

**Sobolev Spaces and Partial Differential Equations**  
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**Eigenvalue problems - Part 1**

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EIGENVALUE PROBLEMS

$\Omega \subset \mathbb{R}^N$  bounded open set  $\Gamma = \partial\Omega$ . We look for  $\lambda \in \mathbb{R}$ ,  $u \neq 0$  s.t.

$$\left. \begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned} \right\}$$

If  $\exists$  soln.  $(\lambda, u)$   $\lambda = \text{eigenvalue}$   $u = \text{eigenfn. / eigenvector}$ .

$\lambda$  fixed.  $\{u \mid -\Delta u = \lambda u\}$  eigenspace corr. to  $\lambda$  (vector sp.)


Thm.  $\Omega \subset \mathbb{R}^N$  bounded open set,  $\Gamma = \partial\Omega$ .  $\exists$  an o.n. basis  $\{u_n\}$  of  $L^2(\Omega)$  and a seq. of positive numbers  $\{\lambda_n\}$  s.t.

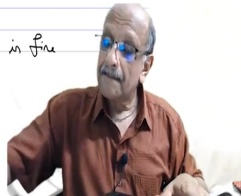
$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n < \lambda_{n+1} < \dots$$

( $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ ),  $u_n \in H_0^1(\Omega) \cap C^\infty(\Omega)$ ,

$$-\Delta u_n = \lambda_n u_n \text{ in } \Omega.$$

Further, the dimension of the eigenspace of each  $\lambda_n$  is finite.





We will now discuss a very important topic : Eigenvalue problems. So,  $\Omega \subset \mathbb{R}^N$  bounded open set bounded domain. Let us say  $\Gamma = \partial\Omega$ . So, we look for  $\lambda \in \mathbb{R}$  and  $u \neq 0$  such that

$$-\Delta u = \lambda u \text{ in } \Omega; u = 0 \text{ on } \Gamma.$$

So, if there exists a solution  $(\lambda, u)$  so then  $\lambda$  is called an eigenvalue and  $u$  is called an eigenfunction. So, the set  $\{u : -\Delta u = \lambda u\}$  is the eigenspace corresponding to  $\lambda$ .

So, if  $\alpha, \beta$  constants  $u_1, u_2$  are Eigen vectors Eigen functions you can also say Eigenvector but usually we say function because we are dealing with functions. And therefore, if  $u_1$  and  $u_2$  are Eigenfunctions  $\alpha u_1 + \beta u_2$  are constants and  $\alpha u_1 + \beta u_2$  is also by linearity and Eigen function corresponding to  $\lambda$  so this is called the Eigenspace.

So, we can pose such problems for different boundary value boundary homogeneous boundary conditions like Neumann condition, Robin condition and so on and so forth. And also you can pose this problem for other elliptic operators but the main flavor of the results will all be shown

in this example, this standing example prototype for all of these cases. And therefore, we will look at this in a little more detail.

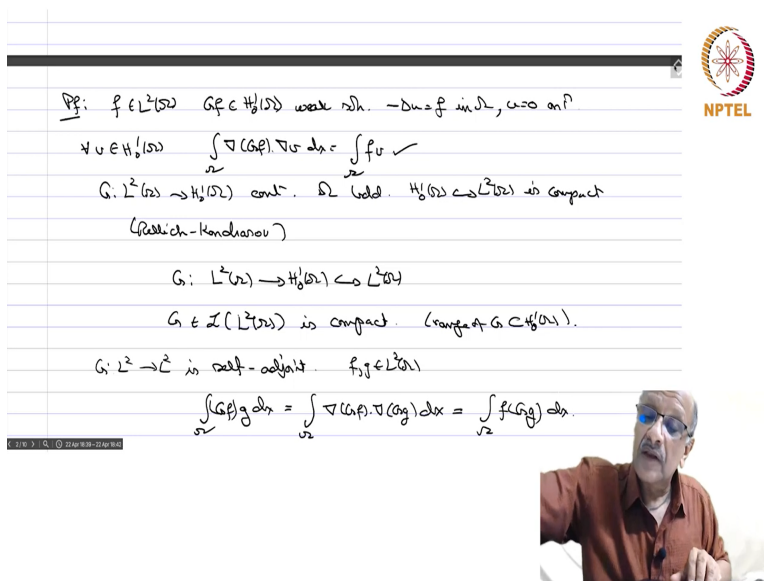
**Theorem:**  $\Omega \subset \mathbb{R}^N$  bounded open set bounded domain. Let us say  $\Gamma = \partial\Omega$ . Then there exists an orthonormal basis  $\{w_n\}$  of  $L^2(\Omega)$  and a sequence of positive numbers  $\{\lambda_n\}$  such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{n+1} \leq \dots, \lambda_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

and  $w_n \in H_0^1(\Omega) \cap C^\infty(\bar{\Omega})$ , and  $-\Delta w_n = \lambda_n w_n$  in  $\Omega$ .

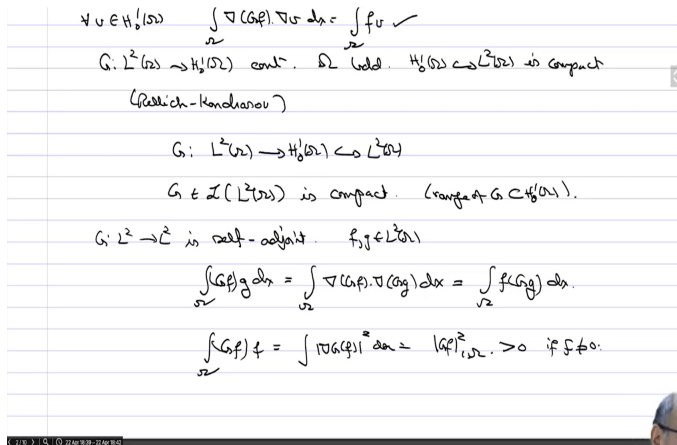
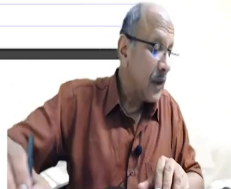
Further the dimension of the Eigenspace of each  $\lambda_n$  is finite.

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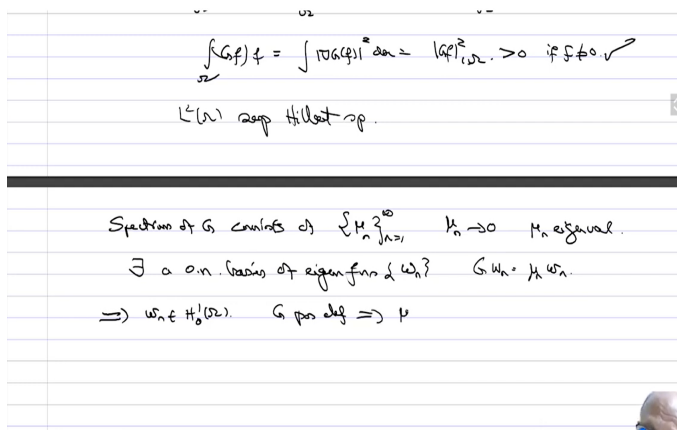



$\text{Pf: } f \in L^2(\Omega) \quad Gf \in H_0^1(\Omega) \text{ weak sol. } \rightarrow \Delta u = f \text{ in } \Omega, u=0 \text{ on } \Gamma$   
 $\forall u \in H_0^1(\Omega) \quad \int_{\Omega} \nabla(Gf) \cdot \nabla u \, dx = \int_{\Omega} f u \, dx \quad \checkmark$   
 $G: L^2(\Omega) \rightarrow H_0^1(\Omega) \text{ cont. } \Delta \text{ is self-adjoint. } H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact}$   
 (Radonik-Kondrakov)  
 $G: L^2(\Omega) \rightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$   
 $G \in \mathcal{L}(L^2(\Omega)) \text{ is compact. (range of } G \subset H_0^1(\Omega)).$   
 $G: L^2 \rightarrow L^2 \text{ is self-adjoint. } f, g \in L^2(\Omega)$   
 $\int_{\Omega} (Gf)g \, dx = \int_{\Omega} \nabla(u_1) \cdot \nabla(u_2) \, dx = \int_{\Omega} f(Gg) \, dx.$

$\forall v \in H_0^1(\Omega) \quad \int_{\Omega} \nabla(Gf) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \checkmark$   
 $G: L^2(\Omega) \rightarrow H_0^1(\Omega)$  cont. b.l. bdd.  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  is compact  
 (Rellich-Kondrakov)  
 $G: L^2(\Omega) \rightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$   
 $G \in \mathcal{L}(L^2(\Omega))$  is compact. (range of  $G \subset H_0^1(\Omega)$ ).  
 $G: L^2 \rightarrow L^2$  is self-adjoint.  $f, g \in L^2(\Omega)$   
 $\int_{\Omega} (Gf)g \, dx = \int_{\Omega} \nabla(Gf) \cdot \nabla(Gg) \, dx = \int_{\Omega} f(Gg) \, dx.$   
 $\int_{\Omega} (Gf)f \, dx = \int_{\Omega} |\nabla(Gf)|^2 \, dx = \|Gf\|_{H_0^1}^2 > 0 \text{ if } f \neq 0.$

$\int_{\Omega} (Gf)f \, dx = \int_{\Omega} |\nabla(Gf)|^2 \, dx = \|Gf\|_{H_0^1}^2 > 0 \text{ if } f \neq 0 \quad \checkmark$   
 $L^2(\Omega)$  resp. Hilbert sp.  
 Spectrum of  $G$  consists of  $\{ \mu_n \}_{n=1}^{\infty}$   $\mu_n \rightarrow 0$   $\mu_n$  eigenval.  
 $\exists$  a o.n. basis of eigenfun  $\omega_n$ ?  $G \omega_n = \mu_n \omega_n.$   
 $\Rightarrow \omega_n \in H_0^1(\Omega)$ .  $G$  pos def  $\Rightarrow \mu$

**proof:** So, if  $f \in L^2(\Omega)$ , define  $Gf \in H_0^1(\Omega)$  weak solution of

$$-\Delta u = \lambda u \text{ in } \Omega; u = 0 \text{ on } \Gamma$$

So, then for every  $v \in H_0^1(\Omega)$  you have that

$$\int_{\Omega} \nabla(Gf) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx.$$

Then  $G: L^2(\Omega) \rightarrow H^1_0(\Omega)$  is continuous, we know this. Further,  $\Omega$  is bounded so  $H^1_0(\Omega) \rightarrow L^2(\Omega)$  is compact (6:30).

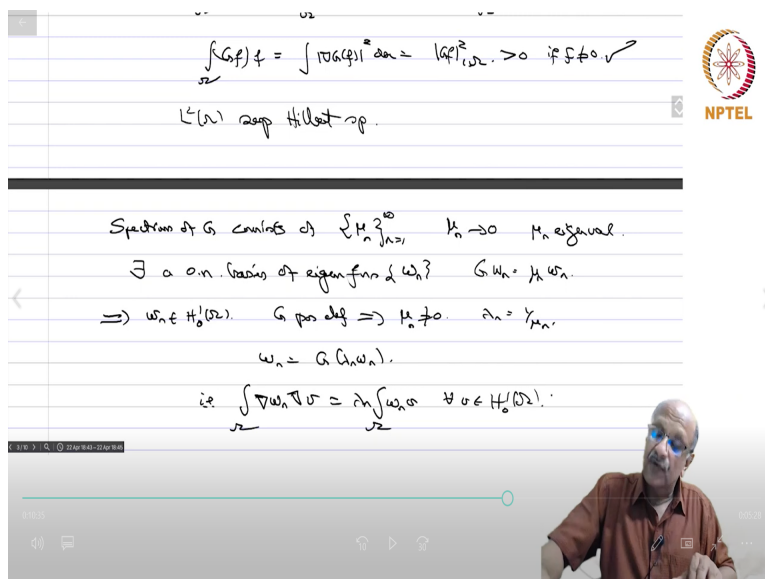
Therefore, you can consider  $G: L^2(\Omega) \rightarrow H^1_0(\Omega) \rightarrow L^2(\Omega)$ ,  $G \in L(L^2(\Omega))$ , range of  $G \subset H^1_0(\Omega)$ .

So,  $G: L^2(\Omega) \rightarrow L^2(\Omega)$  is self adjoint, in fact if  $f$  and  $g$  are in  $L^2(\Omega)$ , then you have

$$\int_{\Omega} (Gf)g \, dx = \int_{\Omega} \nabla(Gf) \cdot (\nabla Gg) \, dx = \int_{\Omega} f(Gg) \, dx$$

$$\int_{\Omega} \nabla(Gf) \cdot \nabla v \, dx = \int_{\Omega} |\nabla Gf|^2 \, dx = \|Gf\|_{1,\Omega}^2 > 0 \text{ if } f \neq 0.$$

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$\int_{\Omega} (Gf)g \, dx = \int_{\Omega} |\nabla Gf|^2 \, dx = \|Gf\|_{1,\Omega}^2 > 0 \text{ if } f \neq 0$

$L^2(\Omega)$  is a Hilbert space.

Spectrum of  $G$  consists of  $\{\lambda_n\}_{n=1}^{\infty}$ ,  $\lambda_n \rightarrow 0$ ,  $\lambda_n$  eigenvalue.

$\exists$  a o.n. basis of eigenfn  $\{w_n\}$ ,  $Gw_n = \lambda_n w_n$ .

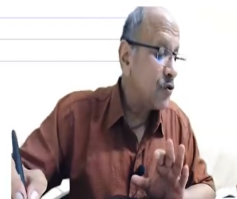
$\Rightarrow w_n \in H^1_0(\Omega)$ ,  $G$  pos def  $\Rightarrow \lambda_n \neq 0$ ,  $\lambda_n = \gamma_{\lambda_n}$ .

$w_n = G(\lambda_n w_n)$ .

i.e.  $\int_{\Omega} \nabla w_n \cdot \nabla v \, dx = \lambda_n \int_{\Omega} w_n v \, dx \quad \forall v \in H^1_0(\Omega)$ .

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$$\begin{aligned} \Rightarrow w_n \in H_0^1(\Omega). \quad G \text{ pos def} \Rightarrow \mu_n \neq 0, \quad \lambda_n = \gamma_{\mu_n}, \\ w_n = G(\lambda_n w_n). \\ \text{i.e. } \int_{\Omega} \nabla w_n \nabla v = \lambda_n \int_{\Omega} w_n v \quad \forall v \in H_0^1(\Omega). \\ \left. \begin{aligned} -\Delta w_n &= \lambda_n w_n \text{ in } \Omega \\ w_n &= 0 \text{ on } \Gamma \end{aligned} \right\} \\ \mu_n \rightarrow 0, \quad \lambda_n \rightarrow \infty. \\ x \in \Omega, \quad r > 0, \quad B(x, r) \subset \Omega. \quad \text{"Interior reg. thm"} \\ w_n \in L^2(B(x, r)) \Rightarrow w_n \in H^2(B(x, r)) \Rightarrow w_n \in H^1(B(x, r)) \Rightarrow \dots \\ \text{By Sobolev, } w_n \in C^\infty(\Omega). \end{aligned}$$



Spectrum of  $G$  consists of only positive elements because of this condition gff is greater than equal to 0 of a sequence  $\{\mu_n\}_{n=1}^\infty$ ,  $\mu_n \rightarrow 0$ ,  $\mu_n$  eigenvalue, and there exists an orthonormal basis of Eigenfunctions  $\{w_n\} \in L^2(\Omega)$ . So,  $Gw_n = \mu_n w_n$ . So, this implies of course the  $w_n \in H_0^1(\Omega)$  and  $G$  is positive definite, namely we have this condition here. So, this implies that  $\mu_n \neq 0$ , so you put  $\lambda_n = \frac{1}{\mu_n}$  and therefore you have  $w_n = G(\lambda_n w_n)$ .

$$\text{i.e., } \int_{\Omega} w_n \nabla v \, dx = \lambda_n \int_{\Omega} w_n v \, dx, \quad \forall v \in H_0^1(\Omega).$$

And therefore, you have

$$-\Delta w_n = \lambda_n w_n \text{ in } \Omega$$

$$w_n = 0 \text{ on } \Gamma.$$

and since  $\mu_n \rightarrow 0$ ,  $\lambda \rightarrow \infty$ , and therefore you can write it in increasing order.


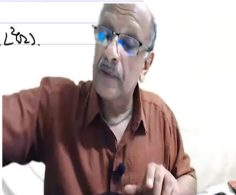
And finally, if  $x \in \Omega$ ,  $r > 0$ ,  $B(x, r) \subset \Omega$ , then by interior regularity theorem,  $w_n \in L^2(B(x, r)) \Rightarrow w_n \in H^2(B(x, r)) \dots \Rightarrow w_n \in C^\infty(\Omega)$  (By Sobolev embedding theorem)

So this completes the proof. We have shown the existence of all these things.



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$\omega_n \in L^2(\Omega) \Rightarrow \omega_n \in H^1(\Omega) \Rightarrow \omega_n \in H^1(\Omega) \Rightarrow \dots$   
 By Sobolev,  $\omega_n \in C^0(\Omega)$ .

**Remark:**  $H_0^1(\Omega)$  inner-product:  $\int \nabla u \cdot \nabla v \, dx = (u, v)$ .  
 Then  $\{\lambda_n^{-1/2} \omega_n\}$  is an o.n. basis for  $H_0^1(\Omega)$ .  
 $\frac{1}{\lambda_n \lambda_m} \int \nabla u_n \cdot \nabla \omega_m \, dx = \left(\frac{\lambda_n}{\lambda_m}\right)^{1/2} \int \omega_n \omega_m \, dx = \delta_{nm} = \int_0^1 \delta(x) \delta(x) \, dx$ .  
 If  $u \in H_0^1(\Omega)$   $(u, \omega_n) = 0 \, \forall n$ .  
 $0 = \int \nabla u \cdot \nabla \omega_n = \lambda_n \int u \omega_n \Rightarrow \int u \omega_n = 0 \, \forall n$ .  
 $\Rightarrow u = 0$  in  $L^2(\Omega)$ .

$\frac{1}{\lambda_n \lambda_m} \int \nabla u_n \cdot \nabla \omega_m \, dx = \left(\frac{\lambda_n}{\lambda_m}\right)^{1/2} \int \omega_n \omega_m \, dx = \delta_{nm} = \int_0^1 \delta(x) \delta(x) \, dx$ .  
 If  $u \in H_0^1(\Omega)$   $(u, \omega_n) = 0 \, \forall n$ .  
 $0 = \int \nabla u \cdot \nabla \omega_n = \lambda_n \int u \omega_n \Rightarrow \int u \omega_n = 0 \, \forall n$ .  
 $\Rightarrow u = 0$  in  $L^2(\Omega)$ .  
 $\Rightarrow \{\lambda_n^{-1/2} \omega_n\}$  o.n. basis for  $H_0^1(\Omega)$ .  
 $u = \sum_{n=1}^{\infty} \left( \int u \omega_n \, dx \right) \omega_n$ .  
 $u = \sum \frac{(u, \omega_n)}{\lambda_n} \omega_n = \sum \frac{1}{\lambda_n} \left( \int \nabla u \cdot \nabla \omega_n \right) \omega_n = \sum \left( \int \nabla u \cdot \nabla \omega_n \right) \omega_n$ .

**Remark:** so you provide  $H_0^1(\Omega)$ , thanks to Poncaré inequality, with the inner product

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = (u, v).$$

So then  $\{\lambda_n^{-\frac{1}{2}} w_n\}$  is an orthonormal basis for  $H_0^1(\Omega)$ .

$$\frac{1}{\sqrt{\lambda_n \lambda_m}} \int_{\Omega} \nabla w_n \cdot \nabla w_m \, dx = \left(\frac{\lambda_n}{\lambda_m}\right)^{\frac{1}{2}} \int_{\Omega} w_n \cdot w_m \, dx = \delta_{mn} = 1 \text{ if } n = m \text{ and } 0 \text{ if } n \neq m.$$

If  $u \in H_0^1(\Omega)$ ,  $(u, w_n) = 0 \, \forall n$ , then you have

$$0 = \int_{\Omega} \nabla u \cdot \nabla w_m \, dx = \lambda_n \int_{\Omega} u w_n \, dx \Rightarrow \int_{\Omega} u w_n = 0 \, \forall n \Rightarrow u = 0 \text{ in } L^2(\Omega).$$

And therefore, so this implies that  $\{\lambda_n^{-\frac{1}{2}} w_n\}$  is an orthonormal basis it is a complete orthonormal set for  $H_0^1(\Omega)$ .

Now if you want to write the Fourier expansion so if you write

$$u = \sum_{n=1}^{\infty} \left( \int_{\Omega} u w_n \right) w_n$$

$$u = \sum_{n=1}^{\infty} \left( u, \frac{w_n}{\sqrt{\lambda_n}} \right) \frac{w_n}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left( \int_{\Omega} \nabla u \cdot \nabla w_n \, dx \right) w_n = \sum_{n=1}^{\infty} \left( \int_{\Omega} u \cdot w_n \right) w_n.$$

And therefore whatever whether it is in  $H^1$  or it is in  $L^2$  this is the expansion for the inner for any function if you have. So, you so you whether it is the  $L^2$  or it does not matter so the for the expansion is always.