Sobolev Spaces and Partial Differential Equations
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Excercises – The Galerkin Method Part 9

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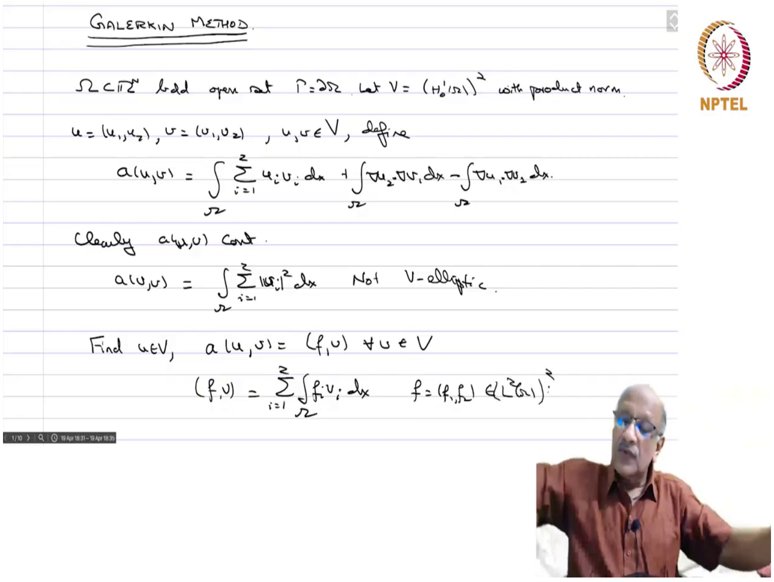

Today we will look at an example of the Galerkin method. The Galerkin method is a method of approximating solutions of functional equations, so on one hand it is useful in numerical analysis, because you produce approximate solutions of equations which you want to solve. On the theoretical side it is also a useful tool, because by producing approximate solutions and showing that they converge in some suitable topology to the solution of a problem, you prove that by the existence of solutions of a problem.

So, we will illustrate the latter in an example here.

So, let $\Omega \subset \mathbb{R}^N$, a bounded open set and $\Gamma = \partial\Omega$. So, let $V = (H^1_0(\Omega))^2$ with the product norm and for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u, v \in V$, define

$$a(u, v) = \int_\Omega \sum_{i=1}^{2} u_i v_i dx - \int_\Omega \nabla u_1 . \nabla v_2 dx - \int_\Omega \nabla u_2 . \nabla v_1 dx .$$

So, by Cauchy-Schwarz inequality, $a(u, v)$ is continuous, but look at

$$a(v, v) = \int_{\Omega} \sum_{i=1}^{2} |v_i|^2 dx$$ – this is not V-elliptic, because we cannot express the L^2 norm greater than or equal to the H^1 norm that is not possible. So, therefore, this is not an elliptic thing, so by the normal course of things you cannot use the Lax-Milgram lemma.

Nevertheless $a(u, v) = (f, v)$, for every v in V where

$$(f, v) = \sum_{i=1}^{2} \int_{\Omega} f_i v_i dx \quad , \quad f = (f_1, f_2) \in (L^2(\Omega))^2 \ .$$

So, we will use the Galerkin; so we will try to prove the existence of a solution to the system of equations. You have a system of equations because you are dealing in a product space you have two unknowns' u1 and u2 to find and therefore you have two equations for them which are coupled because u1 and u2 get mixed up in the equations and you want to solve this and try to prove the existence solution. As I said, the Lax-Milgram lemma is not directly available and therefore we want to show this.

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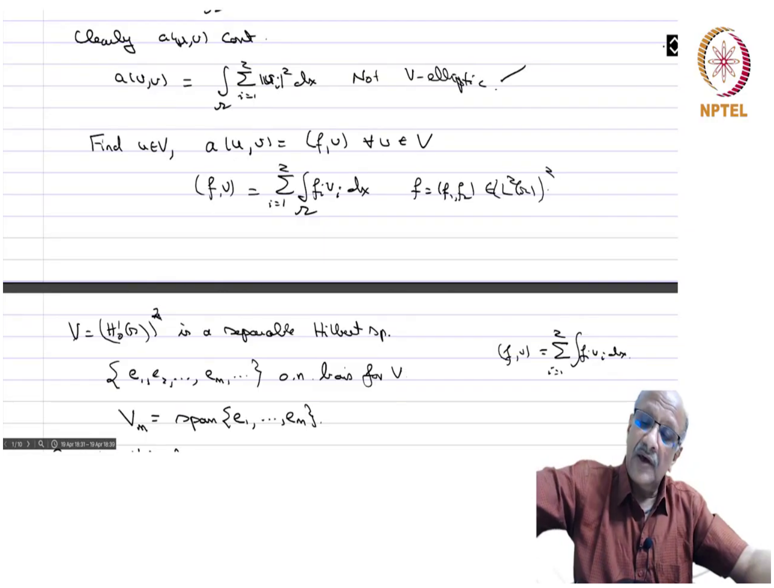

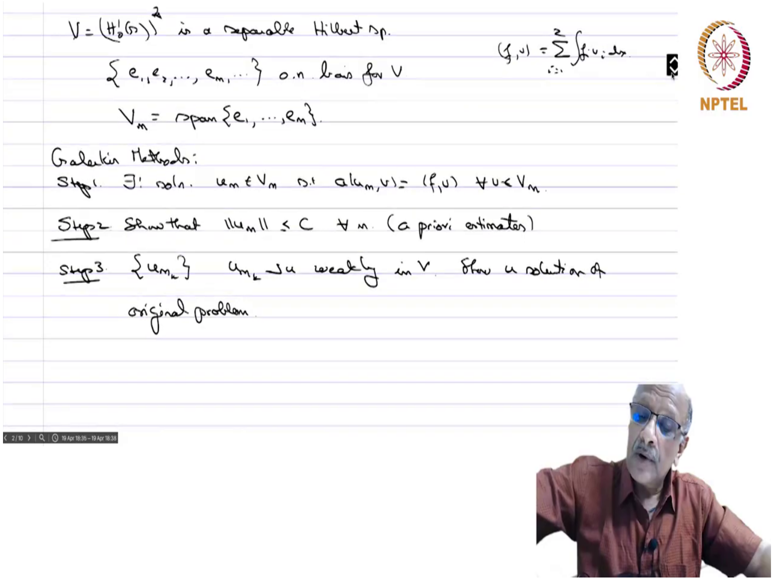

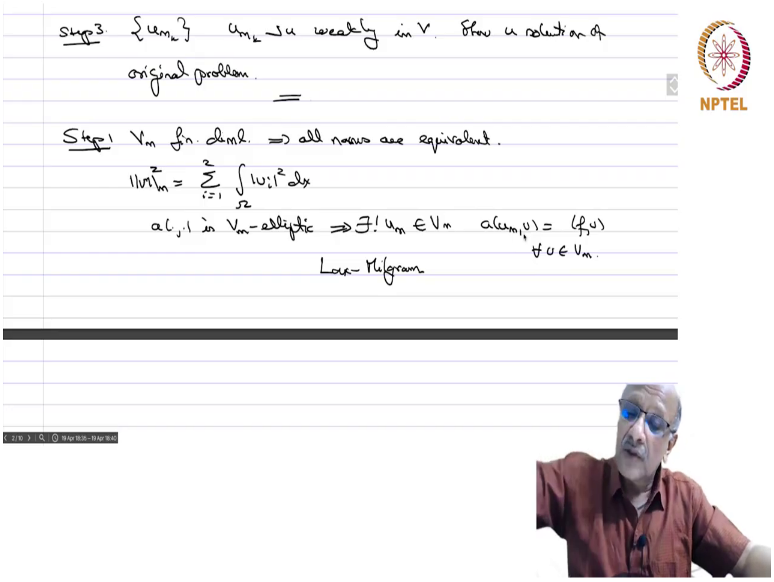

So $V = (H^1_0(\Omega))^2$ is a separable Hilbert space, so therefore let us assume that $\{e_1, e_2, \ldots, e_m\}$ is an orthonormal basis for V. So, Galerkin method, so let us say $V_m = span\{e_1, e_2, \ldots, e_m\}$. So, the Galerkin method has the following stages:

step 1: there exists a unique solution $u_m \in V_m$ such that a of $a(u_m, v) = (f, v)$ for every $v \in V_m$.

Step 2: Show that $||u_m|| \leq C$, for all m. (So, this is a method of a priori estimates).

Step 3: Thirdly, we have in the Hilbert space you have a bounded sequence where there exists a u_{m_k}, such that $u_{m_k} \rightarrow u$ weakly in V, and then show u solution of original problem. So, this is where we will have to use the properties of the bilinear form, linear form, etcetera all these things.

So, let us go and execute this program.

Step 1: V_m is finite dimensional implies all norms are equivalent. So, if you put norm

$||v||_m = \sum_{i=1}^{2} \int_{\Omega} |v_i|^2 \, dx$, this L^2 product norm and this also norm which is equivalent to the original norm. So, a of is now a V_m elliptic we already saw that saw it here and therefore and it is continuous implies there exists a unique $u_m \in V_m$ such that $a(u_m, v) = (f, v)$, for every $v \in V_m$.

Actually you do not even need to show the need to use this Lax-Milgram lemma, so this is by Lax-Milgram lemma. If you write out this equation you have to write it out for each basis function I just give you Alike.

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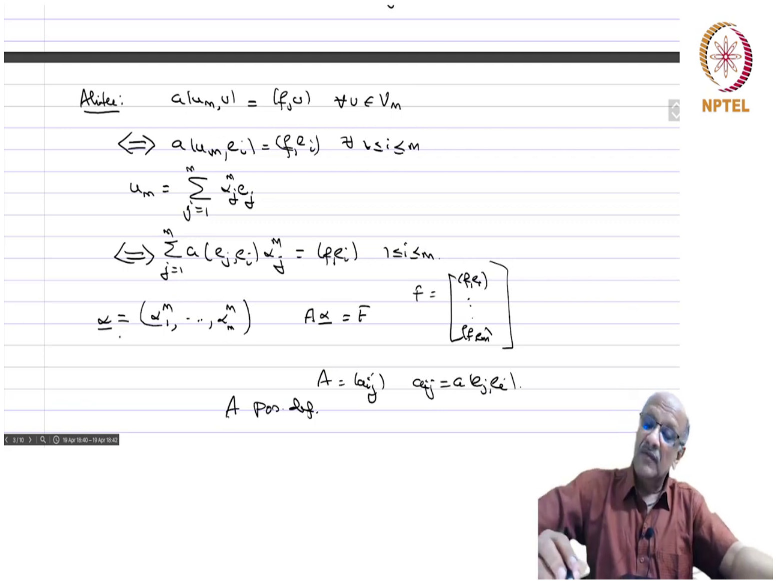

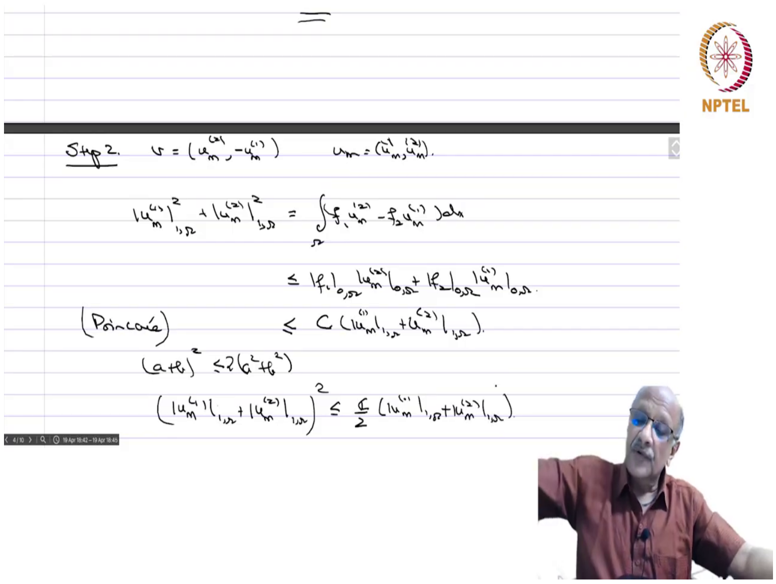

So you want to solve $a(u_m, v) = (f, v)$, for every $v \in V_m$. so this is linear in v therefore it is enough to check for the basis elements. So, this is equivalent to

$$a(u_m, e_i) = (f, e_i), \text{ for every } 1 \leq i \leq m.$$

Now $u_m = \sum_{i=1}^{m} \alpha^m_i e_i$. Therefore, if you substitute it here you get a of

$$\sum_{j=1}^{m} a(e_i, e_j)\alpha^m_i = (f, e_i),\ 1 \leq i \leq m.$$

So, now you have m linear equations in m unknowns namely $\underline{\alpha} = (\alpha^m_1, ..., \alpha^m_m)$ these are the unknowns and if you know these if you can find these then you can find um and that solves the equation.

So, you have a linear equation, you have of the form $A(\underline{\alpha}) = F$, alpha is this vector and therefore, $F = [(f, e_1), ..., (f, e_m)]^T$ and that is known to you. Then A is the matrix (a_{ij}) and $a_{ij} = (e_i, e_j)$ and using the L2 ellipticity and of the bilinear form A it comes that A is positive definite. So, a is positive definite and therefore every positive definite matrix is invertible and therefore there exists a unique alpha vector and therefore you have the solution um, so this is the way of looking at the first step.

step 2: to find an estimate for the solution, you put $v = (u^2_m, - u^1_m)$, where $v = (u^2_m, u^1_m)$. So, if you do that, then the L2 part disappears, if you look at a you have here the L2 part will disappear. These two will add up. You will have grad um 2 squares plus grad um 1 square. So, you will get that

$$|u^2_m|^2_{1,\Omega} + |u^1_m|^2_{1,\Omega} = \int_\Omega (f_1 u^2_m - f_2 u^1_m)dx \leq |f_1|_{0,\Omega}|u^2_m|_{0,\Omega} + |f_2|_{0,\Omega}|u^1_m|_{0,\Omega}$$

$$\text{(Poincare)} \qquad \leq C(|u^2_m|_{1,\Omega} + |u^1_m|_{1,\Omega})$$

and of course you have $(a + b)^2 \leq 2(a^2 + b^2)$, and therefore you have that

$$(|u^2_m|_{1,\Omega} + |u^1_m|_{1,\Omega})^2 \leq \frac{C}{2}(|u^2_m|_{1,\Omega} + |u^1_m|_{1,\Omega})$$

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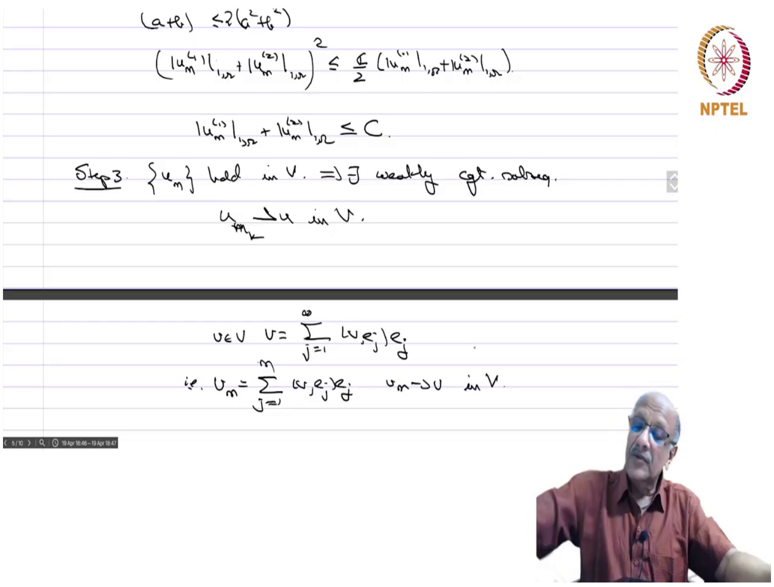

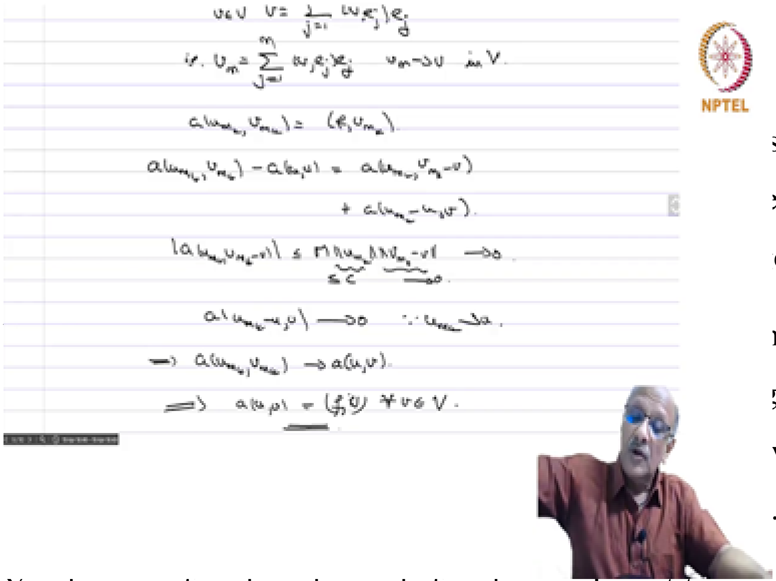

2, 1 omega is less than equal to C

s shows that step 2 is satisfied, so

ed in v implies there exists weakly

es to u in v.

nt v in v, v can be written as mod

gma j equals 1 to infinity, v ej, ej.

vm equals sigma j equals 1 to m v

.

You have a of umk vmk equals f vmk, now I want to pass to limit. So, umk converges to u weekly vmk converges to v in norm and therefore in fact you have, what do you have the a of umk vmk minus auv this equal to a of umk v plus vmk minus v plus a of umk minus u v. Now the first term mod au of umk vmk minus v is less than some m times mod norm umk norm vmk minus v. Now this is bounded because it is a weakly convergent subsequence and this goes to 0 as we know and this goes to 0.

Similarly, if you have a of umk minus uv, this is a continuous bilineal form so action on v gives you a continuous linear functional and since you have weak convergence this goes to 0 since umk weekly converges to u and therefore you have a of umk vmk goes to auv. I have just repeated the fact that if you have a continuous bilinear form in a Hilbert space, then you have one weak convergence, one norm convergence, then the limit will be the correct one which we expect.

Therefore, this implies that auv is equal, this one of course converges to fv because vmk converges to v, therefore auv equals fv for every v and therefore you have a solution to this problem.

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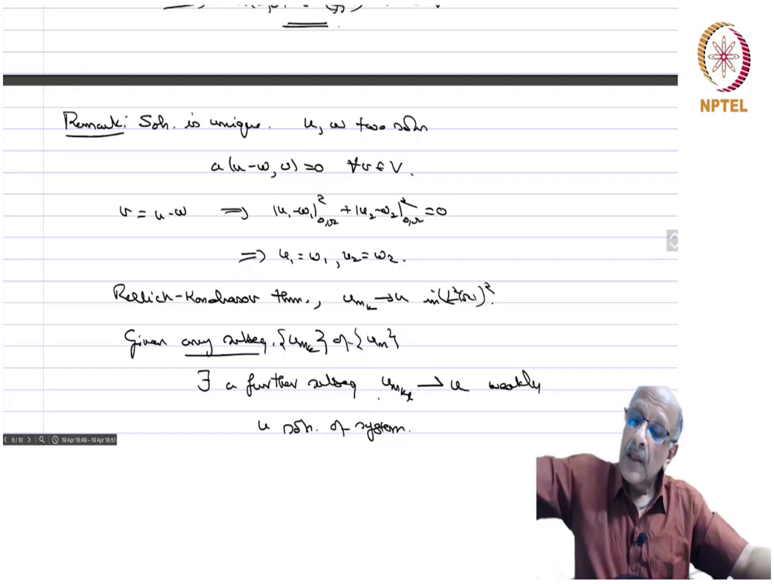

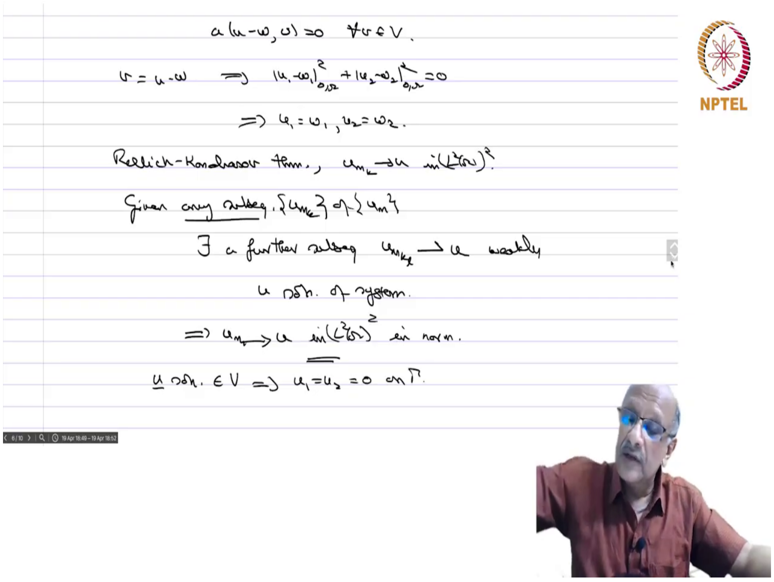

Remark: solution is unique, if you had two solutions u and w, So, you have

$$a(u - w, v) = 0, \forall\, v \in V$$

and now you put $v = u - w$, so then you will get that

$$|u_1 - w_1|^2_{0,\Omega} + |u_2 - w_2|^2_{0,\Omega} = 0 \Rightarrow u_1 = w_1,\ u_2 = w_2 .$$

and therefore the solution is unique.

Now, we also have by the Relic-Kondorov theorem, you have that $u_{m_k} \to u$ in $(L^2(\Omega))^2$ norm. Now given any subsequence u_{m_k} of any subsequence of u_m there exists a further subsequence $u_{m_{k_l}}$ which converges weakly to u, the solution of the system. Therefore, what does it mean? Every every subsequence has a further subsequence and the limit is always the same and therefore you have that $u_m \to u$ in $(L^2(\Omega))^2$ norm.

So, now let us interpret the differential equation as a boundary value problem. So, $\underline{u} \in V$ solution, this implies that $u_1 = u_2 = 0\ on\ \Gamma$. This is the boundary condition which is built into the vector space.

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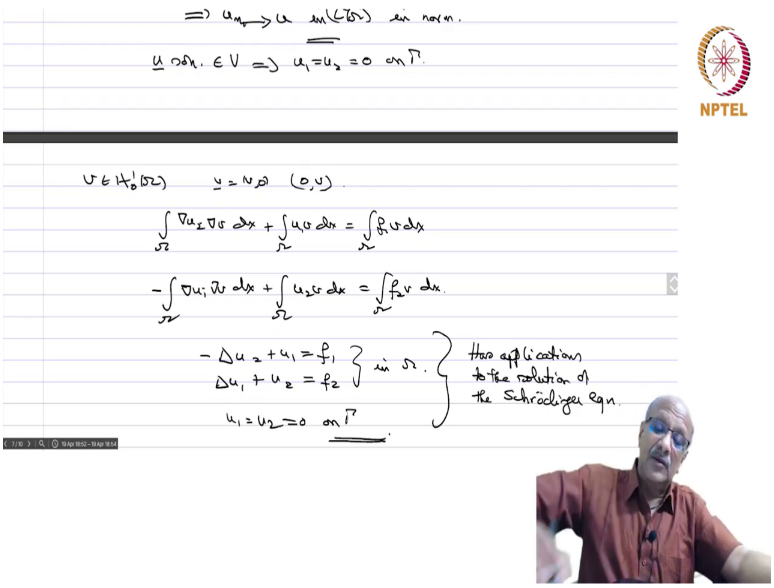

So, now if you take $v \in H^1(\Omega)$ and you take $\underline{v} = (v, 0)\ or\ (0, v)$, so then you will get that

$$\int_{\Omega} \nabla u_2 . \nabla v \, dx + \int_{\Omega} u_1 v \, dx = \int_{\Omega} f_1 v \, dx .$$

And the other equation will give you

$$- \int_{\Omega} \nabla u_1 . \nabla v \, dx + \int_{\Omega} u_2 v \, dx = \int_{\Omega} f_2 v \, dx .$$

So, if we, then you have that from this we can derive easily what is take $v \in D(\Omega)$ so you get

$$- \Delta u_2 + u_1 = f_1 , \quad - \Delta u_1 + u_2 = f_2 \quad in \, \Omega ,$$

$$u_1 = u_2 = 0 \, on \, \Gamma .$$

So, this is the coupled system of linear pdes which we have solved using this method of Galerkin and in fact this has applications to the solution of the Schrodinger equation which is important in quantum mechanics and so on, we will see that later.

So, this is an example of the Galerkin method which is very useful in both. You can use the approximate suitably. In fact the finite element method if you are familiar is a particular case of the Galerkin method. Here we have used an orthonormal basis and use the first span of the first m vectors as the finite dimensional space. The Galerkin method, the finite element method, has a different way of constructing the finite dimensional spaces.

So, that is the only difference which exploits the powers of modern computers. So, that is the importance of the Galerkin method.