_n(w_n, v)$, for all $v \in H^1(\Omega)$.

So, this will imply that

$$\int_{\Omega} \nabla w_n . \nabla v = (\Lambda_n - 1) \int_{\Omega} w_n v \ , \ \forall\, v \in H^1(\Omega) \, .$$

So, you set $\lambda_n = \Lambda_n - 1$ and therefore, since $\int_{\Omega} \nabla w_n . \nabla w_n \geq 0 \Rightarrow \lambda_n \geq 0, \ \forall\, n$.

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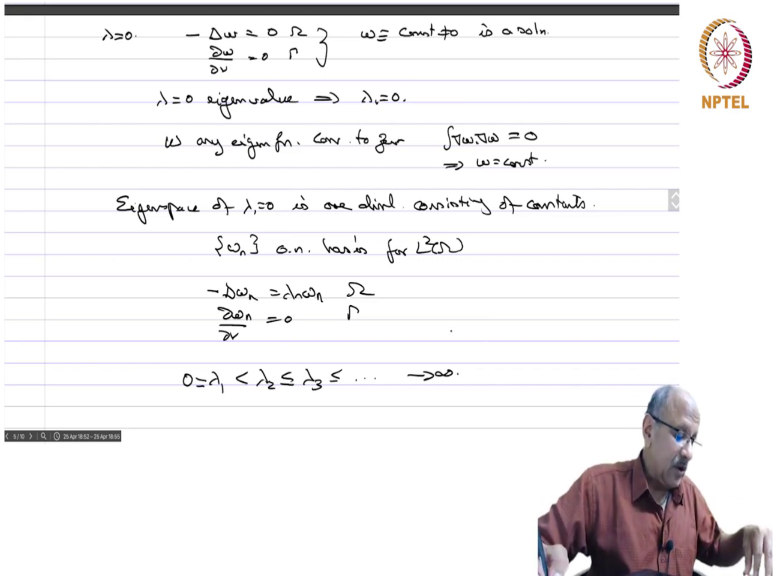

So, what is the take, so assume what about $\lambda = 0$. So, you have

$$-\Delta u = \lambda u \ in\,\Omega\,; \ \frac{\partial u}{\partial \nu} = 0 \ on\,\Gamma\,.$$

and $w \equiv cons. \neq 0$ is a solution obviously and therefore, $\lambda = 0$ is an eigenvalue and this implies that $\lambda_1 = 0$. So, this is the first Eigen value will be 0 and then what is the Eigenspace?

Suppose you have an Eigen function for lambda lambda equals 0. So, minus Laplacian w what happens if w any Eigenfunction corresponding to 0 we have

$$\int_{\Omega} \nabla w \,.\, \nabla w \ = 0 \Rightarrow w = 0\,.$$

So, eigenspace of $\lambda_1 = 0$ is 1 dimensional consisting of constants. So, we have $\{w_n\}$ orthonormal basis for $L^2(\Omega)$ and then $-\ \Delta w_n = \lambda_n w_n \ in\,\Omega\,; \ \frac{\partial w_n}{\partial \nu} = 0 \ on\,\Gamma\,.$

and you have $0 = \lambda_1 < \lambda_2 \leq .... \rightarrow \infty$.

So, this is the thing and you can also do the Rayleigh quotient characterization.

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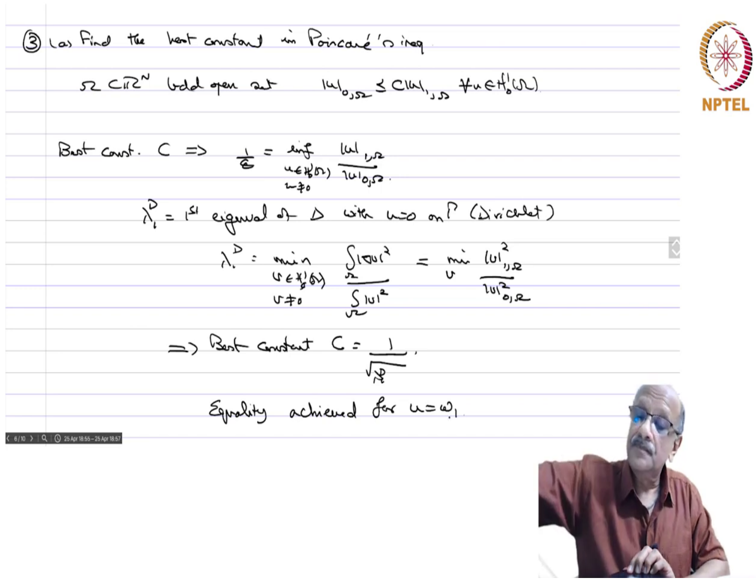

(3) (a), find the best constant in Poincare inequality. $\Omega \subset \mathbb{R}^N$ bounded open set. So, you have

$$|u|_{0,\Omega} \leq C|u|_{1,\Omega}, \ \forall u \in H^1_0(\Omega).$$

So, what is the best possible constant? So, best constant $C \Rightarrow \frac{1}{C} = \inf_{u \in H^1_0(\Omega),\, u \neq 0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$.

But we know that if λ^D_1-first Eigenvalue of Δ with $u = 0 \ on\, \Gamma$, that is the Dirichlet boundary condition. Then, you know that $\lambda^D_1 = \min_{u \in H^1_0(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \min_{u \in H^1_0(\Omega),\, u \neq 0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$.

$$\Rightarrow Best\ constant\ C = \frac{1}{\sqrt{\lambda^D_1}}.$$

equality achieved for $u = w_1$.

So, in any domain the first Eigenvalue of the Dirichlet Laplacian gives you the best constant for the thing.

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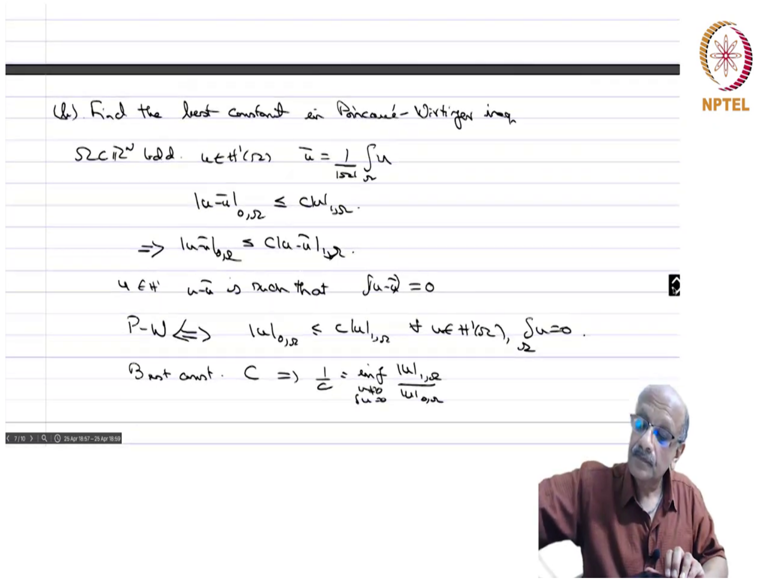

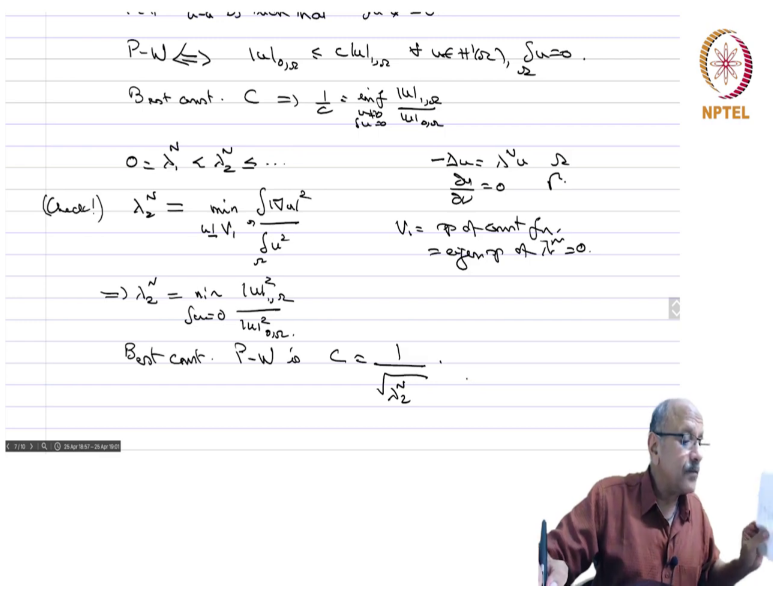

(b) find the best constant in Poincare Wirtinger inequality. So, $\Omega \subset \mathbb{R}^N$ bounded, $u \in H^1{}_0(\Omega)$,

$\bar{u} = \frac{1}{|\Omega|}\int_{\Omega} u\,dx$ and then $|u - \bar{u}|_{0,\Omega} \leq C\,|u|_{1,\Omega}$. Now, if you have this, this can also be written as

$|u - \bar{u}|_{0,\Omega} \leq C\,|u - \bar{u}|_{1,\Omega}$. And $|u| \in H^1$, then $u - \bar{u}$ is such that $\int_{\Omega}(u - \bar{u}) = 0$. So,

Poincare Wirtinger is equivalent to saying $|u|_{0,\Omega} \leq C\,|u|_{1,\Omega}$, $\forall\, u \in H^1(\Omega)$, $\int_{\Omega} u = 0$.

So, this is the same. So, the best constant is C implies $\frac{1}{C} = \inf_{\int u = 0,\, u \neq 0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}$.

But then if you look at the (())(23:08) problem, you have $0 = \lambda^N_1 < \lambda^N_2 \leq$...... So, these are the Neumann Eigenvalue. So,

$$- \Delta u = \lambda^N u \; in\, \Omega\, ; \; \frac{\partial u}{\partial \nu} = 0 \; on\, \Gamma\, .$$

So, these are the Neumann Eigenvalues which he saw and as I told you, you can do the so, then λ^N_2 by the variational characterization you should be so, you have to check this, so check will be

$$\lambda^N_2 = \min_{u \perp V_1} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2}\ ,\ V_1 = \text{is the space of constant functions = eigenspace}$$

of $\lambda^N_1 = 0$.

So, this is the variation characterization which we saw and therefore, this implies that

$$\lambda^N_2 = \min_{\int u = 0} \frac{|u|_{1,\Omega}}{|u|_{0,\Omega}}\, .$$

So, the best constant in Poincare-Wirtinger is $C = \frac{1}{\sqrt{\lambda^N_2}}$. So, this way you have that.