Sobolev Spaces and Partial Differential Equations
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Lecture 60

Excercises Part 9

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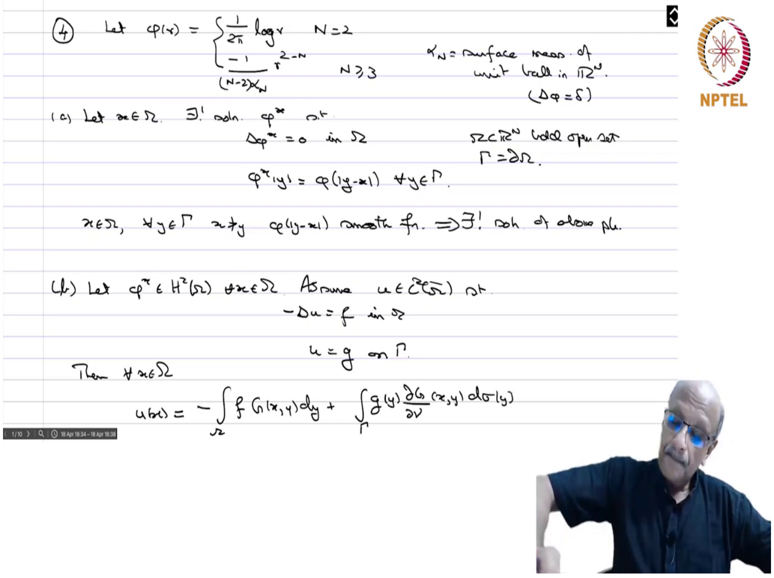

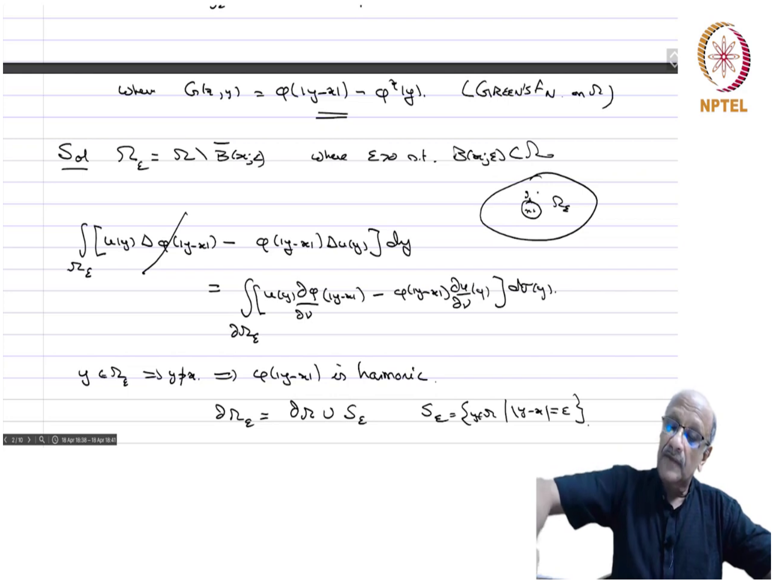

So, we continue with the

Exercises, 4: let

$\varphi(x) = \frac{1}{2\pi} log\, r \qquad , N = 2$

$= \frac{-1}{(N-2)\alpha_N} r^{2-N} \quad , N \geq 3.$

where α_N = equals surface measure of unit ball in $\mathbb{R}^N$. So, recall that this is the fundamental solution of the laplace operator we have delta Fi equal to delta as we have seen earlier in the chapter on distributions.

So **(a)**, let $x \in \Omega$ so there exists a unique solution φ^x such that $\Delta\varphi^x = 0$ in Ω and

$$\varphi^x(y) = \varphi(|y - x|) \text{ in } \Gamma.$$

So, $\Omega \subset \mathbb{R}^N$ bounded open set and gamma equals d omega as usual. So, now, if $x \in \Omega$, then for every $y \in \Gamma$ we have $x \neq y$, so $\varphi(|y - x|)$ is a smooth function, implies there exists a unique solution of above problem, unique weak solution. If omega is smooth, then it will also be a smooth solution.

So **(b),** so let $\varphi^x \in H^2(\Omega)$ for every x in Ω, so this is not very difficult to get if you have a weak solution then by regularity theorems you can easily get it for omega sufficiently smooth. Assume $u \in C^2(\overline{\Omega})$ such that

$$-\Delta u = f \text{ in } \Omega$$

$$u = g \text{ on } \Gamma$$

Then for every $x \in \Omega$ we have the representation formula

$$u(x) = -\int_{\Omega} f G(x, y)\, dy + \int_{\Gamma} g(y) \frac{\partial G(x,y)}{\partial \nu}\, d\sigma(y).$$

Where $G(x, y) = \varphi(|x - y|) - \varphi^x(y)$. so this is called the Greens function. So, we can represent the solution of the Laplace operator in terms of the Greens function, so this is called the Greens function. So, for different domains we try to calculate g then you can get explicit formulae for the solution of the equation. So, it can be done with simple geometries like half plane circle, etcetera.

You might have seen it in your pde courses, classical pde courses, such formulae are familiar. So, now let us try to prove this so the existence of a Greens function is clear, if the green domain is sufficiently smooth and the data is also smooth then you have a representation formula in terms of the Greens function.

So, let, so

solution, $\Omega_\varepsilon = \Omega \backslash \overline{B(x, \varepsilon)}$ where $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \Omega$. So, you have Ω here and you have x and you have a small ball rounded and what you have outside is omega epsilon. So, now we apply green's identity, you have that

$$\int_{\Omega_\varepsilon} u(y)\Delta\varphi(|y - x|)\, dy - \int_{\Omega_\varepsilon} \varphi(y)\Delta u(|y - x|)\, dy.$$

$$= \int_{\partial\Omega_\varepsilon} \{u(y)\, \frac{\partial\varphi(y)}{\partial\nu} - \varphi(y)\frac{\partial u(y)}{\partial\nu}\}\, d\sigma(y)$$

Now $y \in \Omega_\varepsilon$ that means y is not equal to x, because x is the center of this ball which we have excluded from the Ω_ε and this implies $\varphi(|y - x|)$ is harmonic. Therefore, this integral will go off to 0, so then you have d of Ω_ε is nothing but d of Ω_ε S epsilon where S epsilon is set of all $y \in \Omega$ mod y minus x equal to epsilon that means you have the sphere this is the boundary, so this is s epsilon.

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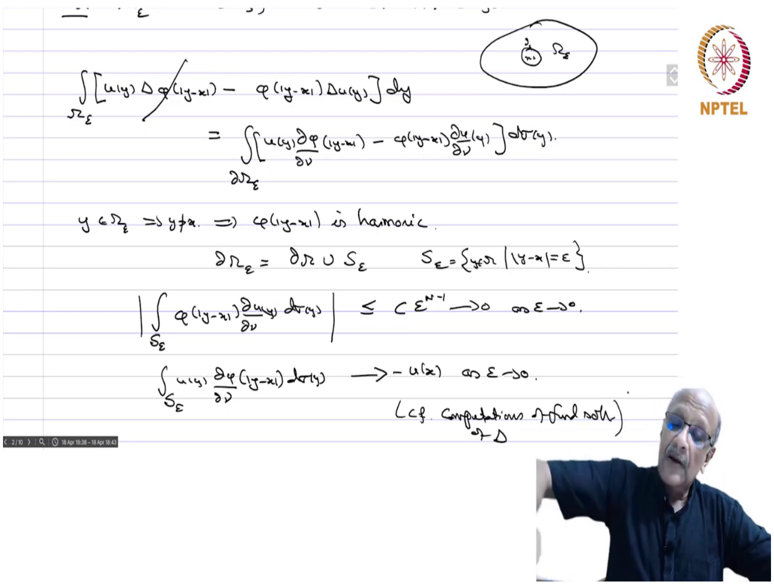

Since everything is smooth integral on S_ε of φ of mod y minus x du by d nu of y d sigma y. Now all these are bounded on S_ε, there is no problem and therefore this is less than or equal to some C times epsilon power n minus 1, because the surface measure of s epsilon is epsilon power N minus 1 times alpha N. So, this whole thing is less than C S_ε and that goes to 0 as epsilon goes to 0.

Now, what about integral of S_ε of u y d φ by d nu mod y minus x d sigma y, if you go back to the calculations which we did when computing the fundamental solution laplacian, then you perform the same calculation so this will converge to minus u of ux as epsilon tends to 0. That is so C computations of fundamental solution of delta.

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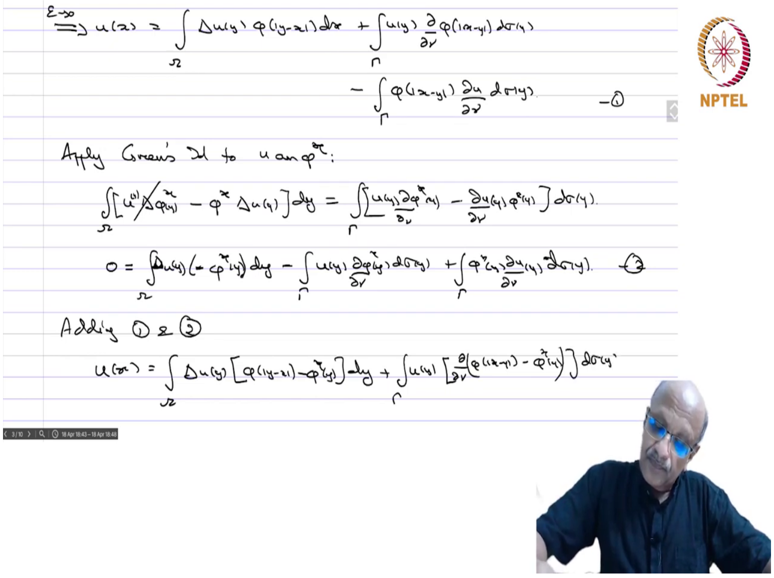

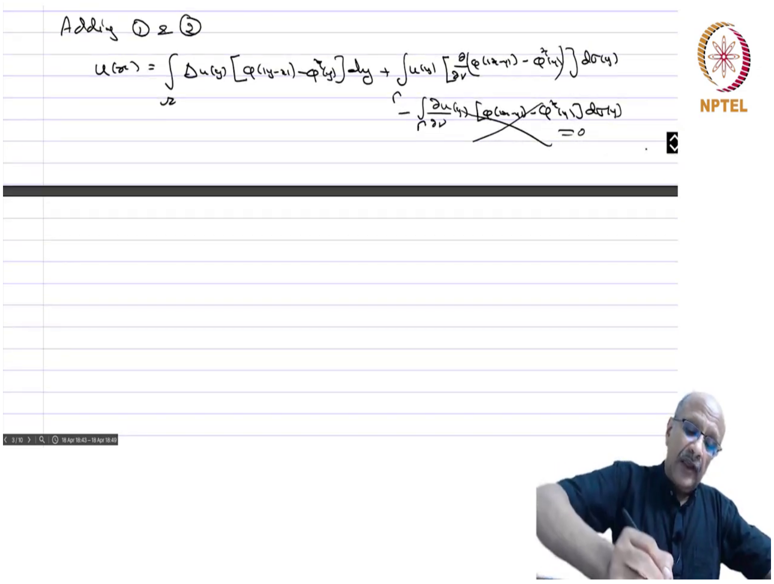

So, if with these two, so if we rewrite the previous thing, so you will get

$$u(x) = -\int_{\Omega} fG(x,y)\,dy + \int_{\Gamma} g(y)\,\frac{\partial G(x,y)}{\partial \nu}\,d\sigma(y).$$

ux will be equal to integral on omega as epsilon goes to 0 delta u(y) φ of mod y minus x dx, so this is s epsilon goes to 0 plus integral on gamma u of y d by d nu φ mod x minus y d sigma y minus integral of gamma φ of mod x minus y du by d nu d sigma y.

So, this is just got by writing integral on omega epsilon omega epsilon is sum of these two, so you have sum of two integrals the integrals on S epsilon we have evaluated taken the limits and then the integral on d omega which is gamma which is remaining and and then you have this extra term here and so if you come use these two properties here then you will get this following relationship.

So, now apply **Green's identity** to u and Fi super x because thats in H^2, I can apply that also, so integral on omega u delta $\varphi(x)$ again that is 0, because its a harmonic function, minus $\varphi(x)$delta u of y uy $\varphi^x(y)$ dy and equal to integral on gamma u y d $\varphi^x(y)$ by d nu minus du by d nu of y $\varphi^x(y)$ d sigma y. So, if we we are in this so you will get 0 equal to integral on omega delta u(y) $\varphi^x(y)$ dy plus integral on gamma u y d $\varphi(x)$ by d nu, sorry this will be with the minus sign, so this is let me rewrite this correctly.

I am bringing everything to the left hand side, so delta u of y into minus $\varphi^x(y)$, for some reason I am going to write like that and then plus minus u y d φ^x by d nu at y d sigma y plus integral on gamma $\varphi^x(y)$ du by d nu y d sigma. Now let us add these two, so adding 1 and 2, this is 1 and this is 2. So, you get u of x equal to integral on omega you have delta u y into… so you have φ of mod y minus x minus $\varphi^x(y)$ dy and then plus integral on gamma.

Here, u y into d by d nu Fi of mod x minus y minus φ^x d, $\varphi^x(y)$, so d by d nu of all this into d sigma y, and then minus integral on gamma, du by d nu at y φ of x minus y minus $\varphi^x(y)$ d sigma. Now we do not know anything about du by d nu, but this is precisely the reason because φ f by x minus y minus $\varphi^x(y)$ is equal to 0, so that is how we defined φ^x. This condition and therefore, so this term vanishes because this is equal to 0 here.

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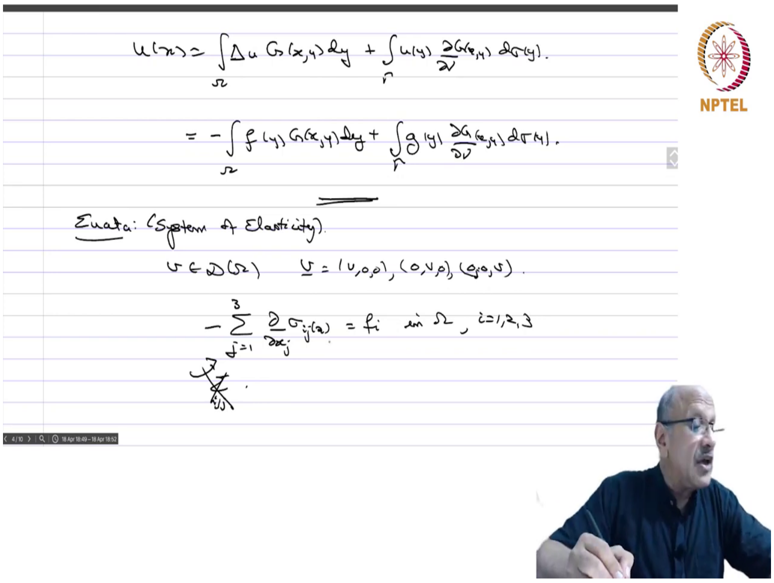

So, now $\varphi(y - x) - \varphi^{x}(y)$ is precisely $G(x, y)$, so

$$u(x) = -\int_{\Omega} \Delta u G(x, y)\, dy + \int_{\Gamma} u(y) \frac{\partial G(x,y)}{\partial \nu}\, d\sigma(y).$$

$$= -\int_{\Omega} f(y) G(x, y)\, dy + \int_{\Gamma} g(y) \frac{\partial G(x,y)}{\partial \nu}\, d\sigma(y).$$

and that is exactly what we wanted to prove.

So, this is the formula for the solution provided the domain is smooth and if you have when you take into account the Greens function which is good enough. So, before quitting this session I just wanted to… I should have done this earlier but probably here some errata which I wanted to correct.

So, system of elasticity, you might have already figured it out, we we had this we had that $v \in D(\Omega)$ and then we took $v\_ = (v, 0, 0)$, $(0, v, 0)$, $(0, 0, v)$ when recovering the differential equations and then we get that

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(u)\, d\sigma = f_i, \; in\ \Omega;\ i = 1, 2, 3$$

you get this.

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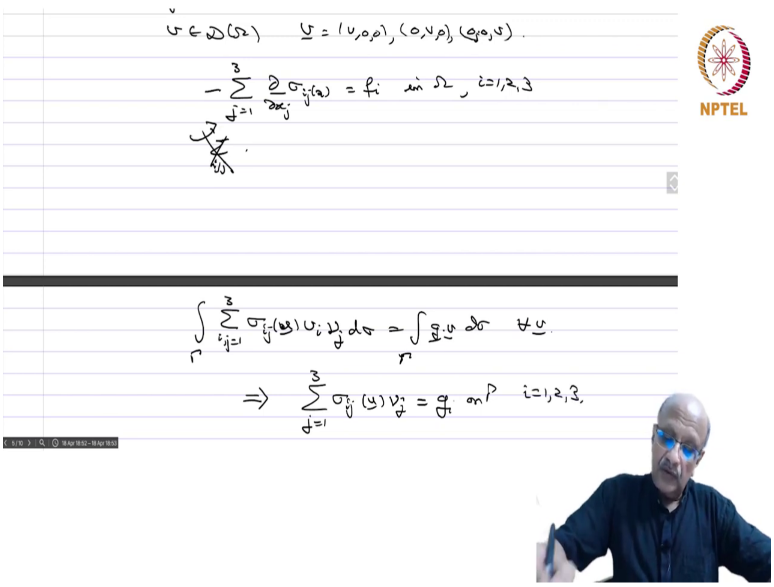

And then similarly, another sigma was missing that was

$$\int_{\Gamma} \sum_{i,j=1}^{3} \sigma_{ij}(u) v_i \mu_j \, d\sigma = \int_{\Gamma} g . v \, d\sigma$$

for every v. and this again from this we deduce that again using the $v =$ (v,0,0), (0,v,0), (0,0,v) in $H^1(\Omega)$, so you get again the sigma was missing this time again.

This is

$$\sum_{i,j=1}^{3} \sigma_{ij}(u) \mu_j = g \quad on \ \Gamma; \ i = 1, 2, 3.$$

There were various sigmas which were missing in that thing, you might have noted it already if not please make correction.