Sobolev Spaces and Partial Differential Equations
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Lecture 06
More operations on distributions

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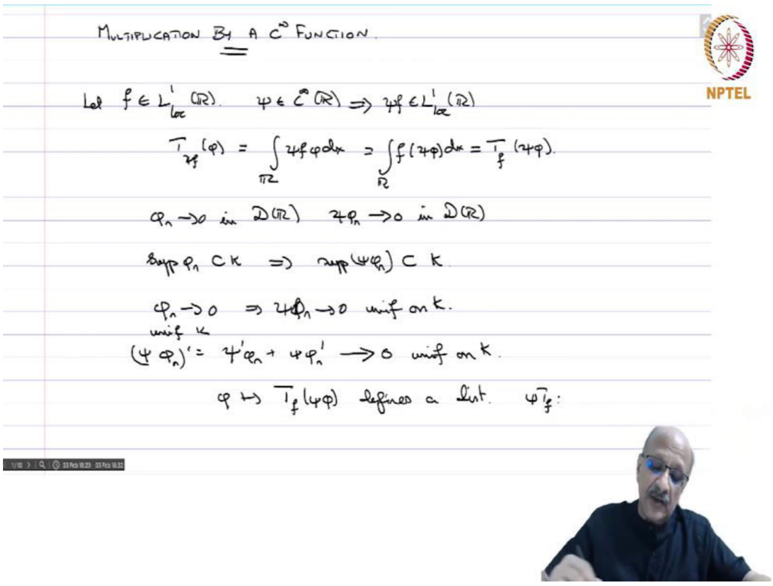

We will now do another operation on distributions. This is multiplication by C^∞ –function. So, let us look at the example of R.

Example: So, let $f \in L^1_{loc}(\mathbb{R})$, so it is a locally integrable function. Let $\psi \in C^\infty(\mathbb{R})$. So, it is infinitely differentiable functions.

Then what about ψf? $\psi f \in C^\infty(\mathbb{R})$. So you can define

$$T_{\psi f}(\varphi) = \int_{\mathbb{R}} \psi f \varphi = \int_{\mathbb{R}} f(\psi\varphi) = T_f(\psi\varphi).$$

Now, if $\phi_n \to 0$ in $\mathcal{D}(\mathbb{R})$, then $\psi\phi_n \to 0$ in $\mathcal{D}(\mathbb{R})$. Why is this so? Because support of all the ϕ_n in a fixed compact K and this means its support of $\psi\phi_n$ is also contained in the same compact with K. And then if $\phi_n \to 0$, since $\psi \in C^\infty(\mathbb{R})$, it is uniformly bounded on a compact set. So, this implies psi phi n also goes to 0 uniformly on K, this is uniformly going on K. Now, what about the first derivative?

Now $(\psi\varphi_n)' = \psi'\varphi_n + \psi\varphi_n' \to 0$ uniformly on K.

So, this we will now copy.

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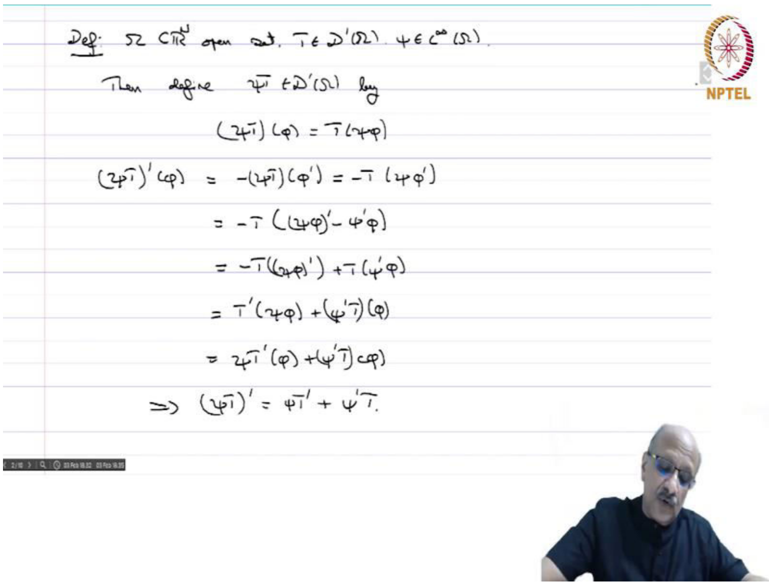

Definition: $\Omega \subset \mathbb{R}^N$ open, $T \in \mathcal{D}'(\mathbb{R})$, $\psi \in C^{\infty}(\Omega)$. Then define $\psi T \in \mathcal{D}'(\Omega)$ by

$(\psi T)(\varphi) = T(\psi\varphi)$.

So, the usual rules of calculus holds.

$$(\psi T)'(\varphi) = -(\psi T)(\varphi') = -T(\psi\varphi') = -T((\psi\varphi)' - \psi'\varphi)$$

$$= -T((\psi\varphi)') + T(\psi'\varphi)$$

$$= T'(\psi\varphi) + (\psi' T)(\varphi)$$

$$= \psi T'(\varphi) + \psi' T(\varphi)$$

$$\Rightarrow (\psi T)' = \psi T' + \psi' T$$

I am just used the product rule because psi dash phi plus psi phi dash minus psi dash phi is psi phi dash, so this is that. And by linearity this becomes minus T of psi phi dash plus T of psi dash phi. Now, what is this, this is minus T of something dash, this is nothing but T dash of psi phi by definition plus psi dash T acting on phi.

Again, I am using the definition because psi dash is a C infinity function. And this is nothing but psi T dash acting on phi plus psi dash T acting on phi. And therefore, this implies that psi T dash is equal to psi T dash plus psi dash T. So, the usual product rule applies also to the case where you multiply by distribution by your C infinity function. So, we can generalize this to RN.

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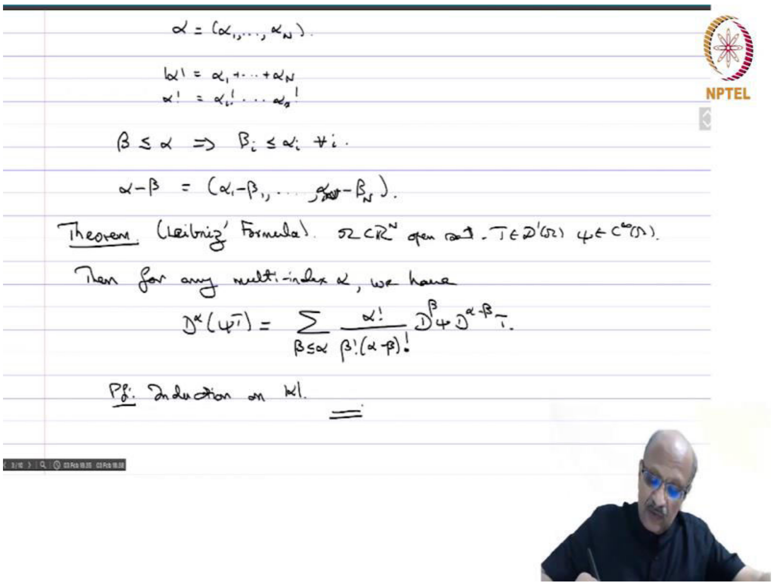

So, before that, so we remember the multi-index notation.

$\alpha = (\alpha_1,,,,,, \alpha_N)$

$|\alpha| = \alpha_1 + \ldots\ldots + \alpha_N.$

$\alpha' = \alpha_1' \ldots \alpha_N'.$

$\beta \leq \alpha \Rightarrow \beta_i \leq \alpha_i, \forall\, i.$

$\alpha - \beta = (\alpha_1 - \beta_1, \ldots\ldots, \alpha_N - \beta_N).$

So, alpha is a multi-index and then I said mod alpha is alpha 1 plus alpha N. And then I said alpha factorial is alpha 1 factorial alpha n factorial and then we said that two indices beta is less than equal to alpha this means that beta i is less than equal to alpha i for all i and then we say alpha minus beta is the index alpha 1 minus beta 1, alpha N minus beta N.

So, these are the notations, so we can generalize the product formula which we just derived. So, this is a theorem.

Leibniz formula: $\Omega \subset \mathbb{R}^N$ open, $T \in \mathcal{D}'(\mathbb{R})$, $\psi \in C^\infty(\Omega)$. Then for any multi-index α of order N, we have

$$D^\alpha(\psi T) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta \psi \, D^{\alpha-\beta} T.$$

proof: Induction on $|\alpha|$.

So, proof of induction on mod alpha you prove it for when you have only one index. So, one partial derivative which we have more or less done in the one-dimensional case, so it is the same proof and then one uses induction on higher and uses the classical Leibniz formula. Anyway, this proof is not very edifying, so we will just, we would not give it.

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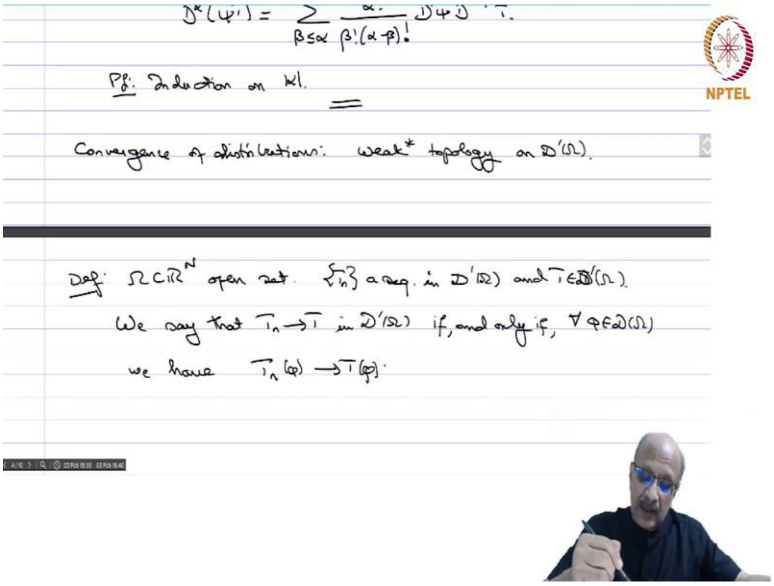

So, now one last thing, before we go to the next topic is convergence of distributions. So, we use what is called the weak star topology to define this convergence.

Definition: $\Omega \subset \mathbb{R}^N$ open, $\{T_n\}$ a sequence in $\mathcal{D}'(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. We say that $T_n \to T$ in $\mathcal{D}'(\Omega)$ if and only if for every $\phi \in \mathcal{D}(\Omega)$, we have $T_n(\phi) \to T(\phi)$.

So, it is really easy. So, this is the weak star topology namely its convergence is governed by the convergence of the action on the base space, that is exactly how in Banach spaces we have weak star convergence, which is different.

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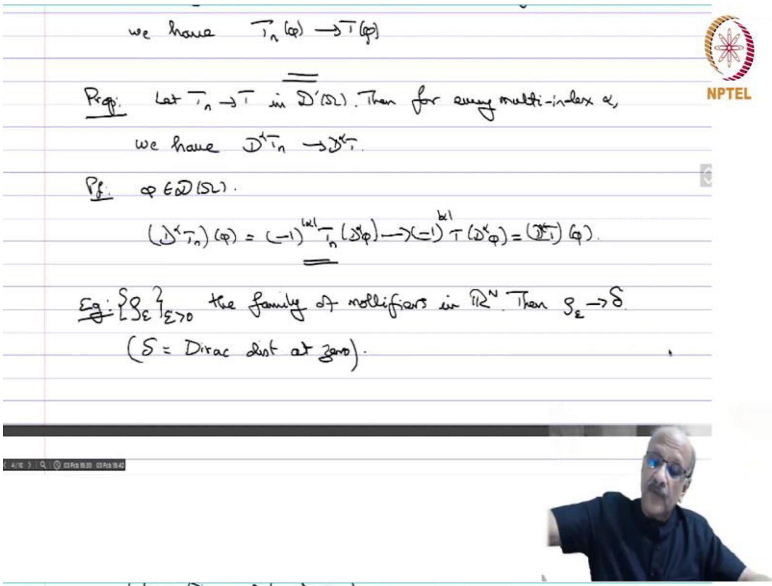

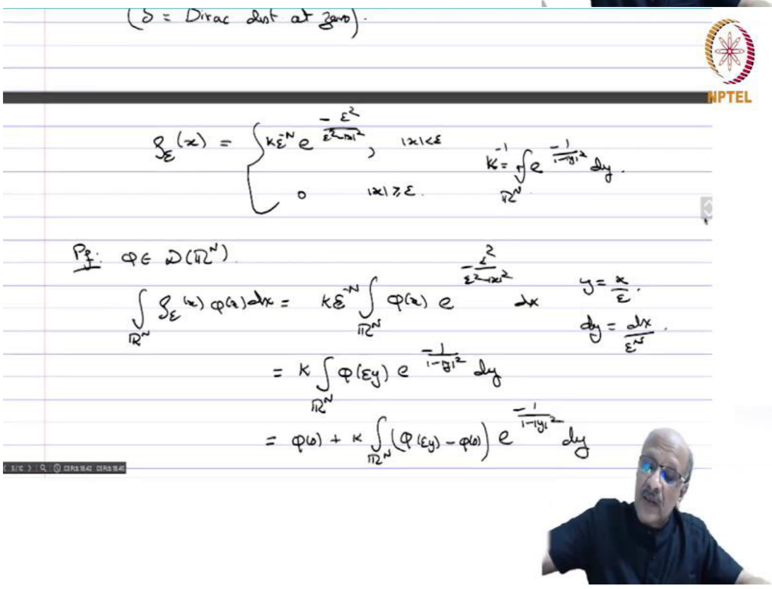

Proposition: Let $T_n \to T$ in $\mathcal{D}'(\Omega)$. Then for every multi-index α, we have $D^\alpha T_n \to D^\alpha T$.

proof: So, let us take $\phi \in \mathcal{D}(\Omega)$.

$$D^\alpha T_n(\phi) = (-1)^{|\alpha|} T_n(D^\alpha \phi) \to (-1)^{|\alpha|} T(D^\alpha \phi) = D^\alpha T(\phi).$$

So, let us look at an example.

Example: $\{\rho_\epsilon\}_{\epsilon \geq 0}$ the family of mollifiers in $\mathbb{R}^N$. Then $\rho_\epsilon \to \delta$. (δ is the Dirac distribution)

$$\rho_\epsilon(x) = k\epsilon^{-N} e^{-\frac{\epsilon^2}{\epsilon^2-|x|^2}}, \text{ if } |x| < \epsilon,$$

$$= 0, \text{ if } |x| \geq \epsilon.$$

proof: Let $\phi \in \square(\mathbb{R}^{\square})$.

$$\int_{\mathbb{R}} \square_{\square}(\square)\square(\square)\square\square = \kappa\epsilon^{-\square}\int_{\mathbb{R}} \square(\square)\square^{\frac{\square^2}{\square^2-|\square|^2}}\square\square =$$

$$\square\int_{\mathbb{R}} \square(\square\square)\square^{\frac{-1}{1-|\square|^2}}\square\square$$

$$= \square(0) + \square\int_{\mathbb{R}} \left(\phi(\square\square) - \square(0)\right) \square^{\frac{-1}{1-|\square|^2}}\square\square$$

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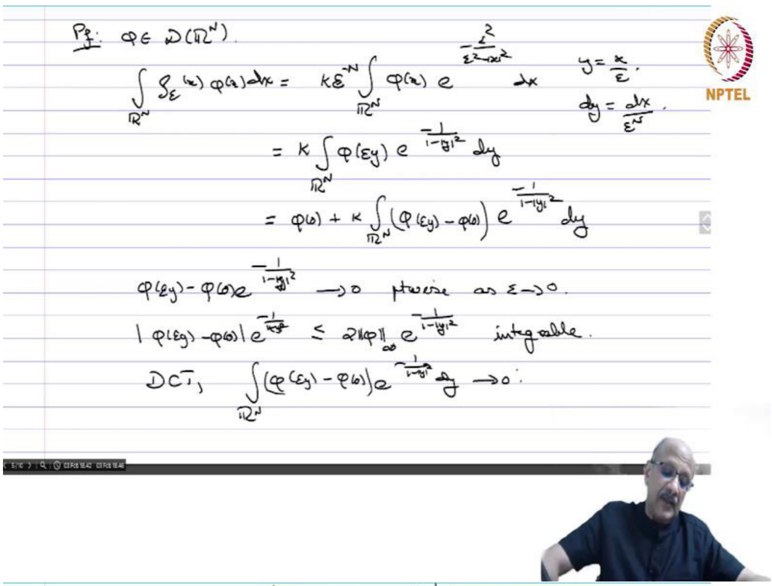

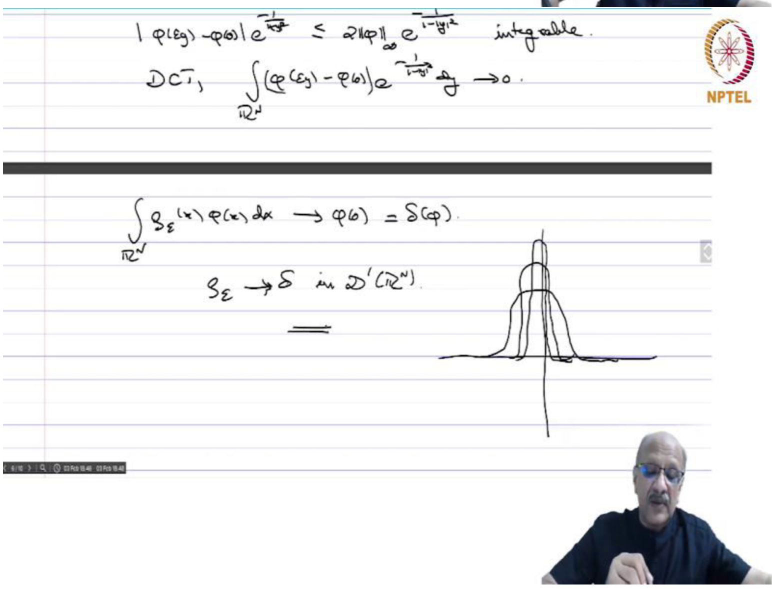

Now, look at this:

$\phi(\square\square) - \square(0)\square^{\frac{-1}{1-|\square|^2}} \to 0$ pointwise as $\epsilon \to 0$.

$|\square(\square\square) - \square(0)|\square^{\frac{-1}{1-|\square|^2}} \leq 2||\square||_{\infty}\square^{\frac{-1}{1-|\square|^2}}$ integrable

DCT $\Rightarrow \int_{\mathbb{R}} \quad (\phi(\square\square) - \square(0))\square^{\frac{-1}{1-|\square|^2}}\square\square \to 0$

Therefore $\int_{\mathbb{R}} \quad \square_{\square}(\square)\square(\square)\square\square \to \square(0) = \square(\square)$. Hence $\rho_\epsilon \to \delta$ in $\square'(\mathbb{R}^{\square})$.

So, if you remember the picture, I drew long ago. So, as an epsilon you have these bell-shaped curves. So, as epsilon becomes smaller and smaller this function becomes steeper and steeper than what I said because the integral inside has to be under the curve to be equal to 1. So, when epsilon goes to 0 this function blows up to infinity and the integral still remains as 1 in some sense under the curve.

This is the Dirac delta function which you must have come across when you are doing Laplace transform and such things in OD course, the Dirac delta function is said to be a function which is 0 outside the origin, infinitely large at the origin and has integral equal to 1. I mean these words are such nonsensical statements which mathematicians could not support and then now, everything is explained very clearly in terms of the distribution thing.

So, this rho epsilon 's are approximations to the Dirac distribution on the Dirac measure. So, this is to be expected. So, with this we will wind up this topic. So, we will start a new one, where we want to discuss what is meant by the support of a distribution. We know what is the support of a function, we want to know extend this definition to a notion of a distribution as well, which we will see next.