

Sobolev Spaces and Partial Differential Equations
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Exercises – Part 10

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EXERCISES

① (Saint Venant Problem)

Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set. $\Gamma_0 = \partial\Omega$.

Let $\{\Omega_i\}_{i=1}^m$ be mutually disjoint subdomains. $\Gamma_i = \partial\Omega_i$ is smooth.

Let $|\Omega_i| = a_i$, $1 \leq i \leq m$. Let $\Omega = \Omega_0 \cup \bigcup_{i=1}^m \Omega_i$. $u = \text{warping fn. on } \Omega$.

Find u s.t.

$$\begin{aligned} -\Delta u &= 2 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_0 \\ u &= c_i \quad \text{on } \Gamma_i, \text{ an unknown const., } 1 \leq i \leq m. \end{aligned}$$

(70)

$$\int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma = 2a_i.$$


$u = 0 \quad \text{on } \Gamma_0$
 $u = c_i \quad \text{on } \Gamma_i, \text{ an unknown const., } 1 \leq i \leq m.$
 (70)

$$\int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma = 2a_i. \checkmark$$

Show that $\exists!$ soln.

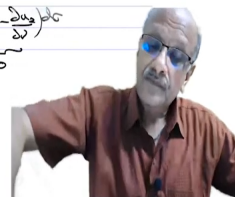
Sol. Step 1: Uniqueness. Let u_1, u_2 be two solns. w.t. $u = u_1 - u_2$.

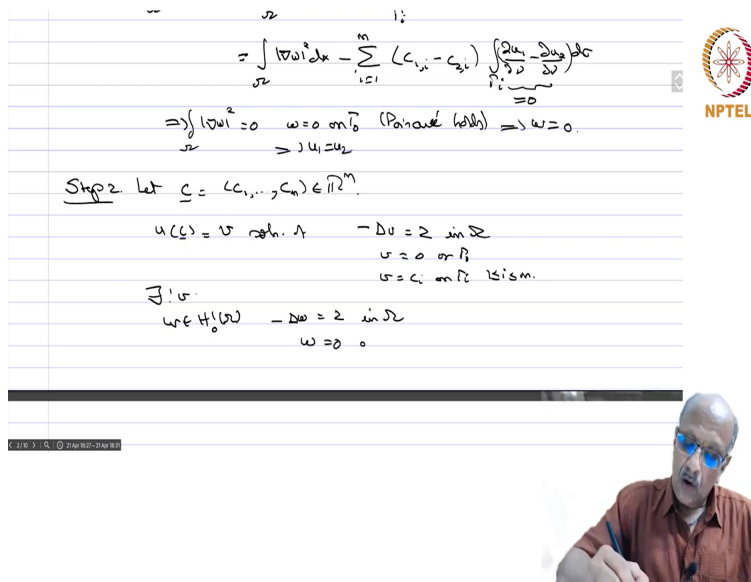
$-\Delta w = 0$ in Ω $w = 0$ on Γ_0 .

$$0 = \int_{\Omega} -\Delta w \cdot w \, dx = \int_{\Omega} |\nabla w|^2 \, dx - \sum_{i=1}^m \int_{\Gamma_i} w \frac{\partial w}{\partial \nu} d\sigma$$

$$= \int_{\Omega} |\nabla w|^2 \, dx - \sum_{i=1}^m (c_{1,i} - c_{2,i}) \int_{\Gamma_i} \left(\frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right) d\sigma$$

$= 0$





Handwritten notes on lined paper showing mathematical derivations. The NPTEL logo is visible in the top right corner. A video inset in the bottom right shows a professor in a red shirt.

Derivation 1:

$$= \int_{\Omega} |\nabla u|^2 dx - \sum_{i=1}^m (c_{i,1} - c_{i,2}) \left(\frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu} \right)_{\Gamma_i} d\sigma$$

Derivation 2:

$$\Rightarrow \int_{\Omega} |\nabla u|^2 = 0 \quad \Rightarrow \quad u = 0 \text{ on } \Gamma_0 \quad (\text{Poincaré's inequality}) \Rightarrow u = 0.$$

Step 2: Let $\underline{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$.

Problem 1:

$$\begin{aligned} u(\underline{c}) &= u \text{ s.t. } \Delta u = 2 \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma_0 \\ u &= c_i \text{ on } \Gamma_i \text{ i.s.m.} \end{aligned}$$

Problem 2:

$$\begin{aligned} u &\in H_0^1(\Omega) \quad \Delta u = 2 \text{ in } \Omega \\ u &= 0 \end{aligned}$$

We will do some more exercises, the first one is called the Saint Venant problem. So, you think of an infinite beam of uniform cross sections, so Ω is a container with some inclusions, think of concrete with steel rods stuck inside. So, the rods are going through along with this beam and so this is the cross-section of the beam and these inclusions here denote the places occupied by the cross sections of the steel rods. So, this is called so we want to study what is called the torsional rigidity of this beam which means how it is, how stiff it is and so on.

(1) Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded open connected set, $\Gamma_0 = \partial\Omega_0$. Let some $\{\Omega_i\}_{i=1}^m$ be open mutually disjoint sub domains that means they are also open sets and they are connected themselves. So, $\Gamma_i = \partial\Omega_i$, $1 \leq i \leq m$ and we write let $|\Omega_i| = a_i$, $1 \leq i \leq m$ and let $\Omega = \Omega_0 \setminus \bigcup_{i=1}^m \overline{\Omega_i}$. That means you remove it from the portion which is other than the black shaded region. Find u such that

$$-\Delta u = 2 \text{ in } \Omega,$$

$$u = 0 \text{ on } \Gamma_0,$$

$$u = c_i \text{ on } \Gamma_i, \quad 1 \leq i \leq m, \quad c_i > 0 \text{ — unknown constant.}$$

$$\int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma = 2a_i, \quad 1 \leq i \leq m.$$

So, u is called a warping function on Ω so this is the thing and the torsional rigidity is determined in terms of the gradient of u . Show that there exists a unique solution.

solution: *step 1 (uniqueness):* So, let u_1, u_2 be two solutions and you said $w = u_1 - u_2$. So

$$-\Delta u = 0 \text{ in } \Omega \text{ and } w = 0 \text{ on } \Gamma_0.$$

$$\begin{aligned} \text{So,} \quad 0 &= - \int_{\Omega} \Delta w \cdot w \, dx = \int_{\Omega} |\nabla w|^2 - \sum_{i=1}^m \int_{\Gamma_i} w \frac{\partial w}{\partial \nu} d\sigma \\ &= \int_{\Omega} |\nabla w|^2 - \sum_{i=1}^m (c_{1,i} - c_{2,i}) \int_{\Gamma_i} \left(\frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right) d\sigma \end{aligned}$$

But then, $\frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} = 0$. So, this means that $\int_{\Omega} |\nabla w|^2 = 0 \Rightarrow w = 0 \text{ on } \Gamma_0 \Rightarrow w = 0$ by

Poincaré inequality. So $u_1 = u_2$.

So now, we want to show the existence of a solution.

Step 2. So, let $\underline{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$. So you define $u(\underline{c}) = v$, solution of

$$\begin{aligned} -\Delta v &= 2 \text{ in } \Omega, \\ v &= 0 \text{ on } \Gamma_0, \\ v &= c_i \text{ on } \Gamma_i, \quad 1 \leq i \leq m. \end{aligned}$$

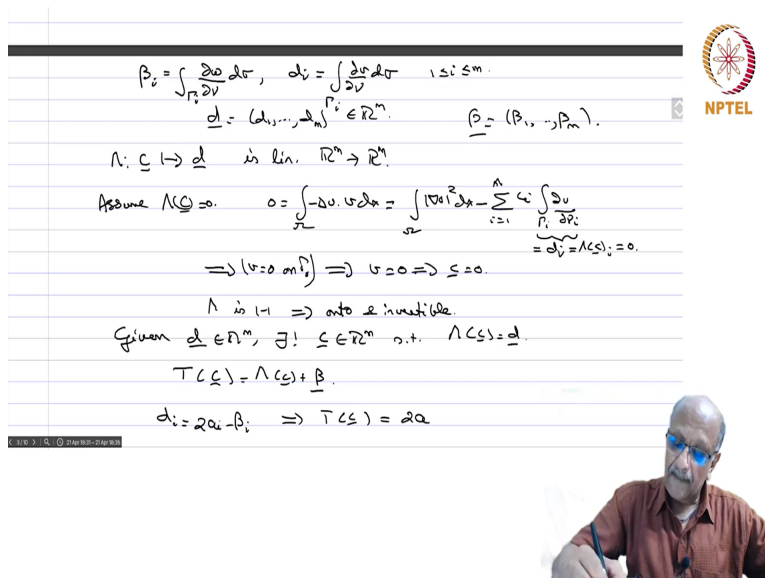
So, this is a straightforward Dirichlet problem and everything is fine therefore, there exists a unique v . And let $w \in H_0^1(\Omega)$ such that

$$-\Delta w = 2 \text{ in } \Omega$$

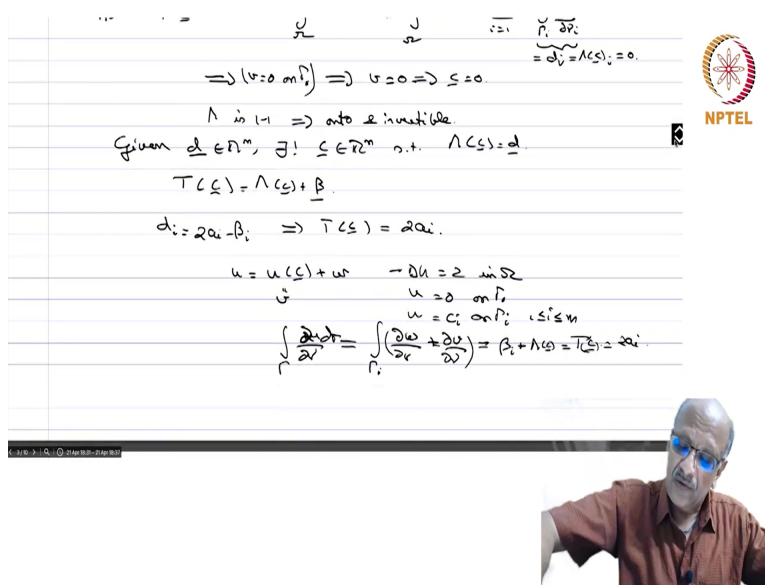
$$v = 0 \text{ on } \Gamma_0$$

this also exists there exists unique w so this also is fine.

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$\beta_i = \int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma, \quad d_i = \int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma \quad 1 \leq i \leq m.$
 $\underline{d} = (d_1, \dots, d_m) \in \mathbb{R}^m, \quad \underline{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m.$
 $\Lambda: \underline{c} \mapsto \underline{d}$ is lin. $\mathbb{R}^m \rightarrow \mathbb{R}^m.$
 Assume $\Lambda(\underline{c}) = 0.$ $0 = \int_{\Gamma} -\Delta u \cdot u d\sigma = \int_{\Gamma} |\nabla u|^2 d\sigma - \sum_{i=1}^m c_i \int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma$
 $= \int_{\Gamma} |\nabla u|^2 d\sigma - \sum_{i=1}^m c_i d_i = 0.$
 $\Rightarrow (u=0 \text{ on } \Gamma_i) \Rightarrow u=0 \Rightarrow \underline{c}=0.$
 Λ is 1-1 \Rightarrow onto & invertible.
 Given $\underline{d} \in \mathbb{R}^m, \exists! \underline{c} \in \mathbb{R}^m$ s.t. $\Lambda(\underline{c}) = \underline{d}.$
 $T(\underline{c}) = \Lambda(\underline{c}) + \underline{\beta}.$
 $d_i = 2a_i - \beta_i \Rightarrow T(\underline{c}) = 2\underline{a}.$



$\Rightarrow (u=0 \text{ on } \Gamma_i) \Rightarrow u=0 \Rightarrow \underline{c}=0.$
 Λ is 1-1 \Rightarrow onto & invertible.
 Given $\underline{d} \in \mathbb{R}^m, \exists! \underline{c} \in \mathbb{R}^m$ s.t. $\Lambda(\underline{c}) = \underline{d}.$
 $T(\underline{c}) = \Lambda(\underline{c}) + \underline{\beta}.$
 $d_i = 2a_i - \beta_i \Rightarrow T(\underline{c}) = 2\underline{a}.$
 $u = u(\underline{c}) + w, \quad -\Delta u = 2 \text{ in } \Omega$
 $\dot{u} = 0 \text{ on } \Gamma_i$
 $w = c_i \text{ on } \Gamma_i \quad 1 \leq i \leq m$
 $\int_{\Gamma} \frac{\partial u}{\partial \nu} d\sigma = \int_{\Gamma} \left(\frac{\partial u}{\partial \nu} + \frac{\partial w}{\partial \nu} \right) d\sigma = \beta_i + d_i = T(\underline{c}) = 2a_i.$

So now, you define $\beta_i = \int_{\Gamma_i} \frac{\partial w}{\partial \nu} d\sigma$, $d_i = \int_{\Gamma_i} \frac{\partial w}{\partial \nu} d\sigma$, $1 \leq i \leq m$, $\underline{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$,

$$\underline{d} = (d_1, \dots, d_m) \in \mathbb{R}^m.$$

So now, if you take $\Lambda: \underline{c} \rightarrow \underline{d}$ is linear from $\mathbb{R}^m \rightarrow \mathbb{R}^m$. Assume that $\Lambda(\underline{c}) = 0$.

That means $0 = - \int_{\Omega} \Delta v \cdot v \, dx = \int_{\Omega} |\nabla v|^2 - \sum_{i=1}^m c_i \int_{\Gamma_i} \frac{\partial v}{\partial \nu_i} \, d\sigma \Rightarrow v = 0 \text{ on } \Gamma_i \Rightarrow v = 0$

$\Rightarrow \underline{c} = 0.$

So, Λ is 1-1 implies onto and invertible. So, given $\underline{d} \in \mathbb{R}^m$ there exists a unique $\underline{c} \in \mathbb{R}^m$ such that $\Lambda(\underline{c}) = \underline{d}$. Now you define $T(\underline{c}) = \Lambda(\underline{c}) + \underline{\beta}$. Now if you take $d_i = 2a_i - \beta_i$. Then this is imply that $T(\underline{c}) = 2a_i$.

So now, if you set u equal to u of c which is equal to v of course plus w , then you have minus Laplacian w equal to 2 in Ω because you see minus Laplacian w is 0. And for this 1 it is 2 so when you add them you will get $2w$ equals 0 on Γ not and w equal to c_i on Γ_i and integral dw by $d\nu$ on Γ_i this is nothing but d_i which is equal to $2a_i$ dw by $d\nu$ plus dv by $d\nu$ this equal to, u . So, integral Γdw by $d\nu$ $d\sigma$ equal to integral Γw by $d\nu$ which is equal to β_i β_i plus Λ of c and that is equal to $T(c)$ equals $2a_i$. So, this is also so you have it satisfies all these conditions 1 less equal to i listening to n .

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Step 3 Claim $c_i > 0$, $1 \leq i \leq m$

Ω smooth $\Rightarrow u \in H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ ($\Omega \subset \mathbb{R}^n$)

$x_0 \in \bar{\Omega}$ s.t. $u(x_0) = \min_{\bar{\Omega}} u$

Suppose $x_0 \in \Omega \Rightarrow u(x_0) = 0$ $\Delta u(x_0) > 0$ \times $\Delta u = -2$

$x_0 \in \Gamma_i$, $1 \leq i \leq m$ $\Rightarrow \min u = c_i$

$= \frac{\partial u}{\partial \nu} \leq 0$ on Γ_i But $\frac{\partial u}{\partial \nu} = 2a_i > 0$

$\Rightarrow x_0 \in \Gamma_0 \Rightarrow u(x_0) > 0$ in $\bar{\Omega} \setminus \Gamma_0$

$\Rightarrow \underline{c_i} > 0$



$$\begin{aligned}
 & x_0 \in \bar{\Omega} \text{ s.t. } u(x_0) = \min_{\bar{\Omega}} u \text{ and} \\
 & \text{Suppose } x_0 \in \bar{\Omega} \Rightarrow \nabla u(x_0) = 0 \text{ due to } \Delta u = -2 \quad \nabla u = -2 \\
 & x_0 \in \Gamma_i, 1 \leq i \leq m. \Rightarrow \min u = c_i. \\
 & = \frac{\partial u}{\partial \nu} \leq 0 \text{ on } \Gamma_i \text{ but } \int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma = 2a_i > 0. \\
 & \Rightarrow x_0 \in \Gamma_0 \Rightarrow u(x) > 0 \text{ in } \bar{\Omega} \setminus \Gamma_0 \\
 & \Rightarrow c_i > 0. \quad (\text{Smoothness needed only in last step}) \\
 & \text{Torsional Rigidity} = S(\Omega) = \int_{\Omega} |\nabla u|^2 dx.
 \end{aligned}$$



step 3: claim: $c_i > 0, 1 \leq i \leq m$.

So, let Ω be smooth enough so this implies that $u \in H^2(\Omega)$ and that since we will be in C of the $\bar{\Omega}$ since we are contained in \mathbb{R}^2 by the Sobolev embedding theorem. So then, if you take $x_0 \in \bar{\Omega}$, then such that $u(x_0) = \inf_{x \in \bar{\Omega}} u(x)$. So, suppose $x_0 \in \Omega$. This implies $\nabla u(x_0) = 0$ and $\Delta u(x_0) \geq 0$, but that is not possible since $\Delta u = -2$.

So, that is not possible so let $x_0 \in \Gamma_i$, then you have this implies that minimum of u is in fact c_i but then if you have so this is the variation so this is γ_i . So, c_i is a minimum of u that means if you approach it along the normal direction here it is a decreasing function so this means

that $\frac{\partial u}{\partial \nu} \leq 0$ on Γ_i . But $\int_{\Gamma_i} \frac{\partial u}{\partial \nu} d\sigma = 2a_i > 0 \Rightarrow x_0 \in \Gamma_0 \Rightarrow u(x) > 0$ on $\bar{\Omega} \setminus \Gamma_0 \Rightarrow c_i > 0$.

So, this proves the smoothness needed only in the last step.

So, the torsional rigidity is defined as integral

$$S(\Omega) = \int_{\Omega} |\nabla u|^2 dx.$$