**Sobolev Spaces and Partial Differential Equations**
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**Exercises – Part 10**

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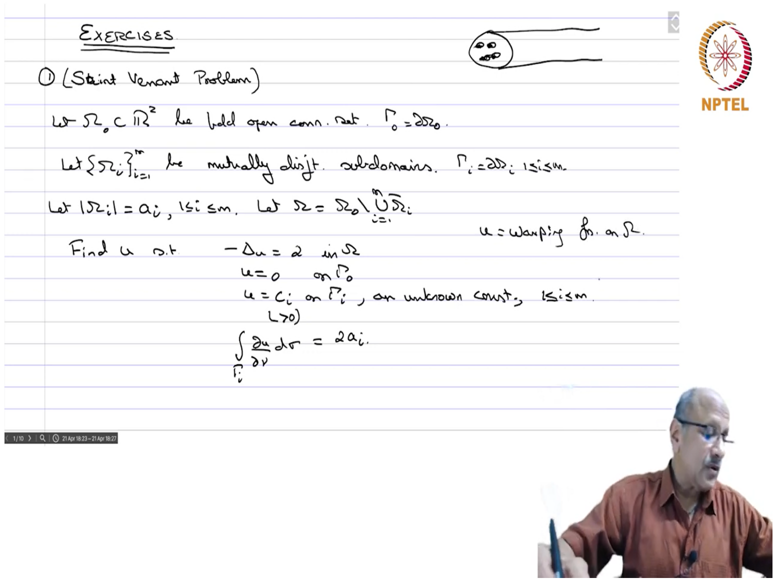


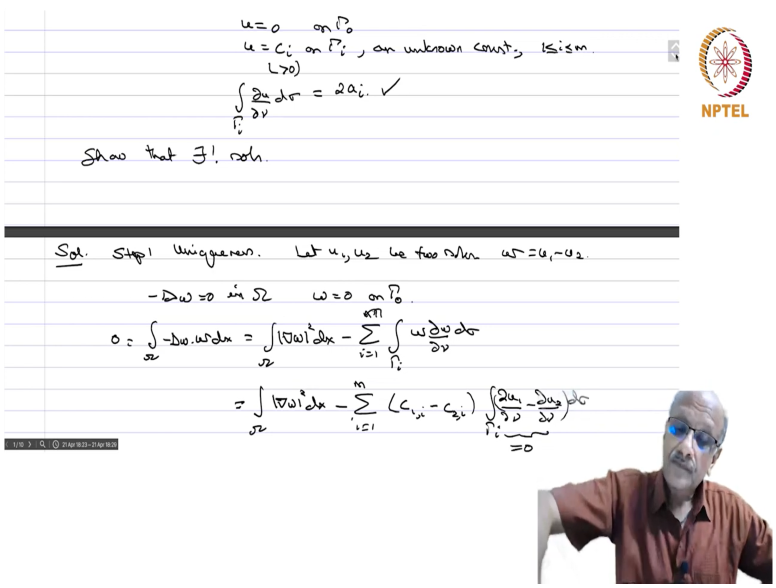

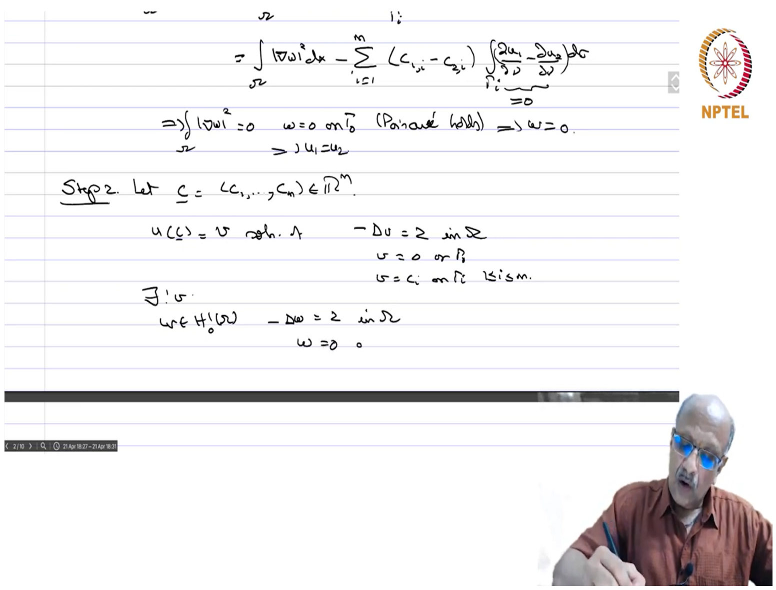


We will do some more exercises, the first one is called the Saint Venant problem. So, you think of an infinite beam of uniform cross sections, so omega is a container with some inclusions, think of concrete with steel rods stuck inside. So, the rods are going through along with this beam and so this is the cross-section of the beam and these inclusions here denote the places occupied by the cross sections of the steel rods. So, this is called so we want to study what is called the torsional rigidity of this beam which means how it is, how stiff it is and so on.

(1) Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded open connected set, $\Gamma_0 = \partial\Omega_0$. Let some $\{\Omega_i\}_{i=1}^{m}$ be open mutually disjoint sub domains that means they are also open sets and they are connected themselves. So, $\Gamma_i = \partial\Omega_i$, $1 \le i \le m$ and we write let $|\Omega_i| = a_i$ , $1 \le i \le m$ and let omega $\Omega = \Omega_0 \setminus \cup_{i=1}^{m} \overline{\Omega_i}$. That means you remove it from the portion which is other than the black shaded region. Find u such that

$$-\Delta u = 2 \quad in\,\Omega,$$

$$u = 0 \quad on\,\Gamma_0,$$

$$u = c_i \quad on\,\Gamma_i\,,\ 1 \le i \le m\,,\ c_i > 0\ -\ unknown\ constant.$$

$$\int_{\Gamma_i} \frac{\partial u}{\partial \nu}\, d\sigma = 2a_i \,,\ 1 \le i \le m.$$

So, u is called a warping function on omega so this is the thing and the torsional rigidity is determined in terms of the gradient of u. Show that there exists a unique solution.

**solution:** *step 1(uniqueness):* So, let $u_1, u_2$ be two solutions and you said $w = u_1 - u_2$. So

$$-\Delta u = 0 \ in\ \Omega\ and\ w = 0\ on\ \Gamma_0.$$

So,
$$0 = -\int_{\Omega} \Delta w \,.\, w\, dx = \int_{\Omega} |\nabla w|^2 - \sum_{i=1}^{m} \int_{\Gamma_i} w \frac{\partial w}{\partial \nu}\, d\sigma$$

$$= \int_{\Omega} |\nabla w|^2 - \sum_{i=1}^{m} (c_{1,i} - c_{2,i}) \int_{\Gamma_i} \left(\frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu}\right) d\sigma$$

But then, $\frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} = 0$. So, this means that $\int_{\Omega} |\nabla w|^2 = 0 \Rightarrow w = 0\ on\ \Gamma_0 \Rightarrow w = 0$ by Poincare inequality. So $u_1 = u_2$.

So now, we want to show the existence of a solution.

*Step 2.* So, let $\underline{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$. So you define $u(\underline{c}) = v,$ solution of

$$-\Delta v = 2 \quad in\ \Omega,$$

$$v = 0 \quad on\ \Gamma_0,$$

$$v = c_i \quad on\ \Gamma_i,\ 1 \le i \le m.$$

So, this is a straightforward Dirichlet problem and everything is fine therefore, there exists a unique v. And let $w \in H^1_0(\Omega)$ such that

$$-\Delta w = 2 \quad in\, \Omega$$

$$v = 0 \quad on\, \Gamma_0$$

this also exists there exists unique w so this also is fine.

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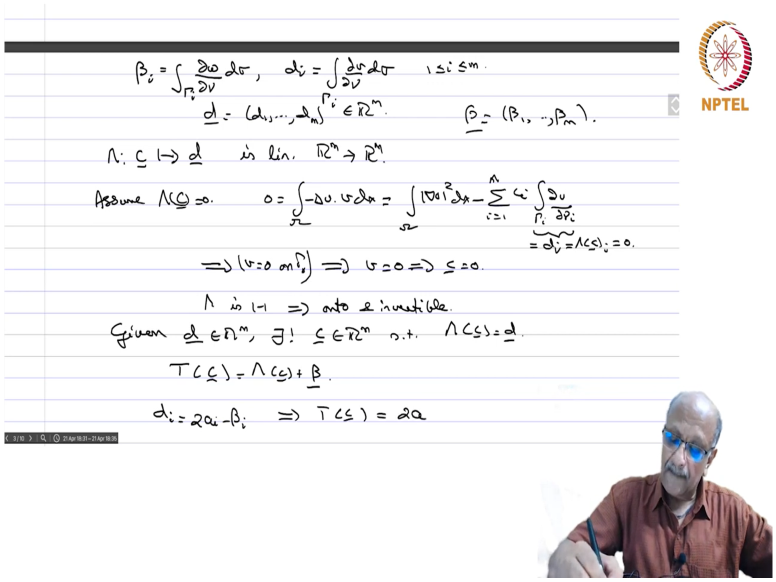

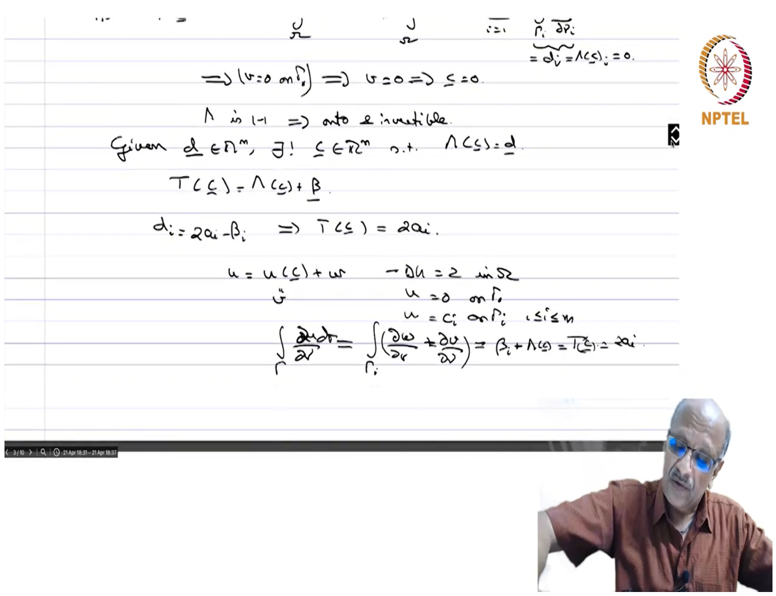

So now, you define $\beta_i = \int\limits_{\Gamma_i} \frac{\partial w}{\partial \nu}\, d\sigma$ , $\quad d_i = \int\limits_{\Gamma_i} \frac{\partial w}{\partial \nu}\, d\sigma \quad$ , $1 \le i \le m$ , $\underline{\beta} = (\beta_1,..., \beta_m) \in \mathbb{R}^m$,

$\underline{d} = (d_1,..., d_m) \in \mathbb{R}^m$ .

So now, if you take $\Lambda$: $\underline{c} \rightarrow \underline{d}$ is linear from $\mathbb{R}^m \rightarrow \mathbb{R}^m$ . Assume that $\Lambda(\underline{c}) = 0$ .

That means $0 = -\int_{\Omega} \Delta v \, . \, v \, dx = \int_{\Omega} |\nabla v|^2 - \sum_{i=1}^{m} c_i \int_{\Gamma_i} \frac{\partial v}{\partial \nu_i} d\sigma \Rightarrow v = 0 \; on \; \Gamma_i \Rightarrow v = 0$

$$\Rightarrow \underline{c} = 0 .$$

So, $\Lambda$ is 1-1 implies onto and invertible. So, given $\underline{d} \in \mathbb{R}^m$ there exists a unique $\underline{c} \in \mathbb{R}^m$ such that $\Lambda(\underline{c}) = \underline{d}$. Now you define $T(\underline{c}) = \Lambda(\underline{c}) + \underline{\beta}$. Now if you take $d_i = 2a_i - \beta_i$. Then this is imply that $T(\underline{c}) = 2a_i$.

So now, if you set u equal to u of c which is equal to v of course plus w, then you have minus Laplacian w equal to 2 in omega because you see minus Laplacian w is 0. And for this 1 it is 2 so when you add them you will get 2 w equals 0 on gamma not and w equal to ci on gamma i and integral dw by d nu on gamma i this is nothing but di which is equal to 2 ai dw by d nu plus dv by d nu this equal to, u. So, integral gamma du by d nu d sigma equal to integral d w by d nu which is equal to beta i beta i plus lambda of c and that is equal to t c equals 2 ai. So, this is also so you have it satisfies all these conditions 1 less equal to i listening to n.

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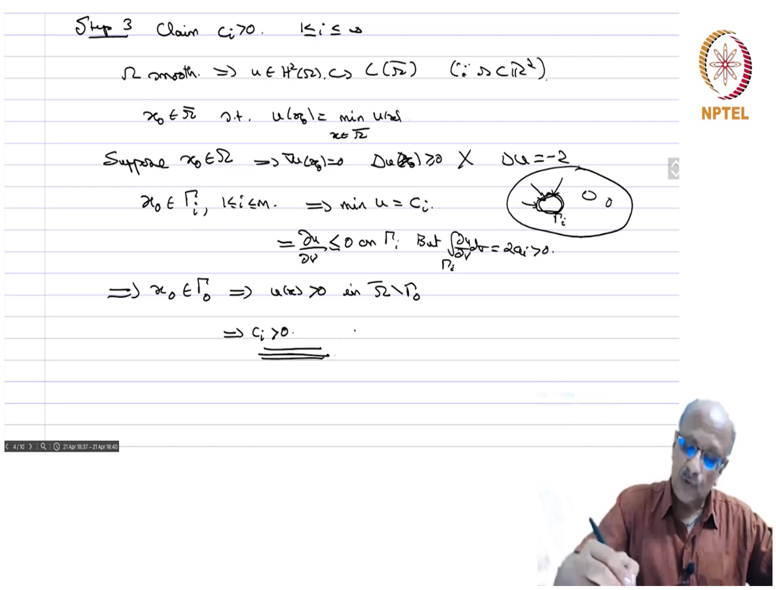

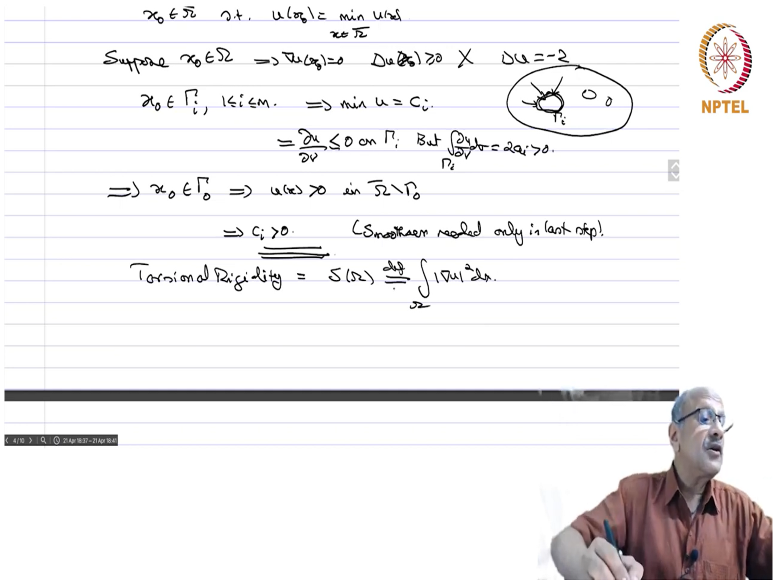


*step 3:* claim: $c_i > 0,\ 1 \le i \le \infty$.

So, let $\Omega$ be smooth enough so this implies that $u \in H^2(\Omega)$ and that since we will be in c of the omega bar since we are contained in R2 by the Sobolev embedding theorem. So then, if you take $x_0 \in \overline{\Omega}$, then such that $u(x_0) = \inf_{x \in \overline{\Omega}} u(x)$. So, suppose $x_0 \in \Omega$. This implies $\nabla u(x_0) = 0$ and $\Delta u(x_0) \ge 0$, but that is not possible since $\Delta u = -2$.

So, that is not possible so let $x_0 \in \Gamma_i$, then you have this implies that minimum of u is in fact ci but then if you have so this is the variation so this is gamma is. So, $c_i$ is a minimum of u that means if you approach it along the normal direction here it is a decreasing function so this means that $\frac{\partial u}{\partial \nu} \le 0 \backslash on\ \Gamma_i$. But $\int_{\Gamma_i} \frac{\partial u}{\partial \nu}\, d\sigma = 2a_i > 0 \Rightarrow x_0 \in \Gamma_0 \Rightarrow u(x) > 0\ on\ \overline{\Omega} \backslash \Gamma_0 \Rightarrow c_i > 0$.

So, this proves the smoothness needed only in the last step.

So, the torsional rigidity is defined as integral

$$S(\Omega) = \int_{\Omega} |\nabla u|^2 dx.$$