Sobolev Spaces and Partial Differential Equations
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Exercises – Part 11

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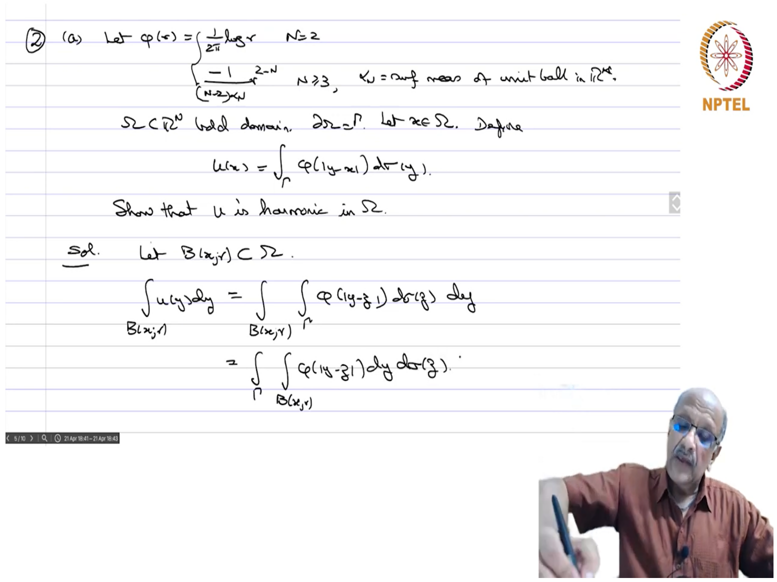

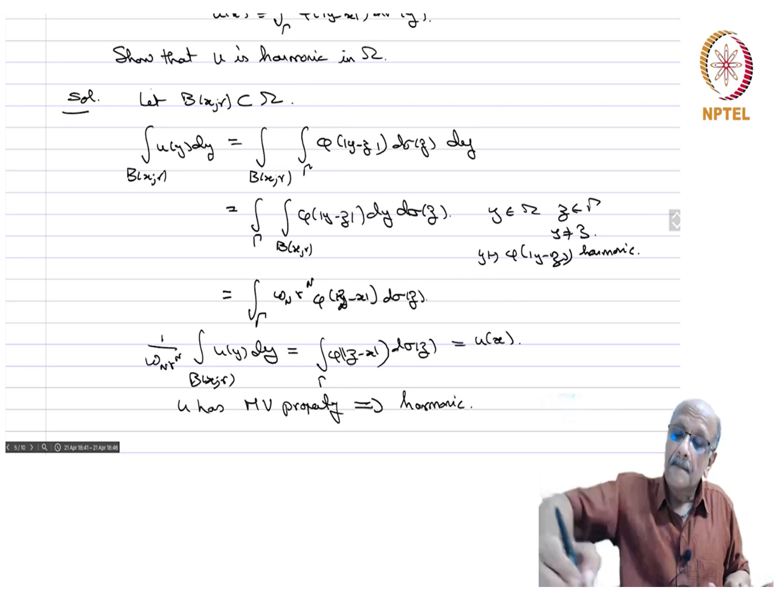

(2) (a) Let $\phi(r) = \frac{1}{2\pi}\log(r)$, $if\ N = 2$

$= -\frac{1}{(N-2)\alpha_N} r^{2-N}$, $if\ N \geq 3$. α_N - equal surface measure of the unit ball.

So, $\Omega \subset \mathbb{R}^N$ bounded domain and $\partial\Omega = \Gamma$. Let, $x \in \Omega$. Define

$$u(x) = \int_{\Gamma} \phi(|y - x|)d\sigma(y).$$

show that u is harmonic.

solution: so let us take $B(x, r) \subset \Omega$. So

$$\int_{B(x,r)} u(y)dy = \int_{B(x,r)} \int_{\Gamma} \phi(|z - x|)d\sigma(z)dy = \int_{\Gamma} \int_{B(x,r)} \phi(|z - x|)dy\, d\sigma(z)$$

$$= \int_{\Gamma} \omega_N r^N \phi(|z - x|)d\sigma(z)$$

$$\Rightarrow \frac{1}{\omega_N r^N} \int_{B(x,r)} u(y)dy = \int_{\Gamma} \phi(|z - x|)d\sigma(z) = u(x)$$

therefore u has mean value property and we know therefore that it means it is harmonic.

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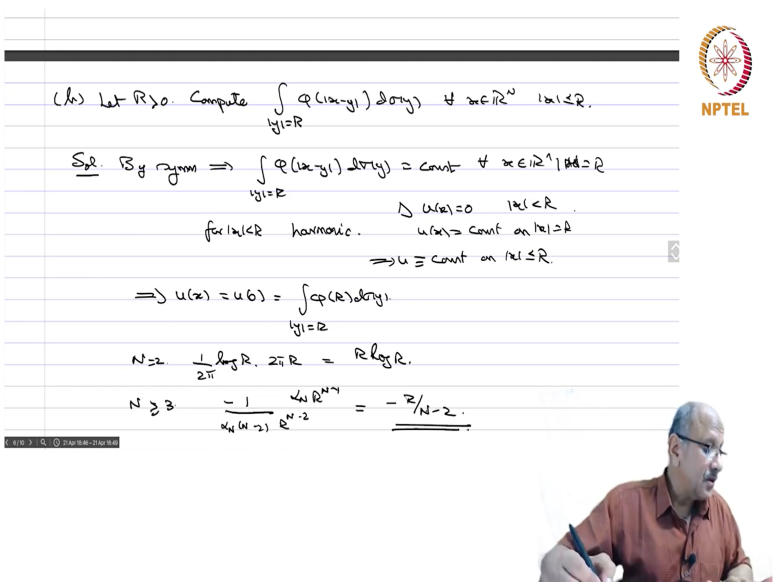

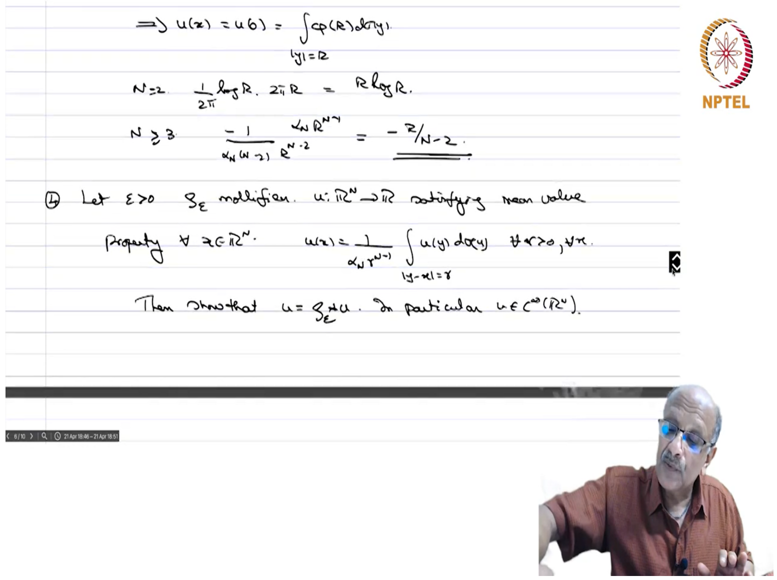

(b) Let $R > 0$. compute the integral $\int\limits_{|x|=R} \phi(|y - x|)d\sigma(y)$, $|x| \leq R$.

solution: by symmetry $\int\limits_{|x|=R} \phi(|y - x|)d\sigma(y)$ = constant , for all $x \in \mathbb{R}^N$, $|x| \leq R$.

And therefore, and for $|x| < R$ harmonic. So, if you call this as u of x so

$$\Delta u\,(x) = 0,\ |x| < R\ ,\ u(x) = constant\ on\ |x| = R\ .$$

$$\Rightarrow u(x) = constant\ for\ |x| \leq R\ .$$

by the uniqueness of the solution to the Dirichlet power. So, implies that

$$u(x) = u(0) = \int\limits_{|y|=R} \phi(R)d\sigma(y)\ .$$

So, if $N = 2,$ you have so that is $\frac{1}{2\pi \log(R)} 2\pi\, R = R \log\ R$. If $N \geq 3,$ so this is equal to

$\frac{-1}{\alpha_N (N-2) R^{N-2}} \alpha_N R^{N-1} = -\frac{R}{N-2}$.

(4) Let $\epsilon > 0$, ρ_ϵ − mollifiers, $u: \mathbb{R}^N \to \mathbb{R}$, satisfying mean value property for every $x \in \mathbb{R}^N$.

That means $u(x) = \frac{1}{\alpha_N r^{N-1}} \int_{|y-x|=r} u(y)d\sigma(y)$, *for every* x. Then show that u equals

$\rho_\epsilon * u = u$. In particular $C^\infty(\mathbb{R}^N)$.

So, a harmonic function is automatically infinitely differentiable from this true in any omega also all you have to do is to take whatever proof you are going to do now we will have take epsilon such that the ball center x radius epsilon is contained in omega that is all we have to do epsilon small enough. Here, we are of course at liberty to take any epsilon we like and therefore the proof is the same so this proof that it is a regularity statement that if you have a harmonic function then it is infinitely differentiable in the interior of the domain.

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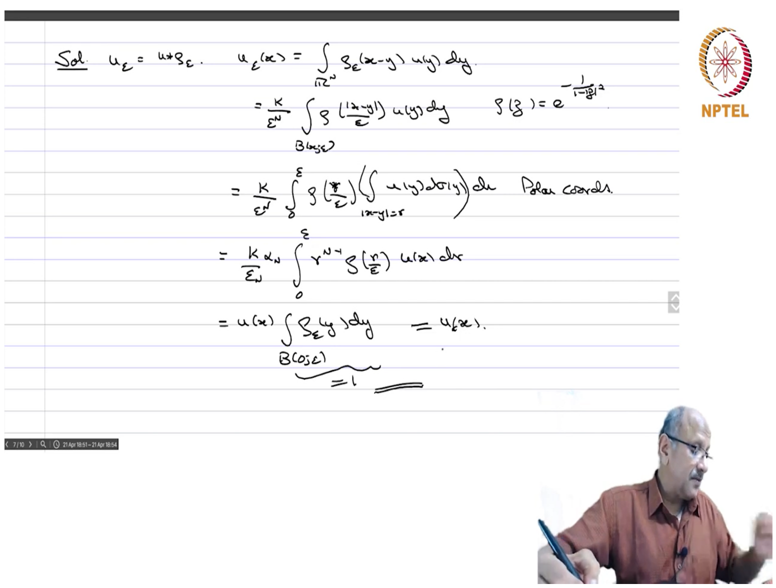


$$= u(x) \underbrace{\int_{B(0;\varepsilon)} \rho_\varepsilon(y)\,dy}_{=1} = u_\varepsilon(x).$$

④ (a) Let $V \subset\subset \Omega$, V, Ω bdd domains in $\mathbb{R}^N$. $r = \frac{1}{4} d(V, \Gamma)$ $\Gamma := \partial\Omega$

Let $x, y \in V$. $|x-y| = r$. Let $u : \Omega \to \mathbb{R}$ be a non-neg. harmonic fn.

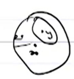
Show that $u(x) \geq \frac{1}{2^N} u(y)$.

Sol. $x, y \in V$, $|x-y| = r$.

$$\underbrace{u(x) = \frac{1}{\omega_N 2^N r^N} \int_{B(x;2r)} u(z)\,dz}_{\text{harmonic}} \underbrace{\geq}_{u \geq 0} \frac{1}{\omega_N 2^N r^N} \int_{B(y;r)} u(z)\,dz$$

Sol. $x, y \in V$, $|x-y| = r$.

$$\underbrace{u(x) = \frac{1}{\omega_N 2^N r^N} \int_{B(x;2r)} u(z)\,dz}_{\text{harmonic}} \underbrace{\geq}_{u \geq 0} \frac{1}{\omega_N 2^N r^N} \int_{B(y;r)} u(z)\,dz$$



$$= \frac{1}{2^N} u(y) \quad \text{Mean val. prop.}$$

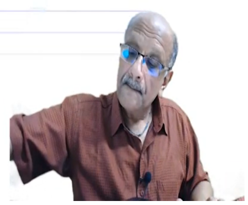
$$u(x) \geq \frac{1}{2^N} u(y). \quad \text{Ill}^y \quad u(y) \geq \frac{1}{2^N} u(x)$$

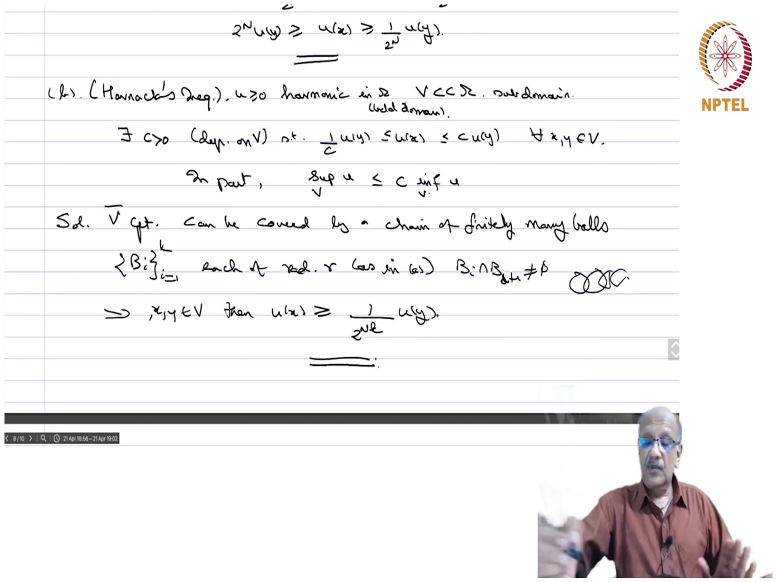
$$2^N u(y) \geq u(x) \geq \frac{1}{2^N} u(y).$$

(b). (Harnack's Ineq.). $u \geq 0$ harmonic in Ω. $V \subset\subset \Omega$. subdomain (bdd domain).

$\exists\, C > 0$ (dep. on V) s.t. $\frac{1}{C} u(y) \leq u(x) \leq C u(y)$ $\forall\, x, y \in V$.

In part, $\sup_V u \leq C \inf_V u$





solution: $\rho_{\epsilon} * u = u_{\epsilon}$. so

$$u_{\epsilon} = \int_{\mathbb{R}^N} \rho_{\epsilon}(x - y)u(y)dy \; = \frac{k}{\epsilon^N} \int_{B(x,\epsilon)} \rho \left(\frac{|x-y|}{\epsilon}\right)u(y)dy$$

$$= \frac{k}{\epsilon^N}\int_0^{\epsilon} \rho\left(\frac{r}{\epsilon}\right) \int_{|y-x|=r} u(y)d\sigma(y)\, dr.$$

$$= \frac{k}{\epsilon^N}\alpha_N \int_0^{\epsilon} r^{N-1}\rho\left(\frac{r}{\epsilon}\right)u(x)dr$$

$$= u(x) \int_{B(x,\epsilon)} \rho_{\epsilon}(y)\, dy \; = u_{\epsilon}(x)$$

(4)(a) Let $V \subset\subset \Omega, V, \Omega$ omega bounded domains in $\mathbb{R}^N$. Let $r = \frac{1}{4}d(V, \Gamma),\ \Gamma = \partial\Omega$. Let $x, y \in V,\ |x - y| = r.$ Let $u\colon \Omega \to \mathbb{R}$ be a non-negative harmonic function. Show that

$$u(x) \geq \frac{1}{2^N}u(y).$$

Solution: so we have x y in v mod x minus y equal to r now u is harmonic so u of x is equal to1 by omega n r power n 2 power n r power n this is the volume of the ball of center x radius 2 r and u of z dz.

Now u is non negative therefore I can write this as greater than equal to integral 1 by omega n 2 power n r power n integral by r uz dz. So, this is because it is harmonic and so this is harmonic so you have a mean value property and this is because u is greater than equal to 0. So, you have a ball of radius 2r so this is x and this is 2r and then you have y here and therefore a ball of radius r is contained in that.

But what is that again by the mean value property this is equal to 1 by 2 power n u of y again mean value property. So, ux is greatly equal to 1 by 2 power N similarly, you have u of y will be similarly 1 by 2 power n u of x. And therefore, you have that 2 power n u of y will be greatly equal to u of x greatly equal to 1 by 2 power n u of y. So, that means you can always compare values between 2 points with some which are close enough.

(b) *Harnack's inequality:* $V \subset\subset \Omega$, sub-domain, $u \geq 0$ harmonic in Ω. There exists $c > 0$ (depending on V) s.t. $\frac{1}{c} u(y) \leq u(x) \leq cu(y)$, $\forall\, x, y \in V$. In particular,

$$\sup_V u \leq C \inf_V u .$$

Solution: so $\overline{V}$ is compact and therefore can be covered by a chain of finitely many balls $\{B_i\}_{i=1}^{k}$ each of radius r (as in (a)). And $B_i \cap B_{j+1} \neq \phi$. so you have to cover it like this. And you have x in b1 and y. So, in each of these bonds you have 1 by 2 power n there are at most k balls that means if $x, y \in V$ then $u(x) \geq \frac{1}{2^{Nk}} u(y)$ and that will be the result.

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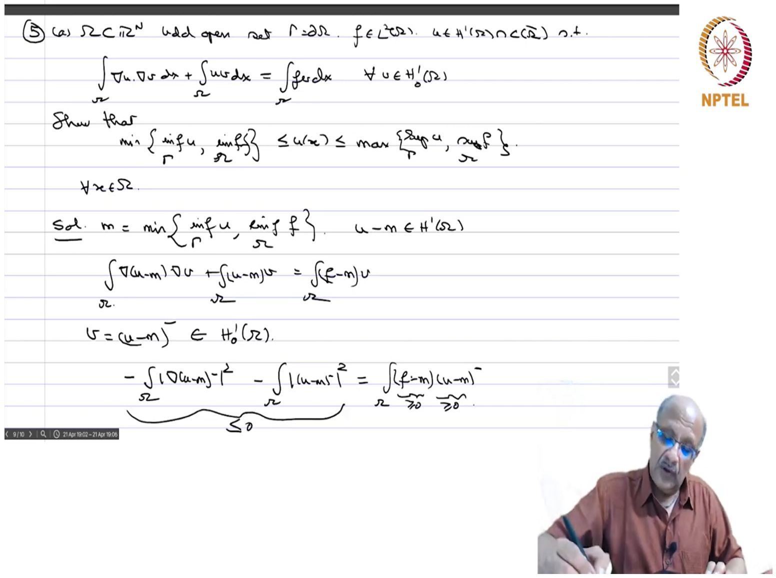

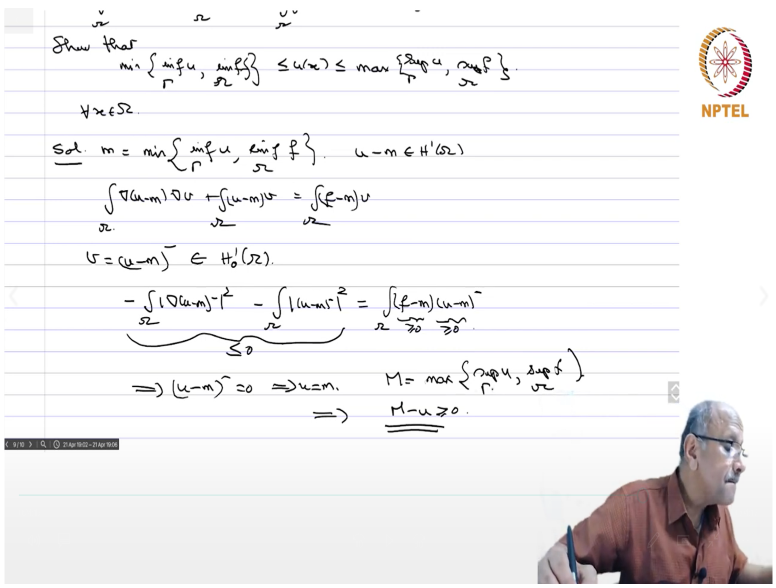

(5) (a): $\Omega \subset \mathbb{R}^N$ bounded open set, $\Gamma = \partial\Omega$, $f \in L^2(\Omega)$, $u \in H^1(\Omega) \cap C(\overline{\Omega})$ $s.t.$

$$\int_{\Omega} \nabla u . \nabla v \, dx + \int_{\Omega} uv dx = \int_{\Omega} fv \, dx \, , \; \forall \, v \in H^1_{\;0}(\Omega) \, .$$

Show that the minimum of $\min \{\inf_{\Gamma} f, \, \inf_{\Omega} f\} \leq u(x) \leq \max \{\sup_{\Gamma} f, \, \sup_{\Omega} f\}$, $\forall \, x \in \Omega$.

solution: you have to take $m = \min \{\inf_{\Gamma} f, \, \inf_{\Omega} f\}$, $u - m \in H^1(\Omega)$ .So, you have

$$\int_{\Omega} \nabla(u - m)\nabla v - \int_{\Omega} (u - m)v = \int_{\Omega} (f - m)v .$$

So, now you take $v = (u - m)^{-} \in H^{1}_{0}(\Omega)$. And therefore you will get

$$- \int_{\Omega} |\nabla(u - m)^{-}|^{2} - \int_{\Omega} |(u - m)^{-}|^{2} = \int_{\Omega} (f - m)(u - m)^{-}$$

Now $f - m$ is non negative so this is also non negative so this everything is non negative and this side everything is less than or equal to 0 and therefore you have equal to 0. And this implies that $(u - m)^{-} = 0 \Rightarrow u = m$. So, similarly you take $M = \max\{\sup_{\Gamma} f, \sup_{\Omega} f\}$. By the same argument $M - u \geq 0$.

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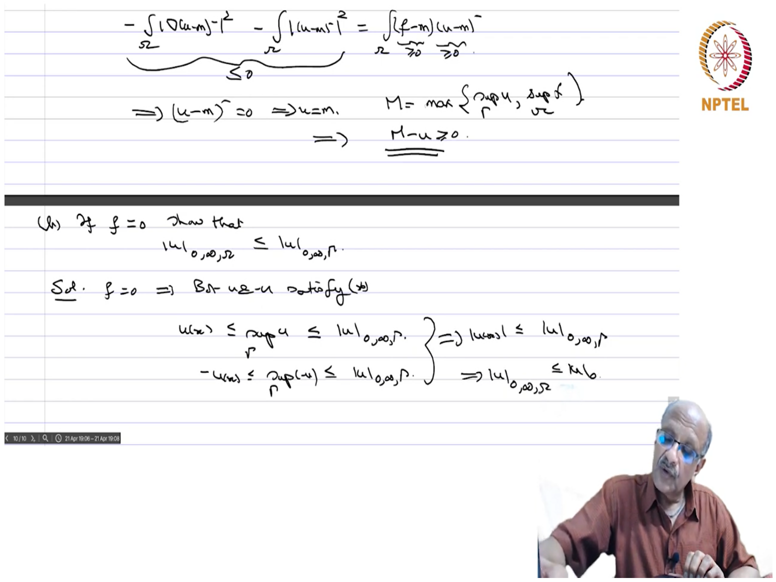

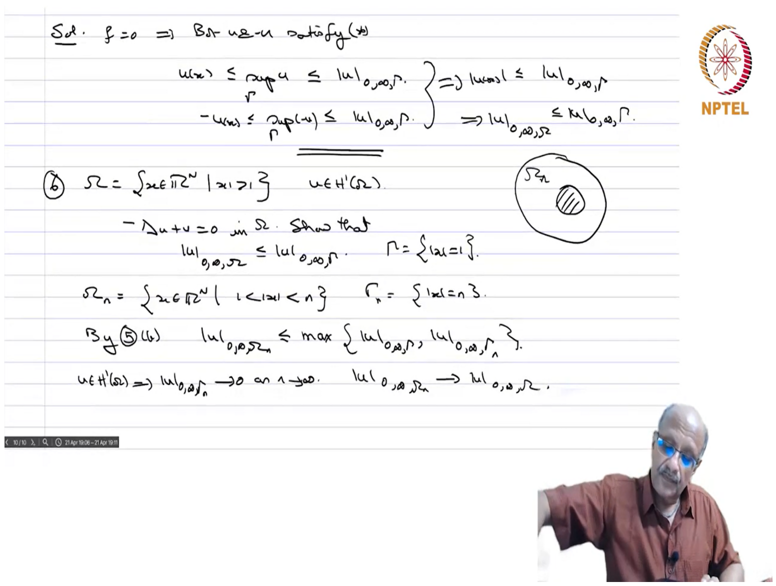

(b): if $f = 0$, show that $|u|_{0,\infty,\Omega} \leq |u|_{0,\infty,\Gamma}$.

solution: $f = 0 \Rightarrow$ both u and -u satisfy (*) , so you have

$$u(x) \leq \sup_{\Gamma} u \leq |u|_{0,\infty,\Gamma} \Rightarrow |u(x)| \leq |u|_{0,\infty,\Gamma}$$

$$-u(x) \leq \sup_{\Gamma} -u \leq |u|_{0,\infty,\Gamma} \Rightarrow |u|_{0,\infty,\Omega} \leq |u|_{0,\infty,\Gamma}$$

(6): Let $\Omega = \{x \in \mathbb{R}^N : |x| > 1\}$, $u \in H^1(\Omega)$, $\Delta u + u = 0$ in Ω. Show that $|u|_{0,\infty,\Omega} \leq |u|_{0,\infty,\Gamma}$.

So, this is like the previous theorem on a problem except that we have an unbounded domain so you take $\Omega_n = \{x \in \mathbb{R}^N : 1 < |x| < n\}$. So, you so now this becomesΩ_n, so that is now a bounded domain and therefore you have that so let $\Gamma_n = \{x : |x| = n\}$.

So, by 5 (b), $|u|_{0,\infty,\Omega} \leq \max\{|u|_{0,\infty,\Omega}, |u|_{0,\infty,\Gamma_n}\}$ and then $u \in H^1(\Omega)$ implies

$|u|_{0,\infty,\Gamma_n} \rightarrow 0$ as $n \rightarrow \infty$. So, it cannot be positive on a set of positive measure as you as the domain grows. So, as this thing and also $|u|_{0,\infty,\Omega_n} \rightarrow |u|_{0,\infty,\Omega}$, from this you conclude that hence the result.