Sobolev Spaces and Partial Differential Equations
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Eigenvalues – Part 2

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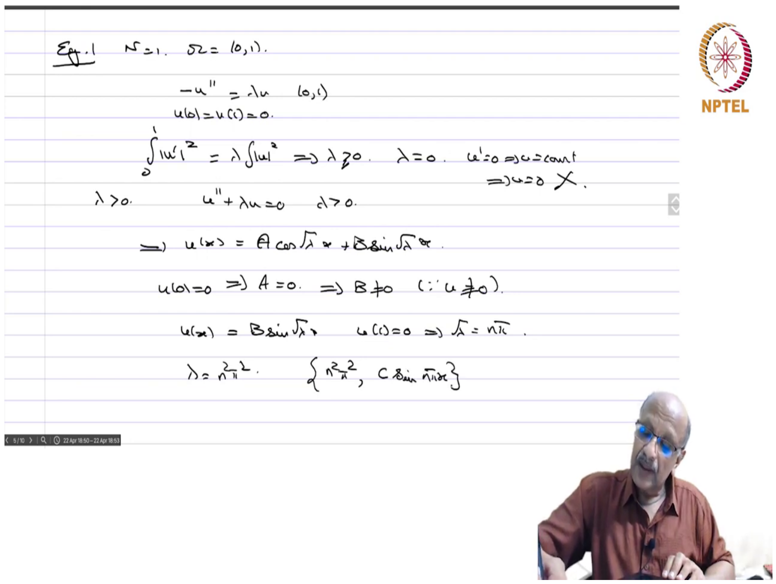

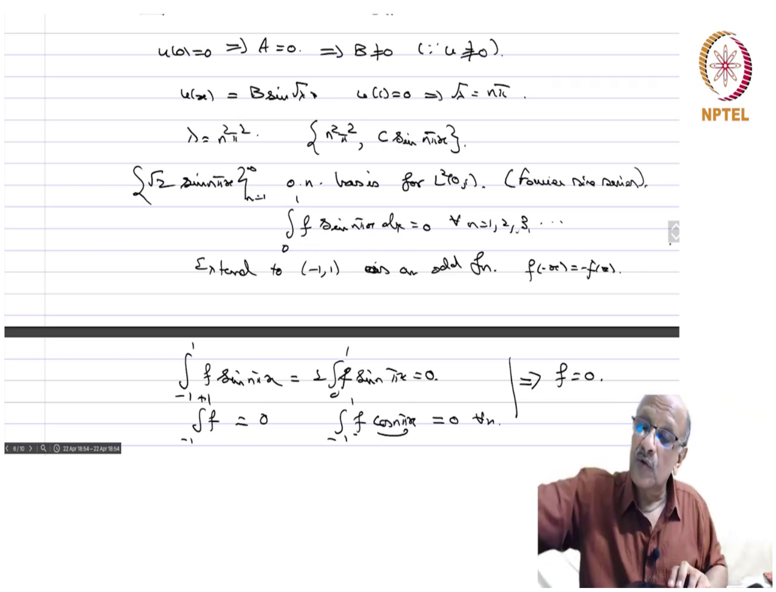

So, let us look at some examples.

(1) So, you take $N = 1,\ \Omega = (0, 1)$. So, you have the problem

$$-u'' = \lambda u \; in\; (0,1) \;;\; u(0) = u(1) = 0.$$

So, multiplying pi u and integrating you get

$$\int_0^1 |u'|^2 dx = \int \lambda u \Rightarrow \lambda \geq 0.$$

If $\lambda = 0$, then $u' = 0 \Rightarrow u = constant$. But it vanishes at the boundary points and therefore, that implies that $u = 0$ and that cannot be an eigenvector. So, $\lambda > 0$. So, in that case the general solution u double dash plus lambda u equal to 0 lambda greater than 0 this implies the more general solution is $u(x) = A \cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$.

So, now you have $u(0) = 0 \Rightarrow A = 0 \Rightarrow B \neq 0 \; (as\; u \neq 0)$. So, $u(x) = B\sin(\sqrt{\lambda}x)$ and then $u(1) = 0$ and this means that $\sqrt{\lambda} = n\pi \Rightarrow \lambda = n^2\pi^2$.

So, the only eigenvalues of this problem are $n^2\pi^2$ and the corresponding Eigenfunctions are some $c\sin n\pi x$. Now, are there any others which we cannot because this is a we know from the Fourier sine series that $\{\sqrt{2}\sin n\pi x\}$ is an orthonormal basis for $L^2(0,1)$. In fact you can do so, this is the Fourier sine series. But you can even prove it other ways: suppose you have the

$$f \in L^2(0,1) \text{ with } \int_0^1 f \sin n\pi x = 0, \; \forall n = 1, 2, 3, ...$$

Then extend it extend to $(-1, 1)$ with as an odd function. So, you just put $f(-x) = -f(x)$.

So, then $\int_{-1}^{1} f \sin n\pi x = 2\int_0^1 f \sin n\pi x = 0 \Rightarrow \int_{-1}^{1} f = 0 \; and \; \int_{-1}^{1} f \cos n\pi x = 0, \; \forall n = 1, 2, ...$

And therefore, from the Fourier, from the Fourier series you know that this implies that f equal to 0 and therefore, that shows that the sin forms a complete orthonormal basis for the $L^2(0,1)$. And therefore, there are no others because you know but from the theory the Eigenfunctions form a complete orthonormal basis and therefore, all the Eigenfunctions Eigenvectors and Eigen functions are given only pi this, there is no other solution.

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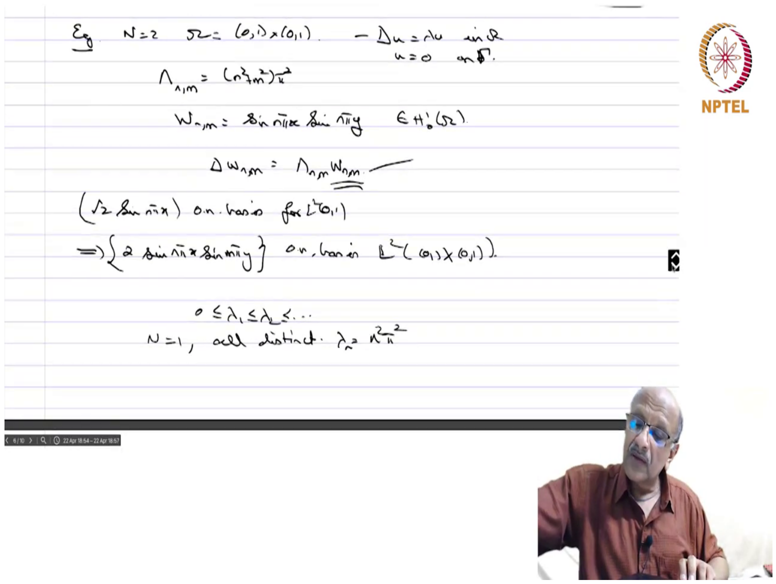

So, the next example you take $N = 2$ and take $\Omega = (0, 1) \times (0, 1)$. Then if you take

$$-\Delta u = \lambda u \; in \, \Omega \; ; \; u = 0 \;\; on \, \Gamma.$$

So, now if you take $\Lambda_{n,m} = (n^2 + m^2)\pi$, $W_{n,m} = \sin n\pi x \, \sin n\pi y \in H^1_0(\Omega)$

and you also have $-\Delta W_{n,m} = \Lambda_{n,m} W_{n,m}$

Also since $\{\sqrt{2}\sin \, n\pi x\}$ is an orthonormal basis for $L^2(0, 1)$, implies that $\{2 \sin \, n\pi x \, \sin \, n\pi y\}$ is an orthonormal basis for $L^2((0, 1) \times (0, 1))$. And therefore, there are no other Eigenvalues or the Eigenfunctions because you already have an orthonormal basis here and therefore, you have these are the only solutions of this. So, when we wrote that

$$0 \leq \lambda_1 \leq \lambda_2 \leq .....$$

If $N = 1$, then all are distinct. Because you have $\lambda_n = n^2\pi^2$.

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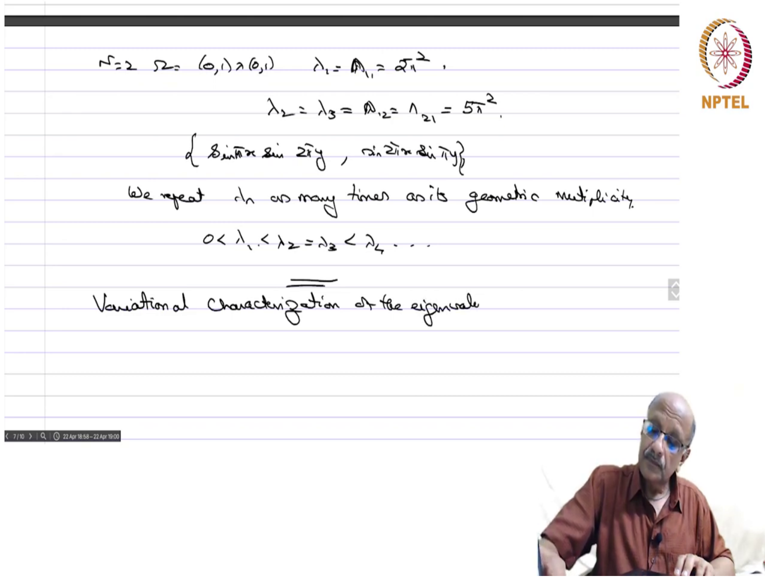

But in the case of example 2, $N = 2$, $\Omega = (0,1) \times (0,1)$, then $\lambda_1 = \Lambda_1 = 2\pi^2$ and

$\lambda_2 = \lambda_3 = \Lambda_{12} = \Lambda_{21} = 5\pi^2$. So, and the corresponding Eigen so it means this Eigen space has dimension 2 and the Eigen vectors which span the dimension are

$\{\sin \pi x \ \sin 2\pi y,\ \sin 2\pi x \ \sin \pi y\}$. So, these are the two basis functions for the Eigenspace connected for $5\pi^2$. And therefore, when the number of so, as so, be number we repeat lambda n as many times as its geometric multiplicity. So, you have

$$0 < \lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 .....$$

So, now we want to give a variational characterization of the eigenvalues.

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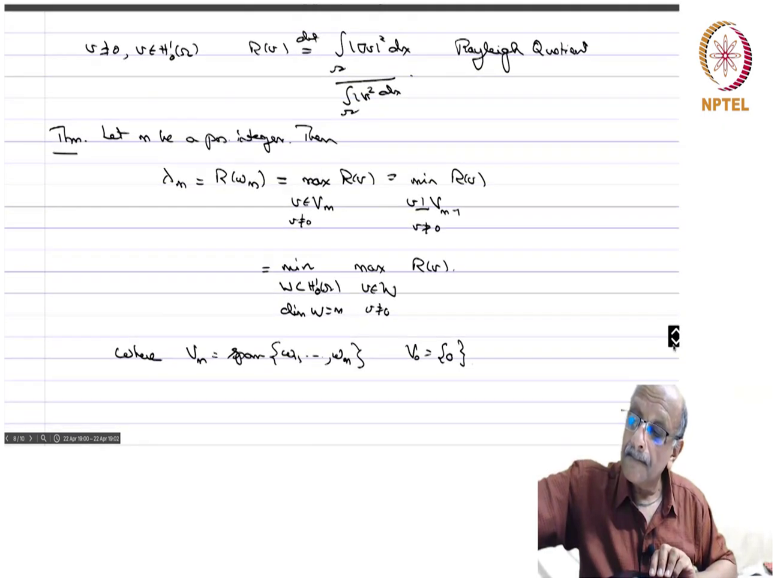

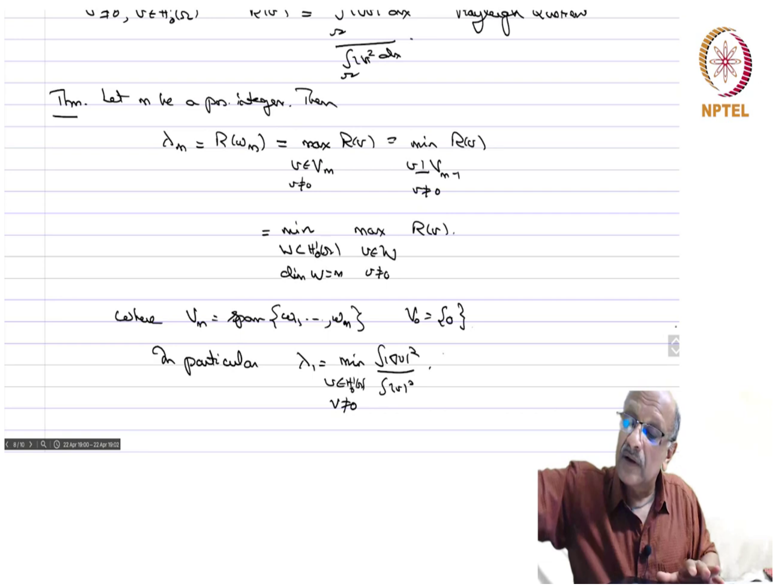

So, we define for $v \neq 0$, $v \in H^1_0(\Omega)$, $R(v) = \dfrac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}$ – This is called the Rayleigh quotient.

So, we have the following theorem.

Theorem: Let m be a positive integer, then

$$\lambda_m = R(w_m) = \max_{v \in V_m, v \neq 0} R(v) = \min_{v \perp V_{m-1}, v \neq 0} R(v) = \min_{W \subset H^1_0(\Omega), \dim(W) = m} \max_{v \in W, v \neq 0} R(v),$$

where $V_m = span\{w_1, w_2, ..., w_m\}$, $V_0 = \{0\}$. In particular $\lambda_1 = \min_{v \in H^1_0(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}$.

So, this is called the Rayleigh quotient characterization. It is called a variational characterization because it talks of minima or Maxima of the Rayleigh quotient over various constraint minima and maxima.

So, the first three are clear what we are saying the last one this is called an intrinsic characterization this min max the first three depends on the Eigenvectors which you have chosen because the choice of Eigenvectors can be of an orthonormal basis can be different union then it is not there is nothing standard about them because you can choose w m or minus wm whatever you like.

So, it does not matter, but the last one does not depend it is a frame independent thing it does not depend on the choice of the Eigen function, it says you take any m dimensional space and take the maximum of the value question there and minimize that maximum over all m dimensional spaces that will give you the lambda m. So, this is a very beautiful characterization of the Eigenvalues and therefore, in particular when m equals 1 you have that it is just a minimum over the entire space. So, this is the. So, we will prove this.

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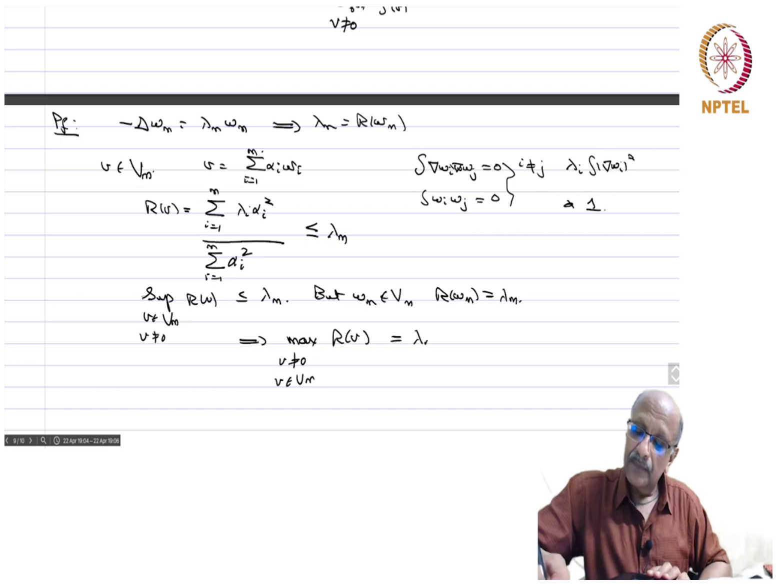

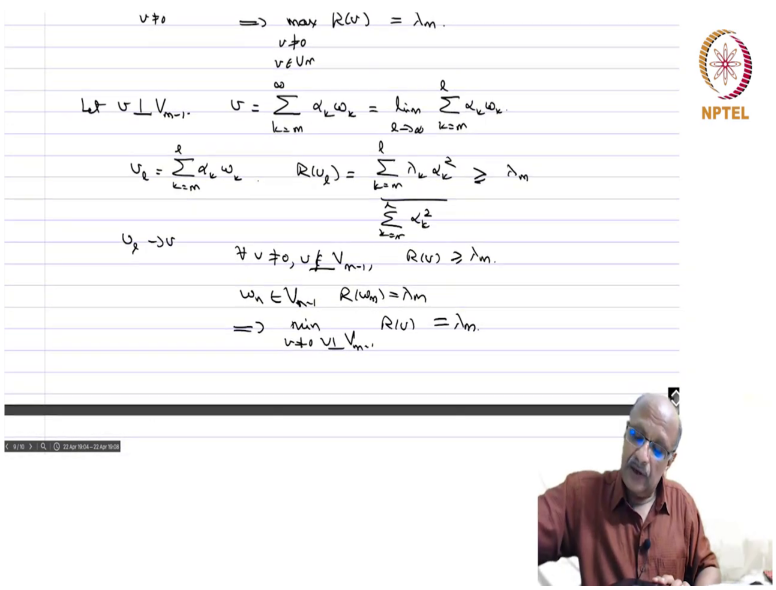

proof: So, $-\Delta w_m = \lambda_m w_m \Rightarrow \lambda_m = R(w_m)$. Now, let us take any $v \in V_m$. So, v can be written as $v = \sum_{i=1}^{m} \alpha_i w_i$. Remember that $\int_\Omega \nabla w_i . \nabla w_j = 0$ *and* $\int_\Omega w_i . w_j = 0$ *for* $i \neq j$.

So, this is what we have. So, if you compute the Rayleigh quotient on this using this expansion then $R(v) = \frac{\sum_{i=1}^{m} \lambda_i \alpha_i^2}{\sum_{i=1}^{m} \alpha_i^2} \leq \lambda_m$. So $\sup_{v \in V_m, v \neq 0} R(v) \leq \lambda_m$. But $w_m \in V_m$ *and* $R(w_m) = \lambda_m$.

$$\Rightarrow \max_{v \in V_m,\, v \neq 0} R(v) = \lambda_m .$$

Now, let $v \perp V_{m-1}$. If you write the Fourier expansion, this will be equal to

$v = \sum_{k=m}^{\infty} \alpha_k w_k = \lim_{l \to \infty} \sum_{k=m}^{l} \alpha_k w_k$. So, if we set $v_l = \sum_{k=m}^{l} \alpha_k w_k$. Then

$$R(v_l) = \frac{\sum_{k=m}^{l} \lambda_k \alpha_k^2}{\sum_{k=m}^{l} \alpha_k^2} \geq \lambda_l .$$

And therefore, since $v_l \to v$, we have that for all $v \neq 0$, $v \perp V_{m-1}$, $R(v) \geq \lambda_m$. And

$w_m \in V_{m-1}$ *and* $R(w_m) = \lambda_m \Rightarrow \min_{v \perp V_{m-1},\, v \neq 0} R(v) = \lambda_m$.

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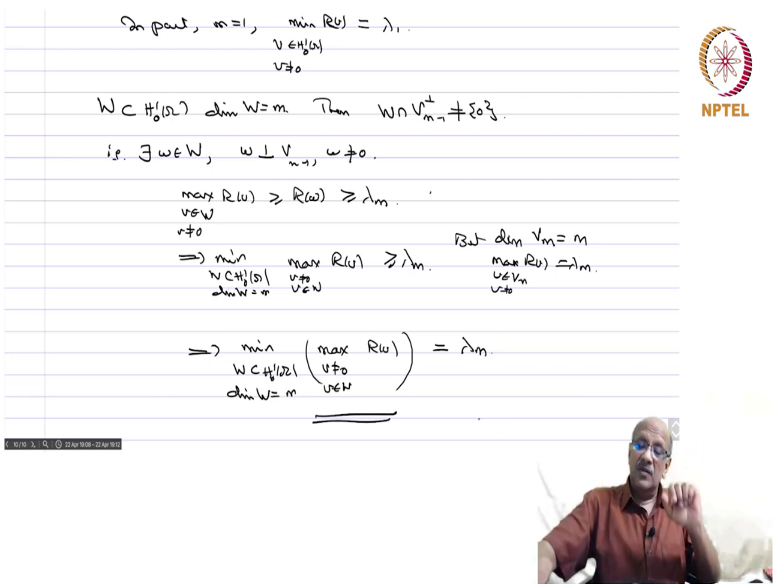

And in particular, if $m = 1$, we get that $\lambda_1 = \min_{v \in H^1_0(\Omega), v \neq 0} R(v)$.

So, finally, let $W \subset H^1_0(\Omega)$, $\dim W = m$. So, now, $W \cap V^{\perp}_{m-1} \neq \{0\}$. Why is this so? So, that means there exists a $w \in W$, so such $w \perp V_{m-1}$, $w \neq 0$. Therefore,

$$\max_{v \in W, v \neq 0} R(v) = R(w) \geq \lambda_m \Rightarrow \min_{W \subset H^1_0(\Omega), \dim W = m} \max_{v \in W, v \neq 0} R(v) \geq \lambda_m$$. So, this implies that

$$\min_{W \subset H^1_0(\Omega), \dim W = m} \max_{v \in W, v \neq 0} R(v) = \lambda_m .$$

min W in H1 0 of omega dimension w equals m max v not equal to 0 V in W of R of v, this is greater than or equal to lambda m.

So, we prove this theorem completely. So, as I said in the last one, the min max principle is an intrinsic one it does not depend on the Eigenfunctions at all and therefore, it is a very powerful result and we will see several applications of this presently.