Sobolev Spaces and Partial Differential Equations
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Maximum Principles – Part 1

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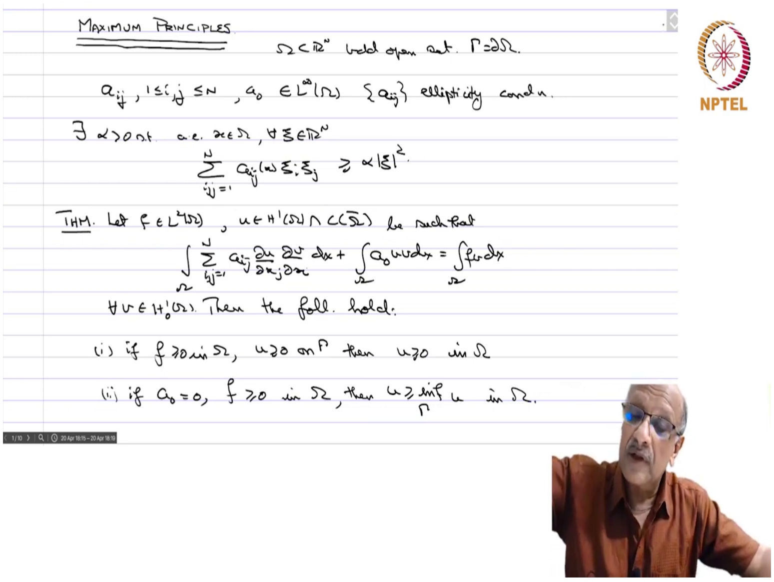

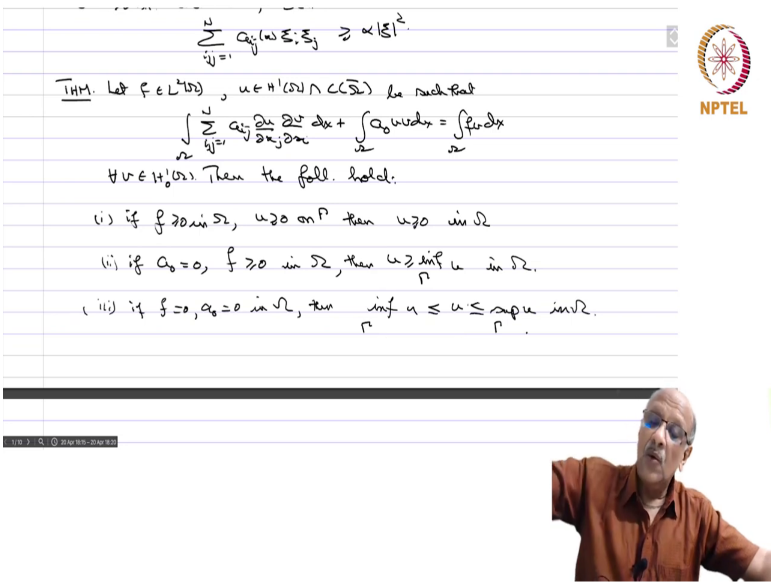

We will now study a very important property of second order elliptic equations. These are called Maximum Principles, you might have come across this word already when studying complex analysis and that is because the real and imaginary parts of analytic functions satisfy they are harmonic. Therefore, they satisfy $\Delta u = 0$ and that is why you have maximum principles there also.

So, throughout this section we will assume that $\Omega \subset \mathbb{R}^N$ bounded set and $\Gamma = \partial\Omega$ $a_{i,j}$, $1 \leq i, j \leq N$, $a_0 \in L^\infty(\Omega)$. So, all these functions are the same and $a_{i,j}$ satisfies an ellipticity condition.

So, let me remind you of that, so for every, almost every $x \in \Omega$ and $\xi \in \mathbb{R}^N$ there exists an $\alpha > 0$,

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i \xi_j \geq \alpha ||\xi||^2.$$

So, this is called the uniform ellipticity condition for these operators, so we have the following theorem.

Theorem: Let $f \in L^2(\Omega)$, $u \in H^1(\Omega) \cap C(\overline{\Omega})$ be such that

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\, dx + \int_\Omega a_0 uv\, dx = \int_\Omega fv\, dx . \text{ -----------(*)}$$

for every $v \in H_0^1(\Omega)$. Then the following holds

(i) if $f \geq 0$ in Ω, $u \geq 0$ in Γ, then $u \geq 0$ in Ω.

(ii) if $a_0 = 0$ and $f \geq 0$ in Ω, then $u \geq \inf_\Gamma u$ in Ω .

(iii) if $a_0 = 0$ and $f = 0$ in Ω, then $\inf_\Gamma u \leq u \leq \sup_\Gamma u$ in Ω .

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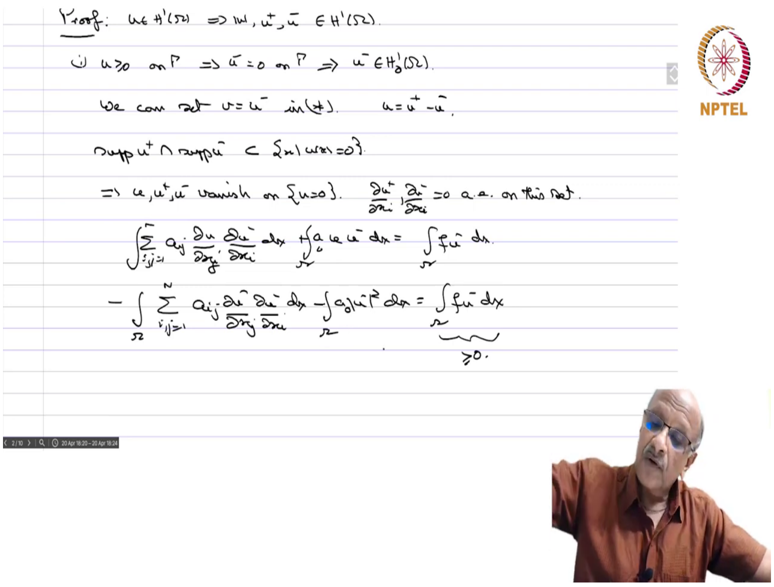

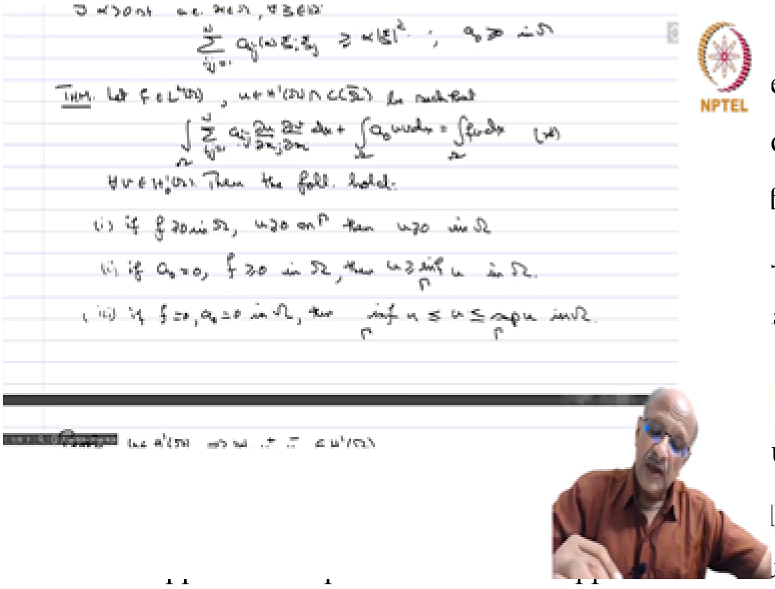

ega, so this implies mod u, u plus,
o and even then I commented that
f u is greater than or equal to 0 on
ıse u equals u plus minus u minus
als mod u, since u is greater than
minus belongs to H1 0 of omega.

ut u equals…we can set v equals i
ıuals u plus minus u minus. What
ıs intersect only on the set that is
contained in, in fact it is contained in the set of all x sets ux equal to 0. And u, u^{+}, u^{-} vanish on $\{u = 0\}$ and $\frac{\partial u^{+}}{\partial x_i}, \frac{\partial u^{-}}{\partial x_i} = 0$ almost everywhere on the set $\{u = 0\}$.

So, these are all properties which we prove whenever you have u equals constant, then you have u plus du by d xi du minus by dx i there of course, the set may be of measure 0 that is a different point, but then even if it is not of measure 0 this is always true. And therefore, if you substitute, you have

$$\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u^{-}}{\partial x_i}\, dx + \int_{\Omega} a_0 u u^{-}\, dx = \int_{\Omega} f u^{-}\, dx$$

$$- \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u^-}{\partial x_j} \frac{\partial u^-}{\partial x_i}\, dx \;+ \int_{\Omega} a_0 |u^-|^2\, dx = \int_{\Omega} f u^-\, dx$$

Now this f is non negative u minus is non negative, so this integral is non negative, a naught…. I forgot to say that already, so this additional condition is a naught greater than equal to 0 in omega, so these two conditions are necessary.

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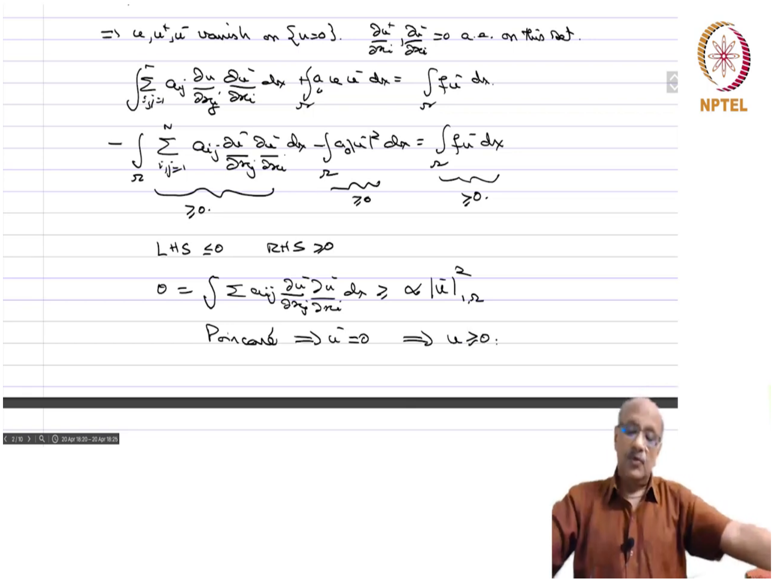

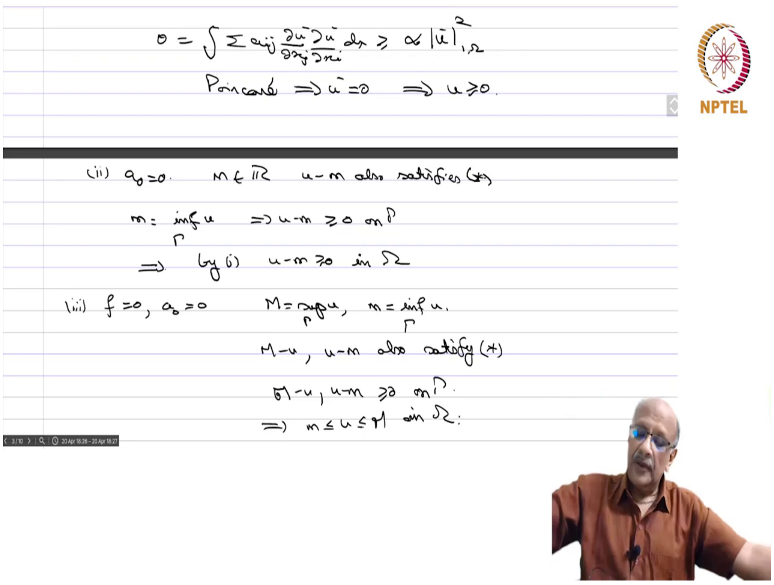

So, a naught is non-negative u minus square this non negative, so this integral is also non negative and by the ellipticity condition this integral is also non negative. Therefore, the left hand side is less than or equal to 0, LHS is less than equal to 0, RHS is greater than equal to

0, so the whole thing has to be 0. If that is 0, then each of the terms on the left hand side is equal to 0 and by the ellipticity condition,

$$0 = \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u^-}{\partial x_j} \frac{\partial u^-}{\partial x_i}\, dx \geq \alpha |u^-|^2_{1,\Omega}.$$

Poincare implies that $u^- = 0$ and therefore this implies that $u \geq 0$. This is the very simple proof based on these sign arguments here. We get the first thing, this proves (i).

Now for (ii); let $a_0 = 0$, then if $m \in \mathbb{R}$ is constant then $u - m$ also satisfies (*). So recall star, here this term has now disappeared and therefore you have fv which has nothing to do with the u so if you put u minus m the derivative does not change and therefore u minus m is also a solution of star. So, now if you put $m = \inf_{\Gamma} u$, this implies that $u - m \geq 0$ on Γ and by (i), $u - m \geq 0$ on Ω.

(iii) if $f = 0$, $a_0 = 0$, then you have if $M = \sup_{\Gamma} u$ and $m = \inf_{\Gamma} u$, then you have $M - u$ and $u - m$ also satisfies (*), and $M - u$ and $u - m$ are greater than equal to 0 on Γ and therefore this implies that $m \leq u \leq M$ on Ω, so this proves this theorem completely.

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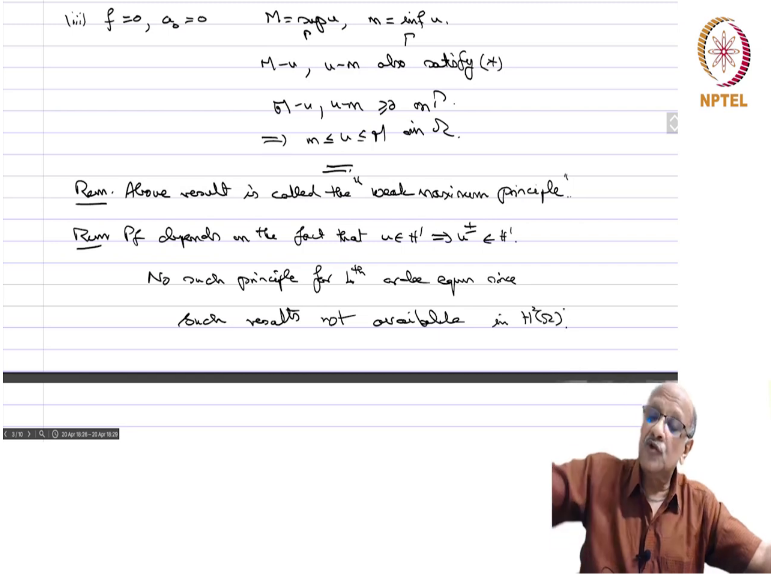

Remark: The above result is called the "weak maximum principle".

You just have the sign of the data and sign of the solution on the boundary that determines the sign in the domain. This is called the weak maximum principle.

Remark: proof depends on the fact that $u \in H^1 \Rightarrow u^{+,-} \in H^1$ and therefore, no such principle for fourth order equations since such results are not available in $H^2(\Omega)$.

This is purely a property of H^1 and that is why second order equations have this nice property.

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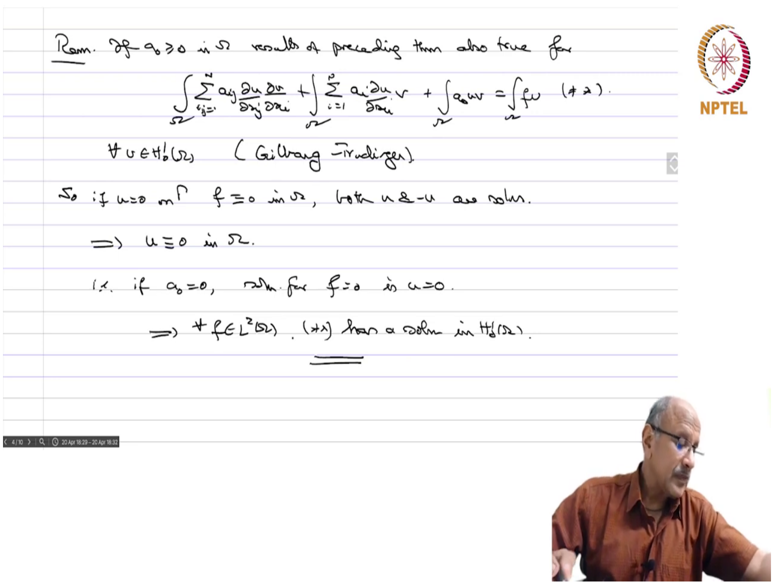

Remark: if $a_0 \geq 0 \ in \ \Omega$, results of preceding theorem also true for

$$\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\, dx \ + \int_{\Omega} \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i} v\, dx \ + \int_{\Omega} a_0 uv = \int_{\Omega} fv\, dx\,. \text{ ----------(**)}$$

for every $v \in H^1_0(\Omega)$.

So, this is a slightly difficult thing so we refer to the book by Gilberg and Trudinger. So, if $u = 0 \ on \ \Gamma$ then of course it is both non negative, non positive and $f \equiv 0 \ on \ \Omega$, then we have then both u and -u are solutions. And this will be proven by the previous theorem since the maximum and minimum are both on the u equals 0 should occur on the boundary and the boundary the value is 0.

So, this implies $u \equiv 0 \ on \ \Omega$. So, if you remember we said that the solution of this will be a d dimensional subspace and the the general equation will have when $f \neq 0$, so when $f = 0$ the solution is a d-dimensional subspace of $H^1 \ (\Omega)$ and $H^1_0(\Omega)$ and the solution will exist for the general f only if it belongs to the orthogonal complement of another d-dimensional subspace in L^2.

But now, this shows that if $a_0 = 0$, so solution for $f = 0$ is $u = 0$. Therefore, for every $u \in L^2(\Omega)$, (**) has a solution. So, by means of the maximum principle we have if you go back to the Dirichlet problem for this equation we said that f should belong to the orthogonal complement of a d-dimensional subspace, where d is the dimension of the subspace of solutions, when $f = 0$.

And now that if $f = 0$ the maximum principle tells you that the solution is only $u = 0$ and that means $d = 0$ and therefore all of you have a solution for space. Now, the condition that $u \in C(\overline{\Omega})$ that depends of course on the regularity theorems, which use the Sobolev embedding theorem.