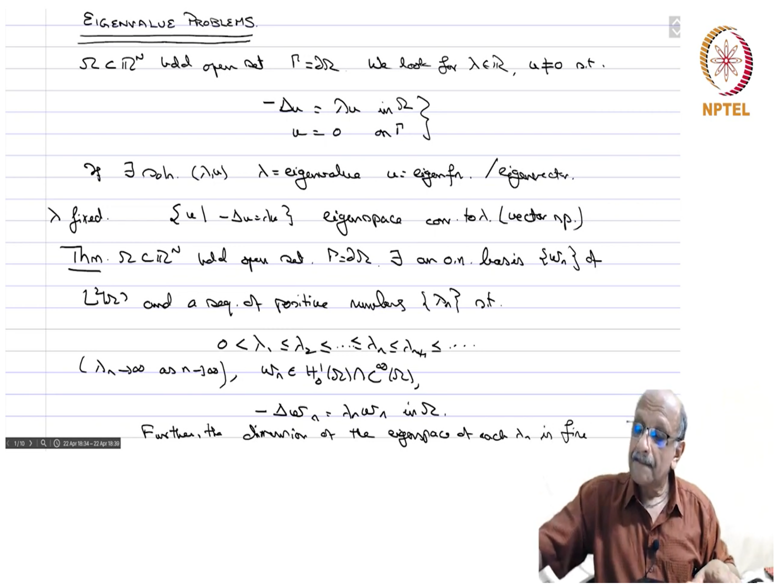
**Sobolev Spaces and Partial Differential Equations**
**Professor. S. Kesavan**
**Department of Mathematics**
**Institute of Mathematical Sciences**
**Eigenvalue problems - Part 1**

(Refer Slide Time: 00:18)




We will now discuss a very important topic : Eigenvalue problems. So, $\Omega \subset \mathbb{R}^N$ bounded open set bounded domain. Let us say $\Gamma = \partial\Omega$. So, we look for $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$- \Delta u = \lambda u \; in \, \Omega \, ; \; u = 0 \; on \, \Gamma \, .$$

So, if there exists a solution $(\lambda, u)$ so then λ is called an eigenvalue and u is called an eigenfunction. So, the set $\{u : \; - \Delta u = \lambda u\}$ is the eigenspace corresponding to λ.

So, if alpha, beta constants u1, u2 are Eigen vectors Eigen functions you can also say Eigenvector but usually we say function because we are dealing with functions. And therefore, if u1 and u2 are Eigenfunctions lam alpha beta are constants and alpha u1plus beta u2 is also by linearity and Eigen function corresponding to lambda so this is called the Eigenspace.

So, we can pose such problems for different boundary value boundary homogeneous boundary conditions like Neumann condition, Robin condition and so on and so forth. And also you can pose this problem for other elliptic operators but the main flavor of the results will all be shown in this example, this standing example prototype for all of these cases. And therefore, we will look at this in a little more detail.

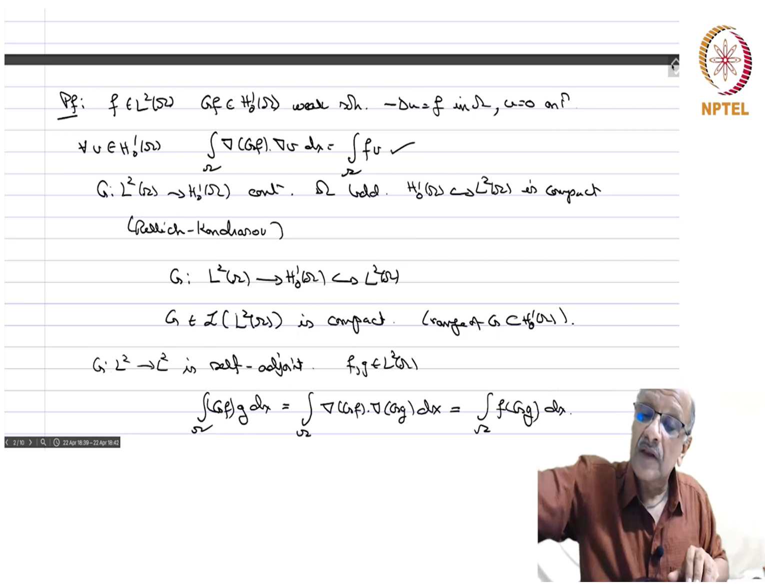
**Theorem:** $\Omega \subset \mathbb{R}^N$ bounded open set bounded domain. Let us say $\Gamma = \partial\Omega$. Then there exists an orthonormal basis $\{w_n\}$ of $L^2(\Omega)$ and a sequence of positive numbers $\{\lambda_n\}$ such that

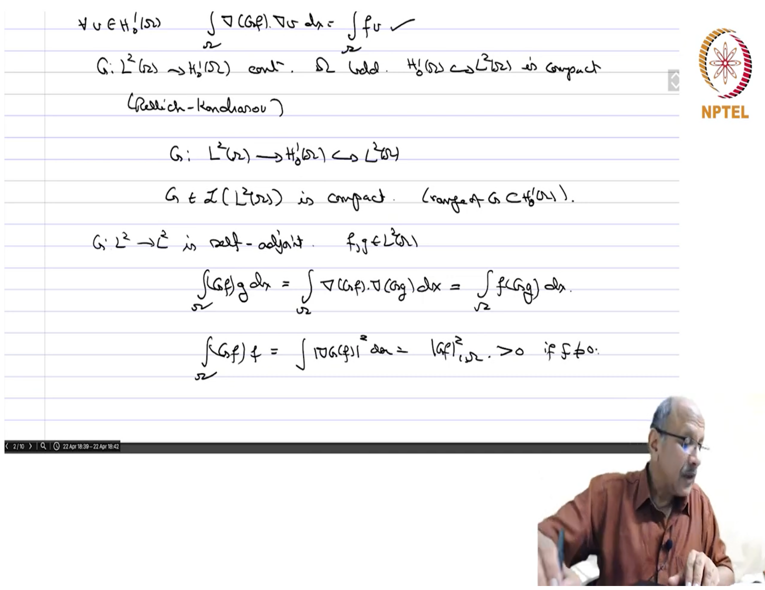
$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq .... \leq \lambda_{n+1} \leq ...., \quad \lambda_n \to \infty \; as \; n \to \infty$$

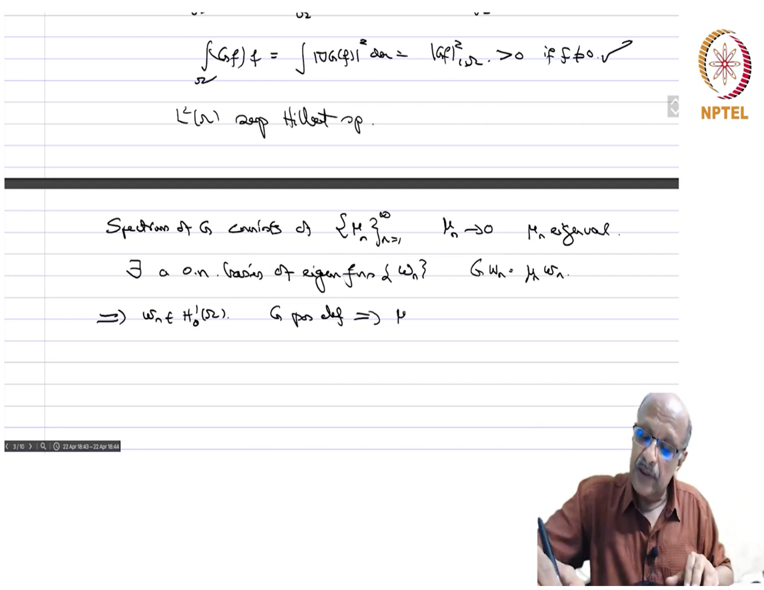
and $w_n \in H^1_0(\Omega) \cap C^\infty(\overline{\Omega})$, and $-\Delta w_n = \lambda_n w_n \; in \; \Omega$.

Further the dimension of the Eigenspace of each $\lambda_n$ is finite.

(Refer Slide Time: 5:22)






**proof:** So, if $f \in L^2(\Omega)$, *define* $Gf \in H^1_0(\Omega)$ weak solution of

$$-\Delta u = \lambda u \text{ in } \Omega;\ u = 0 \text{ on } \Gamma$$

So, then for every $v \in H^1_0(\Omega)$ you have that

$$\int_\Omega \nabla(Gf).\nabla v\,dx = \int_\Omega fv\,dx.$$

Then $G: L^{2}(\Omega) \rightarrow H^{1}_{0}(\Omega)$ is continuous, we know this. Further, omega is bounded so $H^{1}_{0}(\Omega) \rightarrow L^{2}(\Omega)$ is compact (())(6:30).

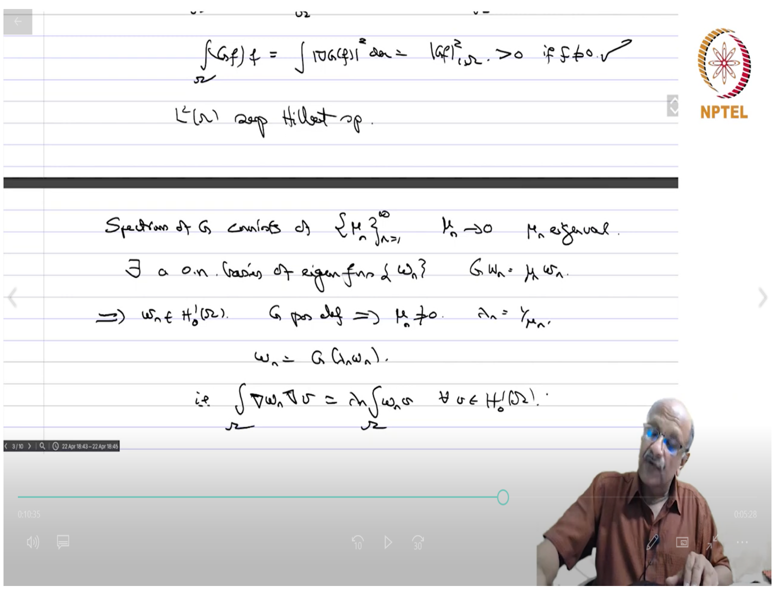
Therefore, you can consider $G: L^{2}(\Omega) \rightarrow H^{1}_{0}(\Omega) \rightarrow L^{2}(\Omega)$ , $G \in L(L^{2}(\Omega))$, range of G $\subset H^{1}_{0}(\Omega)$.

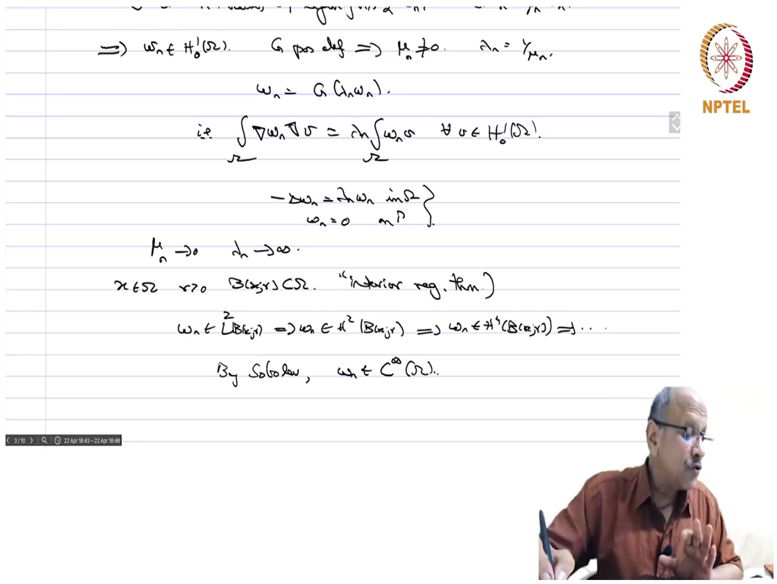
So, $G: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ is self adjoint, in fact if f and g are in $L^{2}(\Omega)$, then you have

$$\int_{\Omega}(Gf)g dx = \int_{\Omega} \nabla(Gf).(\nabla Gg)\, dx = \int_{\Omega} f(Gg)\, dx$$

$$\int_{\Omega} \nabla(Gf). \nabla v\, dx = \int_{\Omega} |\nabla Gf|^{2} = |Gf|^{2}_{1,\Omega} > 0 \; if\; f \neq 0 .$$

(Refer Slide Time: 08:44)





Spectrum of G consists of only positive elements because of this condition gff is greater than equal to 0 of a sequence $\{\mu_n\}_{n=1}^{\infty}$, $\mu_n \rightarrow 0$, $\mu_n$ eigenvalue, and there exists an orthonormal basis of Eigenfunctions $\{w_n\} \in L^2(\Omega)$. So, $Gw_n = \mu_n w_n$. So, this implies of course the $w_n \in H^1_0(\Omega)$ and G is positive definite, namely we have this condition here. So, this implies that $\mu_n \neq 0$, so you put $\lambda_n = \frac{1}{\mu_n}$ and therefore you have $w_n = G(\lambda_n w_n)$.

$$i.e., \int_{\Omega} w_n \nabla v \, dx = \lambda_n \int_{\Omega} w_n v \, dx \ , \ \forall v \in H^1_0(\Omega).$$

And therefore, you have

$$-\Delta w_n = \lambda_n w_n \quad in\, \Omega$$

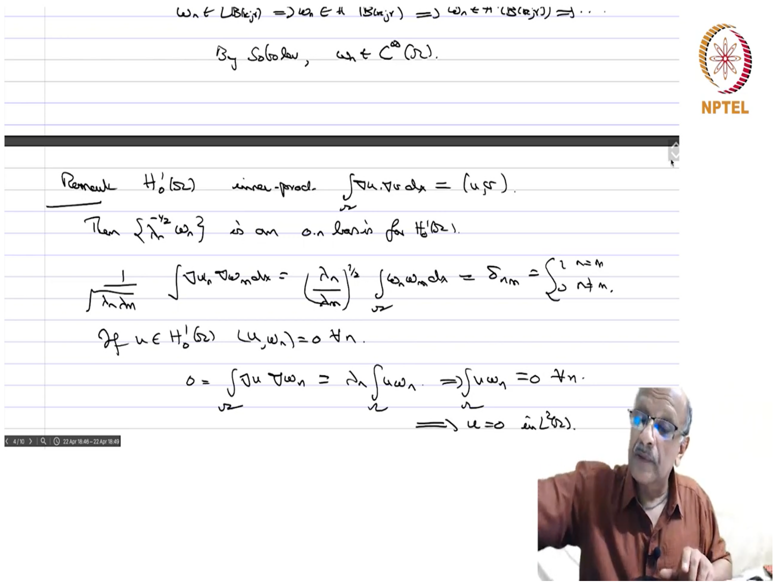
$$w_n = 0 \, on \, \Gamma.$$

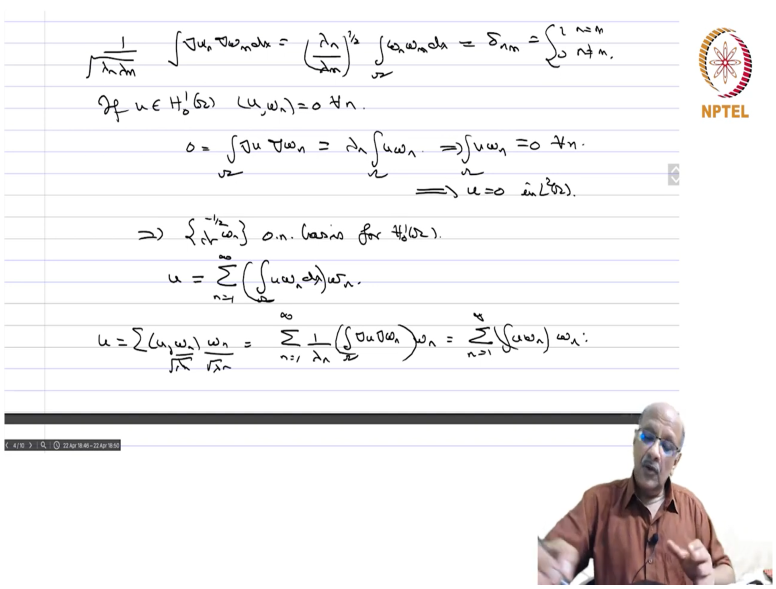
and since $\mu_n \rightarrow 0$, $\lambda \rightarrow \infty$, and therefore you can write it in increasing order.

And finally, if $x \in \Omega$, $r > 0$, $B(x, r) \subset \Omega$, then by interior regularity theorem, $w_n \in L^2(B(x, r)) \Rightarrow w_n \in H^2(B(x, r))$..... $\Rightarrow w_n \in C^{\infty}(\Omega)$ (By Sobolev embedding theorem)

So this completes the proof. We have shown the existence of all these things.

(Refer Slide Time: 12:12)




**Remark:** so you provide $H^1_0(\Omega)$, thanks to Poncari inequality, with the inner product

$$\int_\Omega \nabla u . \nabla v \, dx = (u, v) .$$

So then $\{\lambda_n^{-\frac{1}{2}} w_n\}$ is an orthonormal basis for $H^1_0(\Omega)$ .

$$\frac{1}{\sqrt{\lambda_n \lambda_m}} \int_\Omega \nabla w_n . \nabla w_m \, dx = (\frac{\lambda_n}{\lambda_m})^{\frac{1}{2}} \int_\Omega w_n . w_m \, dx = \delta_{mn} = 1 \; if \; n = m \; and \; 0 \; if \; n \neq m .$$

If $u \in H^1_0(\Omega),\ (u, w_n) = 0 \ \forall\, n,$ then you have

$$0 = \int_\Omega \nabla u . \nabla w_m \, dx = \lambda_n \int_\Omega u \, w_n \, dx \Rightarrow \int_\Omega u \, w_n = 0 \ \forall\, n \Rightarrow u = 0 \; in \; L^2(\Omega) .$$

And therefore, so this implies that $\{\lambda_n^{-\frac{1}{2}} w_n\}$ is an orthonormal basis it is a complete orthonormal set for $H^1_0(\Omega)$.

Now if you want to write the Fourier expansion so if you write

$$u = \sum_{n=1}^{\infty} (\int_\Omega u \, w_n) w_n$$

$$u = \sum_{n=1}^{\infty} (u, \frac{w_n}{\sqrt{\lambda_n}}) \frac{w_n}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (\int_\Omega \nabla u . \nabla w_n \, dx) w_n = \sum_{n=1}^{\infty} (\int_\Omega u . w_n) w_n .$$

And therefore whatever whether it is in $H^1$ or it is in $L^2$ this is the expansion for the inner for any function if you have. So, you so you whether it is the $L^2$ or it does not matter so the for the expansion is always.