Sobolev Spaces and Partial Differential Equations

Professor. S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences

Eigenvalue problems – Part 3

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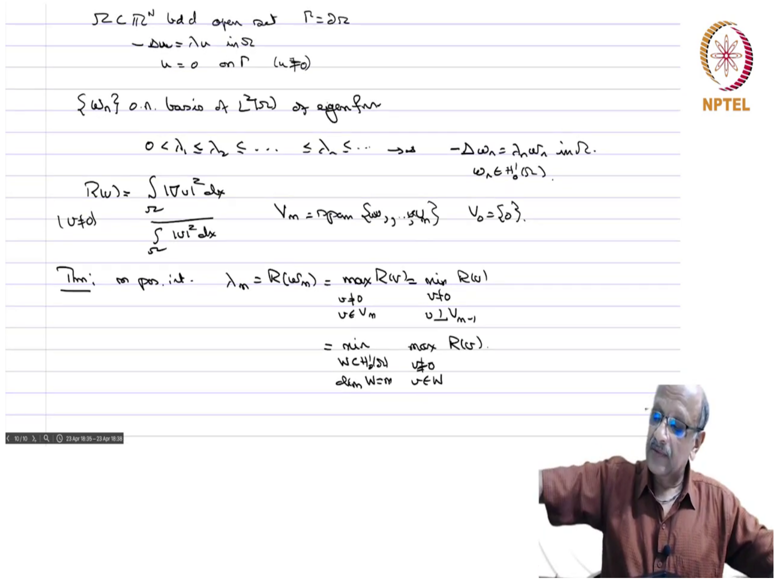

So, we were looking at eigenvalue problems. So, $\Omega \subset \mathbb{R}^N$ bounded open set and $\Gamma = \partial\Omega$ and we were looking at

$$- \Delta u = \lambda u \ in\, \Omega\, ,$$

$$u = 0\, on\, \Gamma\, . \quad (u \neq 0)$$

This is the eigenvalue problem. So, then we saw that there is $\{w_n\}$ orthonormal basis of $L^2(\Omega)$ (also $H^1{}_0(\Omega)$) with some factor in the front orthogonal the L2 for omega i of Eigenfunctions and

$$0 < \lambda_1 \leq \lambda_2 \leq ... \lambda_n \leq \ ..... \rightarrow \infty$$

So, $- \ \Delta w_n = \lambda w_n \ in\, \Omega \ , \ w_n \in H^1{}_0(\Omega)\, .$

So, we repeat these values according to the geometric multiplicity of the dimension of the Eigenspace. So, if lambda 2 has a 2 dimensional eigenspace, then lambda 2 and lambda 3 will be called the same. So, then we prove the following theorem variational characterization : the Rayleigh quotient $R(v) = \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}$.

And then we had the following theorem that m is a positive integer. So, $V_m = span\{w_1, w_2, ..., w_m\}$, $V_0 = \{0\}$.

Theorem: we have that $\lambda_m = R(w_m) = \max_{v \in V_m, v \neq 0} R(v) = \min_{v \perp V_{m-1}, v \neq 0} R(v)$

$$= \min_{W \subset H^1_0(\Omega), \dim W = m} \ \max_{v \in W, v \neq 0} R(v).$$

So, this was the min max or the intrinsic characterization. So, this was the theorem which we proved last time.

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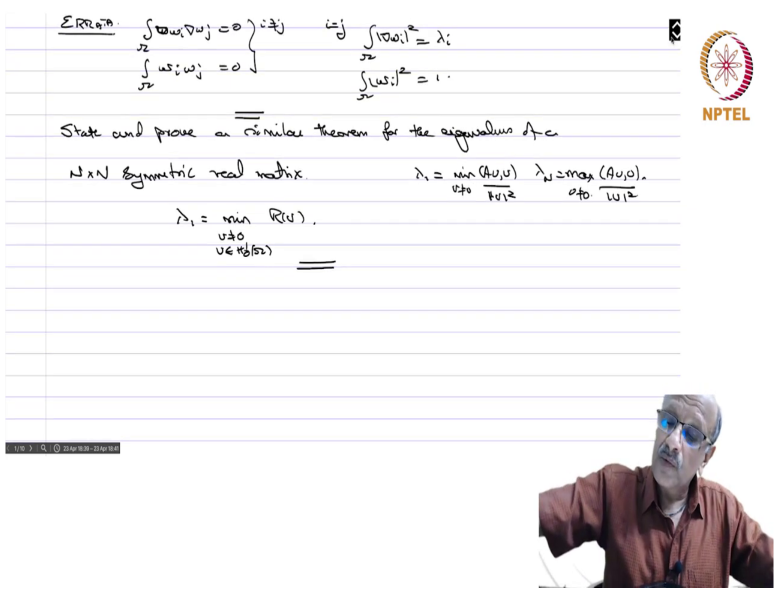

So, there was some errata:

$$\int_\Omega \nabla w_i . \nabla w_j = 0 \ and \int_\Omega w_i . w_j = 0 \ for \ i \neq j.$$

For $i = j$, $\int_\Omega \nabla w_i . \nabla w_i = \lambda_i$ and $\int_\Omega w_i . w_j = 1$.

So, here I wrote an integral grad Wi square or something like that. So, there was something wrong here and (())(04:19).

Now state and prove a similar theorem for the eigenvalues of $N \times N$ symmetric real matrices. So, you know that for symmetric real matrix the Eigenvalues are all real so you can write them in ascending in order ascending order for lambda 1 to lambda n and of course, the Eigenvalues will be orthogonal in the usual sense in Euclidean space and therefore, you can state and prove exactly the same kind of theorem which you have.

So, we also had that $\lambda_1 = \min_{v \neq 0} \frac{(Av,v)}{|v|^2}$, $\lambda_N = \max_{v \neq 0} \frac{(Av,v)}{|v|^2}$.

So, this is the corresponding result you have for the eigenvalue and this is very useful in computing the Eigenvalues of symmetric matrices.

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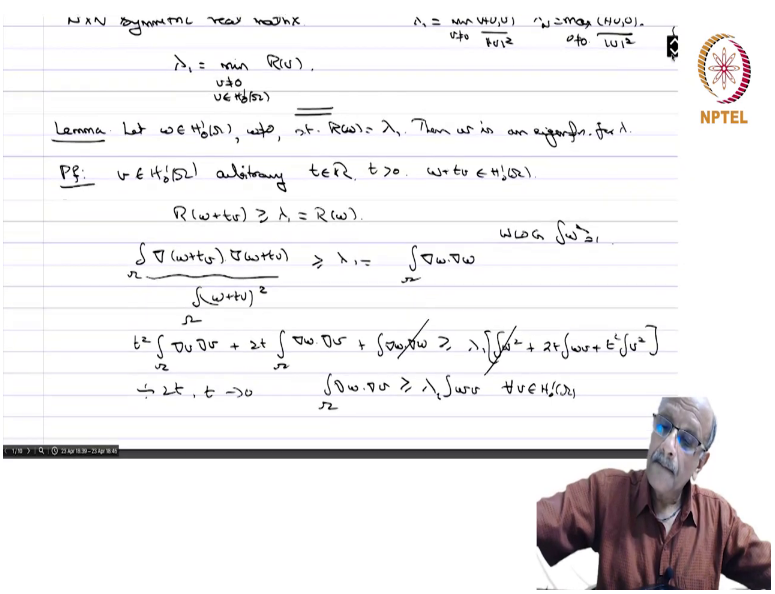

So, let us now continue with the (())(06:08) problem. So, we have the following lemma.

Lemma : let $w \in H^1_0(\Omega)$, $w \neq 0$ $s.t.$ $R(w) = \lambda_1$. Then w is an Eigenfunction for λ.

proof: So, let $v \in H^1_0(\Omega)$, $t \in \mathbb{R}, t > 0$. Then $w + tv \in H^1_0(\Omega)$ and you have

$$R(w + tv) \geq R(w) = \lambda_1 .$$

which is because it is a minimum which is equal to R of w. So, now that means

$$\frac{\int_\Omega \nabla(w+tv).\nabla(w+tv)}{\int_\Omega (w+tv)^2} \geq \lambda_1 = \int_\Omega \nabla w . \nabla w \, , \quad assume\ that \int_\Omega w^2 = 1.$$

So, we expand and cross multiply and simplify so, you get

$$t^2 \int_\Omega \nabla v . \nabla v + 2t \int_\Omega \nabla w . \nabla v + \int_\Omega \nabla w . \nabla w \geq \lambda_1 [\int_\Omega w^2 + 2t \int_\Omega wv + t^2 \int_\Omega v^2] .$$

But grad v grad w equals lambda 1 w square, so, these two terms will get canceled so, divide by 2t and let t tend to 0. So, if you do that, then you will get the

$$\int_\Omega \nabla w . \nabla v \geq \lambda_1 \int_\Omega w . v \, , \ \forall \, v \in H^1_0(\Omega) .$$

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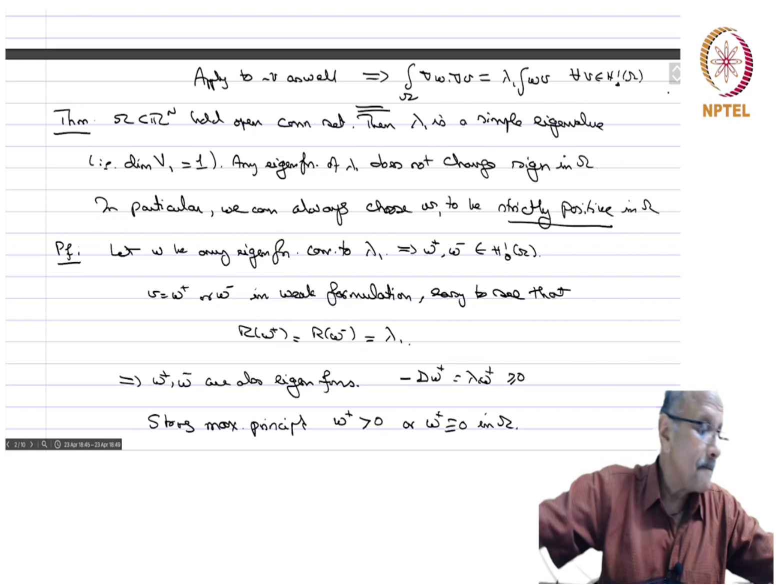

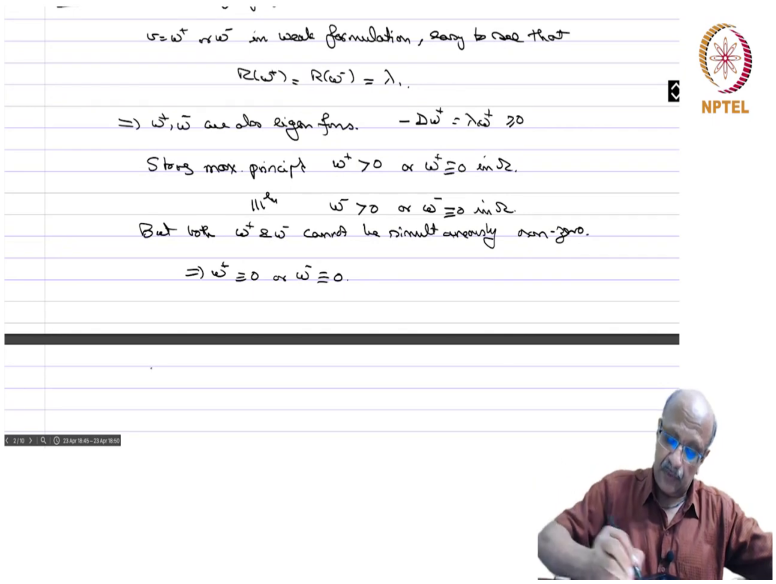

Apply to minus V as well and that will imply you get the opposite inequality so, you did integral grad w grad v equals lambda 1 integral wv for every v in H1 0 of omega and therefore it is an Eigenfunction because it satisfies the weak formulation of the eigenvalue problem. So, now we have a very nice theorem:

Theorem: $\Omega \subset \mathbb{R}^N$ bounded open connected set, then λ_1 is a simple eigenvalue. Any Eigenfunction of λ_1 does not change sign in Ω. In particular, we can always choose w_1 to be strictly positive in Ω.

proof: So, let w be any Eigenfunction corresponding to λ_1, then $w^+, w^- \in H^1_0(\Omega)$. So, now if you choose $v = w^+ \, or \, w^-$ in weak formulation. So, easy to see that $R(w^+) = R(w^-) = \lambda_1$.

So, this implies that w^+, w^- are also Eigen functions. So, $-\Delta w^+ = \lambda_1 w^+ \geq 0$ and therefore, by the strong maximum principle we have $w^+ > 0$ or $w^+ \equiv 0$ in Ω.

Similarly, w^- is strictly positive or w^- is identically 0 in w^-. But both w^+ and w^- cannot be simultaneously non-zero because they are the positive and negative parts of the w and therefore, they cannot be simultaneously non-zero. So, therefore, you have that $w^+ \equiv 0$ or $w^- \equiv 0$.

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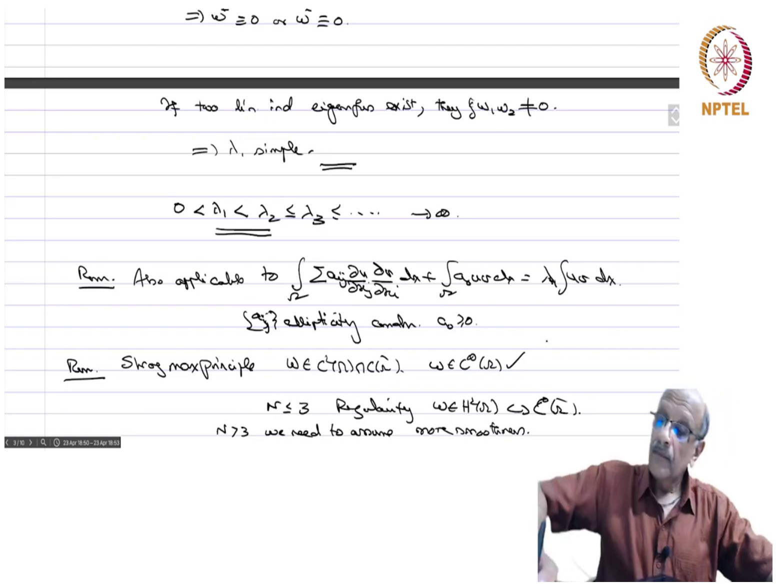

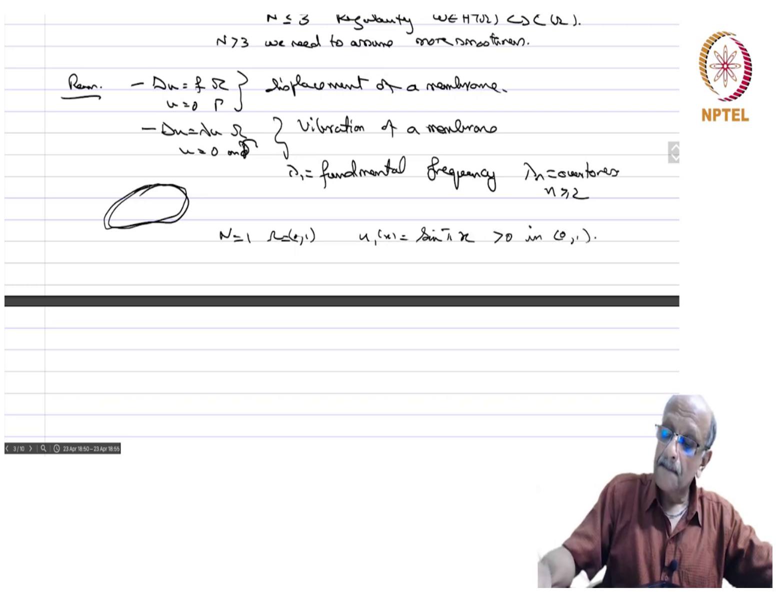

And now so, you have that the Eigenfunction does not change in fact you can have it strictly positive therefore, if two linearly independent Eigen functions exist, they are orthogonal, then $\int w_1 . w_2 \neq 0$. But we always know we can find orthogonal basis of Eigenvectors and Eigen functions and therefore, that would not be possible because any two of them will always have constant sign. So, the integral cannot vanish and therefore, you have that λ_1 is simple.

So, this means what? So, you have $0 < \lambda_1 < \lambda_2 \leq ... \lambda_n \leq \; ..... \rightarrow \infty$.

So, you have here because of the strong maximum of principles. So, remark.

Remark: also applicable to

$$\int_{\Omega} \sum a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \int_{\Omega} a_0 uv = \lambda_1 \int_{\Omega} uv\,.$$

λ_1 let us say, the eigenvalue problem for the elliptic operators where a_{ij} satisfy ellipticity condition and $a_0 \geq 0$.

Remark: So, this theorem is, now we have applied the strong maximum principle. So, which we usually need u should be so, we needed the $w \in C^2(\Omega) \cap C(\overline{\Omega})$. But we already know that w belongs to $C^{\infty}(\Omega)$. Now what about $C(\overline{\Omega})$ well if $N \leq 3$ by regularity $w \in H^2(\Omega) \rightarrow C(\overline{\Omega})$.

But this for almost every simple domain. But if you want $N \geq 3$ then we need to assume more smoothness.

Remark: we said that if you have $-\Delta u = f,\ u = 0$ on the boundary , then this is the displacement of a membrane.

So, if you have $-\Delta u = \lambda u,\ u = 0\ on\, \Gamma$, then this is nothing but vibration of a membrane. So, λ_1 is called the fundamental frequency and λ_n are called the overtones. So, if you have a drum, so a drum is what is a drum it is a membrane which is stretched over a frame and then you beat this membrane then you get so, the vibration will be given by these things.

And so, if you look at the n equals 1 omega equals 01 of course, here all the eigenvalues are simple and you have u1 of x is sine by x which is of course, strictly positive in 01.

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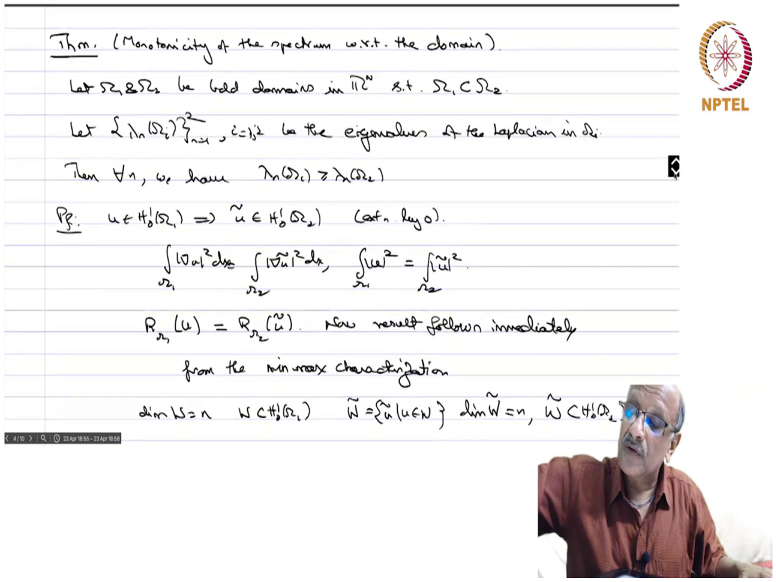

Theorem: (monotonicity of the spectrum with respect to the domain). So, let Ω_1 *and* Ω_2 be bounded domains in $\mathbb{R}^N$ such that $\Omega_1 \subset \Omega_2$. So, let $\{\lambda_n(\Omega_i)\}^2_{i=1}$ be the Eigenvalues of the Laplacian in Ω_i, then for every n we have $\lambda_n(\Omega_1) \geq \lambda_n(\Omega_2)$.

proof: Proof is just very simple. So, if you have u is in H10 of omega 1, so this implies that you tilde is in H1 0 of omega 2 extension by 0 and further integral on omega 1 of mod grad u square equals integral on omega 2 mod grad u tilde square dx and integral on omega 1 of mod u square equals integral of omega 2 u tilde square. Therefore, the Rayleigh quotient with respect to omega 1 of u is the same as the Rayleigh quotient with respect to omega 2 of your tilde.

Now, the result follows immediately from the min max characterization. So, if W dimension W equals n W in H10 of omega 1, then W tilde is set value tilde u in W, then dimension of W tilde equals n and W tilde is contained in H10 of omega 2. So, every and Rayleigh quotients are the same. So, the maximum over W is the same as the maximum over W tilde but the more spaces in omega 2 which are of M dimension therefore, the minimum of the maximum will be less for the bigger domain. So, it immediately follows from the characterization of the thing.

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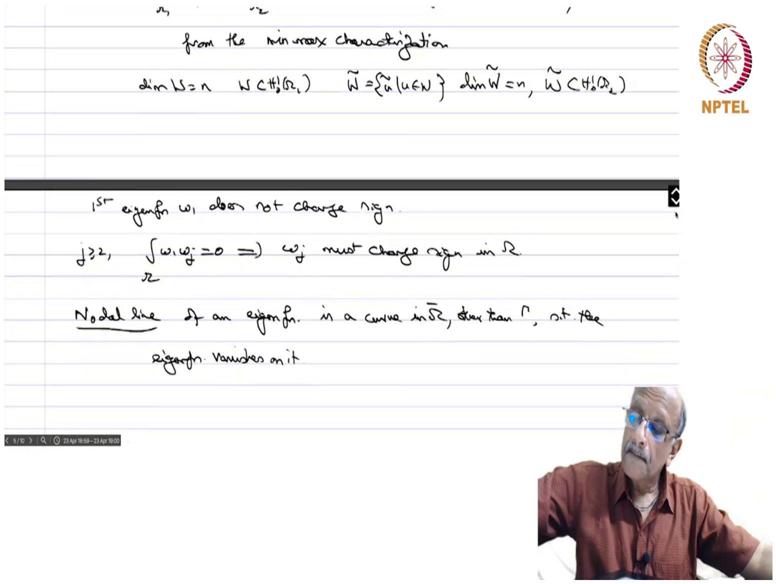

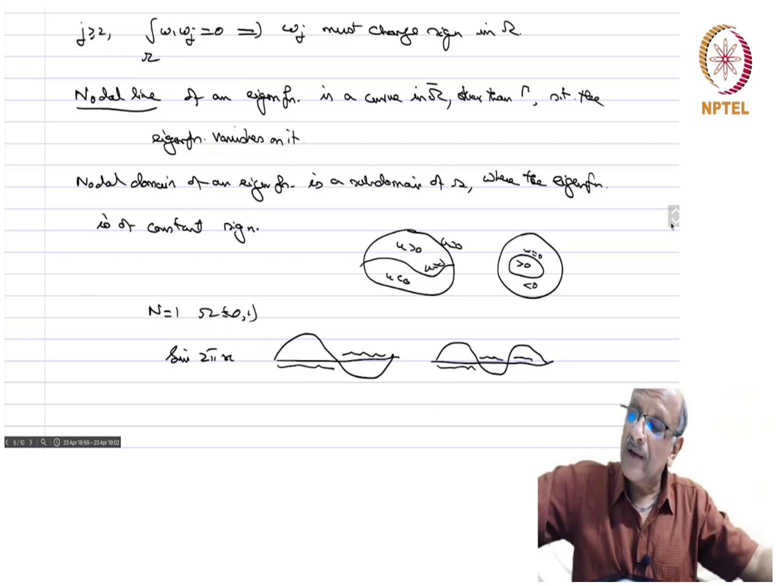

So, now, we saw that the first Eigenfunction w1 does not change sin omega connected. So, if j is greater than or equal to 2 integral w1 wj is 0. So, this implies that wj must change sign; you cannot have something which is a constant sign anyway. So, we have a nodal line of an Eigenfunction is a curve in omega other than gamma such that the Eigen function vanishes on it. Nodal domain of an Eigenfunction is a sub domain of omega where the Eigenfunction is of constant sign.

So, I have omega here so, suppose, u vanishes along a curve like this then u will be positive here u will be negative here u will be 0 here u will be 0 on the boundary also. And these two are called nodal domains. You could also have nodal domains like this. So, u equal to 0 is a closed curve like this, then it could be positive here it could be negative here and so on. So, you could have the mini Nodal domain. So, if you take in n equals 1 omega equals 01.

So, if you take sin 2 pi x, then it will have it will be like this sin 3 pi x will be like this and so on. So, you have these are all Nodal domains this is Nodal domain this is Nodal domain and you have. So, now we have a very beautiful theorem.

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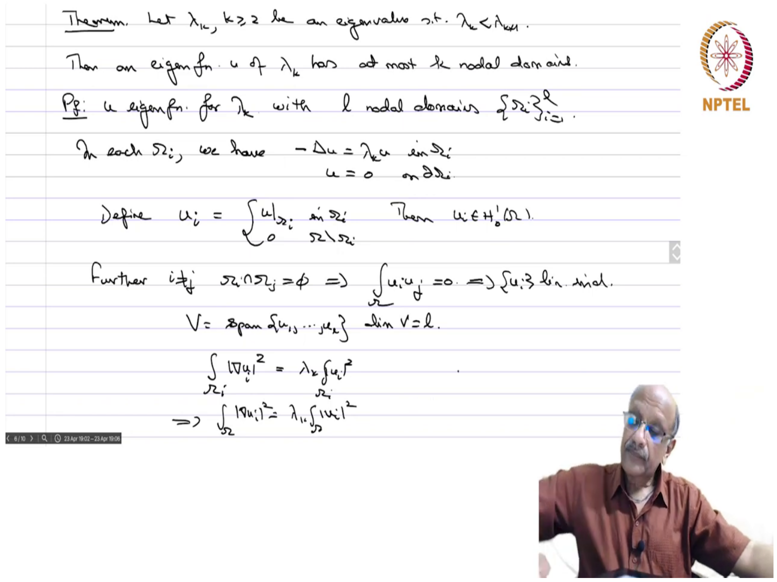

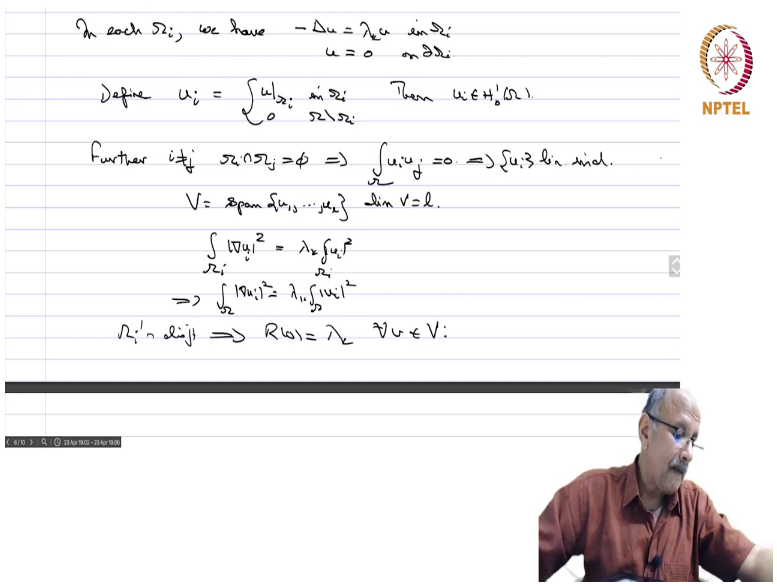

Theorem: Let λ_k , $k \geq 2$ be an eigenvalue such that $\lambda_k < \lambda_{k+1}$, then an Eigenfunction u of λ_k has at most k nodal domains.

proof : so, u Eigen function for λ_k with l nodal domains $\{\Omega_i\}_{i=1}^{l}$. In each Ω_i, we have

$$- \Delta u = \lambda_k u \ in\, \Omega_i \, , \, u = 0 \; on\, \partial\Omega \, .$$

So, define $u_i = u|_{\Omega_i}$ in Ω_i and $u_i = 0$ in $\Omega \backslash \Omega_i$. Then $u_i \in H^1_0(\Omega)$. Further for $i \neq j$, $\Omega_j \cap \Omega_j = \phi$, $\Rightarrow \int_\Omega u_j u_i = 0 \Rightarrow \{u_i\}$ are linearly independent.

And this implies that $V = span\,\{u_1,\dots u_l\}$ and $\dim V = l$. So,

$$\int_{\Omega_i} |\nabla u_i|^2 = \lambda_k \int_{\Omega_i} |u_i|^2 \Rightarrow \int_\Omega |\nabla u_i|^2 = \lambda_k \int_\Omega |u_i|^2 .$$

And since all the Ω_i are disjoint, this implies that $R(v) = \lambda_k$, $\forall\, v \in V$. So, now we are through.

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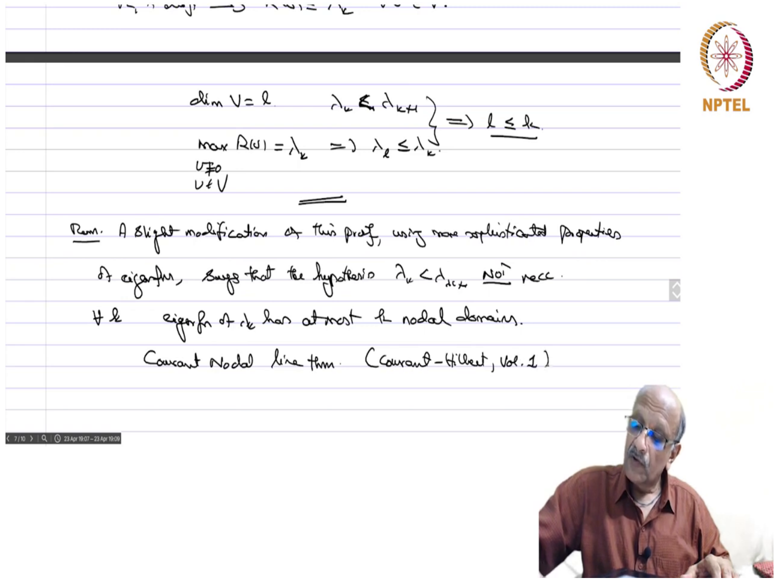

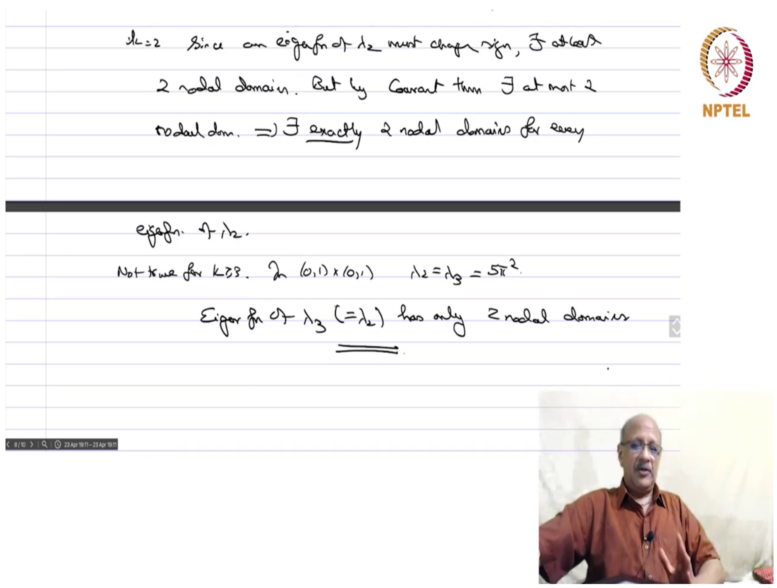

So, $\dim V = l$ and $\lambda_k < \lambda_{k+1}$ and you have that the $\max_{v \in V, v \neq 0} R(v) = \lambda_k$ and therefore, the $\dim V = l$ and therefore, this implies that $\lambda_l \leq \lambda_k$ and lambda k is k plus 1 is bigger and this these two together implying since we are writing in increasing order that $l \leq k$.

That proves the theorem.

So, that proves that at most k nodal domain.

Remark: A slight modification of this proof using more sophisticated properties of Eigenfunctions says that the hypothesis $\lambda_k < \lambda_{k+1}$ not necessary. Therefore, for every k we have that the Eigen function of lambda k has at most k nodal domains.

This is called the Courant Nodal line theorem and you can see this for instance in Courant and Hilbert volume 1. Methods of mathematical physics, that is the title of the book. So, Courant nodal line theorem.

So, for instance if you take $k = 2$, since an Eigenfunction of λ_2 must change sign there exists at least 2 nodal domains. But, by Courant theorem now, there exists at most 2 nodal domains, which implies there exist exactly 2 nodal domains for every Eigen function of λ_2.

So, for λ_1 does exactly 1 nodal domain for λ_2 the exactly 1 Nodal domain is not true for k greater equal to 3.

For instance in the square in $(0, 1) \times (0, 1)$, we have $\lambda_2 = \lambda_3 = 5\pi^2$.

So, Eigen function of λ_3, which is also equal to λ_2 has only 2 nodal domains. So, we just see that the min max has many applications. These are just some of the applications.

And the spectrum of the Laplacian has several very, very interesting properties. It gives us a lot of geometric information about the domain. And it is a very fascinating subject, which is in the confluence of geometry, functional analysis and partial differential equations.