Sobolev Spaces and Partial Differential Equations
Professor. S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 59

Excercises Part 8

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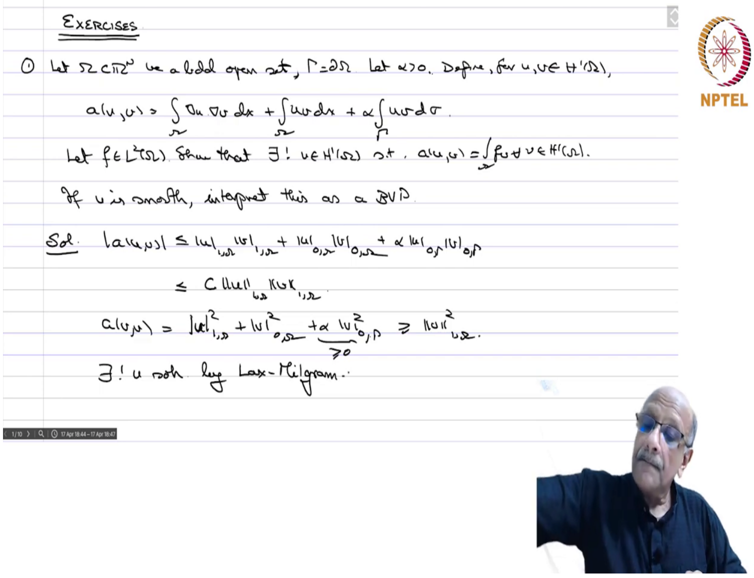

We will now do some

Exercises. 1: let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $\Gamma = \partial\Omega$, let $\alpha > 0$. Define for $u, v \in H^1(\Omega)$

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u\, v \, dx + \alpha \int_\Gamma uv \, d\sigma$$

Let $f \in L^2(\Omega)$. Show that there exists a unique $u \in H^1(\Omega)$

such that

$$a(v, v) = \int_\Omega f v \, gx, \; v \in H^1(\Omega).$$

If u is smooth interpret it as a boundary value problem.

Solution: So,

$$|a(u,v)| \leq |u|_{1,\Omega}|v|_{1,\Omega} + |u|_{0,\Omega}|v|_{0,v} + \alpha|u|_{0,\Gamma}|v|_{0,\Gamma}$$

$$\leq C||u||_{1,\Omega}||v||_{1,\Omega}.$$

By the **Trace theorem**, so it is continuous and then

$$|a(v,v)| = |v|^2_{1,\Omega} + |v|^2_{0,v} + \alpha|v|^2_{0,\Gamma} \geq ||v||^2_{1,\Omega}.$$

So, this is greater than equal to 0 because alpha and therefore this becomes equal to norm v square 1 omega. And therefore, this is H^1 omega elliptic, so there exist unique u solution by **Lax Milgram.**

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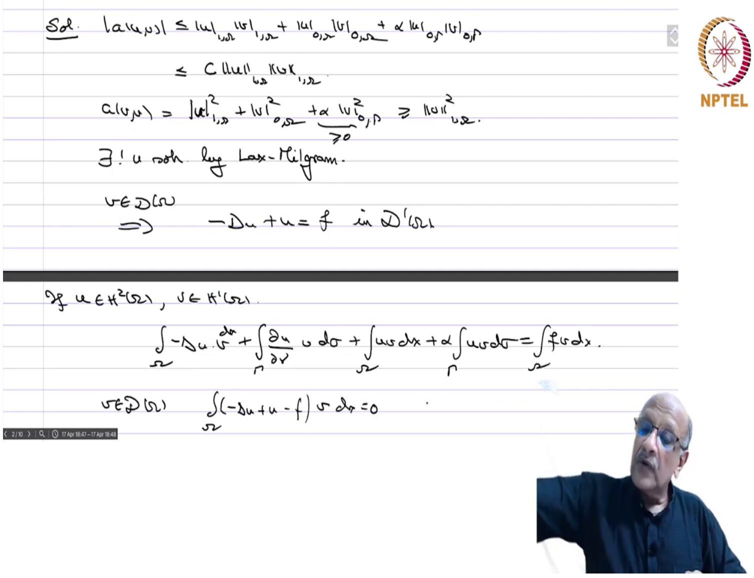

Now we want to interpret it as a boundary value problem, so if we have $v \in D(\Omega)$ then of course if you look at it immediately this will give you

$$-\Delta u + u = f$$

$f \in D'(\Omega)$. We let $v \in H^1(\Omega)$ of so assume $u \in H^2(\Omega)$ if and $v \in H^1(\Omega)$ then you get that

$$\int_\Omega -\Delta u . v \, dx + \int_\Gamma \frac{\partial u}{\partial \nu} v \, d\sigma + \int_\Omega uv \, dx + \alpha \int_\Gamma uv \, d\sigma = \int_\Omega fv \, dx$$

$$\int_\Omega (-\Delta u + u - f) v \, dx = 0$$

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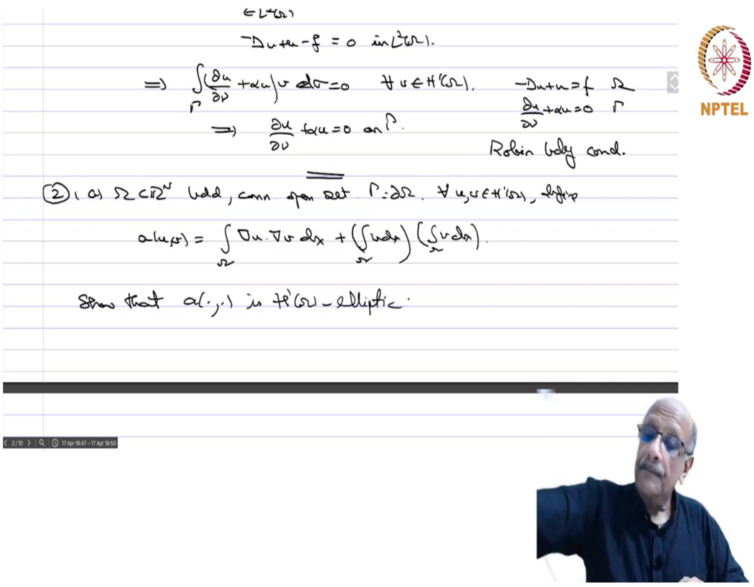

So everything is now in L^2 and we have d omega dense in L^2 and therefore, you have $-\Delta u + u = f$ in $L^2(\Omega)$. This will now imply that integral on gamma du by dnu plus alpha u times v equal d sigma equal to 0, for every $v \in H^1(\Omega)$. This we know we have seen during the Neumann problem case this implies that du by d nu plus alpha u equal to 0 on gamma. So, we have the minus laplacian u plus u equal to f in omega and du by d nu plus alpha u equal to 0 on gamma. So, this is the Robin boundary condition.

2(a): Let $\Omega \subset \mathbb{R}^N$ bounded connected open set and $\Gamma = \partial\Omega$, for all$u, v \in H^1(\Omega)$ define

$$a(u, v) = \int_\Omega \nabla u . \nabla v + \left(\int_\Omega u\, dx\right)\left(\int_\Omega v\, dx\right)$$

This is well defined because u is in $L^2(\Omega)$ is bounded so $u \in L^1(\Omega)$ so the functions are integrable. So, show that a is $H^1(\Omega)$ elliptic.

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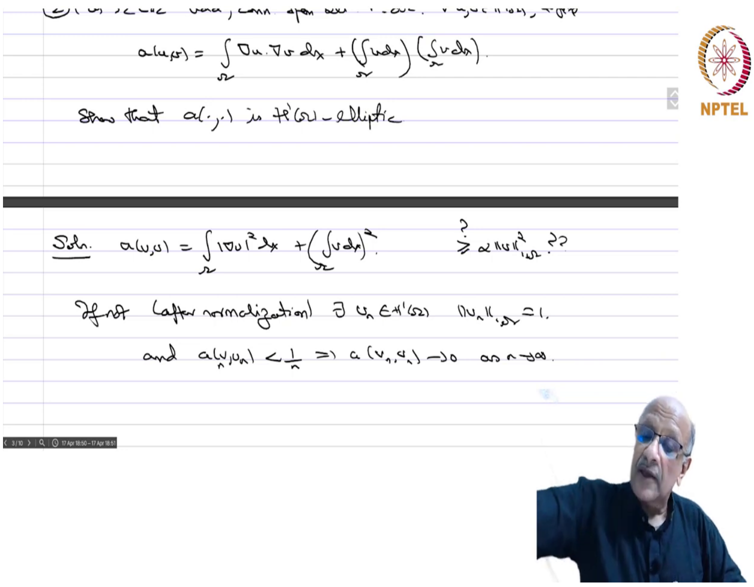

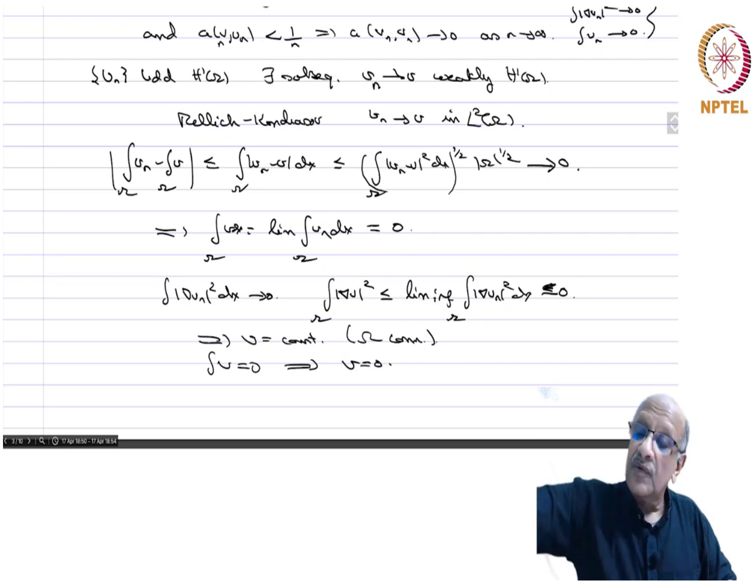

Solution,

$$a(v, v) = \int_{\Omega} |\nabla v|^2 + \left(\int_{\Omega} v\, dx\right)^2.$$

We want to show that this is greater than equal to alpha times norm v square 1 omega. If not however small alpha you take you can always contradict this therefore by after normalization there exists $v_n \in H^1(\Omega)$, norm v_n 1 omega equal to 1 and $a(v_n, v_n)$ is less than 1 by n times norm vn with 1 omega and implies that $a(v_n, v_n)$ goes to 0 as n tends to infinity.

So, $v_n \in H^1(\Omega)$ $is\ bounded$, so $v_n \in H^1(\Omega)$ so there exists a sub sequence hence fourth I will only deal with that subsequence, I wont write v_{n_k}. So, vn converges to b weakly $H^1(\Omega)$. Then by Rellick because omega is bounded contras of $v_n \in L^2(\Omega)$. Integral v_n minus integral v is less than equal to integral omega mod vn minus v and that is less than equal to integral on omega mod vn minus v square dx power half mod omega power half by Koshy-Schwartz and of course that goes to 0.

Therefore, we have integral vn itself goes to 0, so integral v equals limit integral v dx equal to 0. Integral grad vn square dx also goes to 0 because a v, n vn goes to 0 that means integral grad vn square goes to 0 and integral v n goes to 0 both these strings go to 0. Integral mod grad v square on omega is less than equal to limins integral mod grad v_n square dx that is because of the weak lower semi continuity of the L^2 in the integral and that is equal to 0.

Integral grad v square equal to 0 and that implies that v equal to constant omega is connected and since integral v equal to 0, v equals constant, so this implies that v itself is equal to 0.

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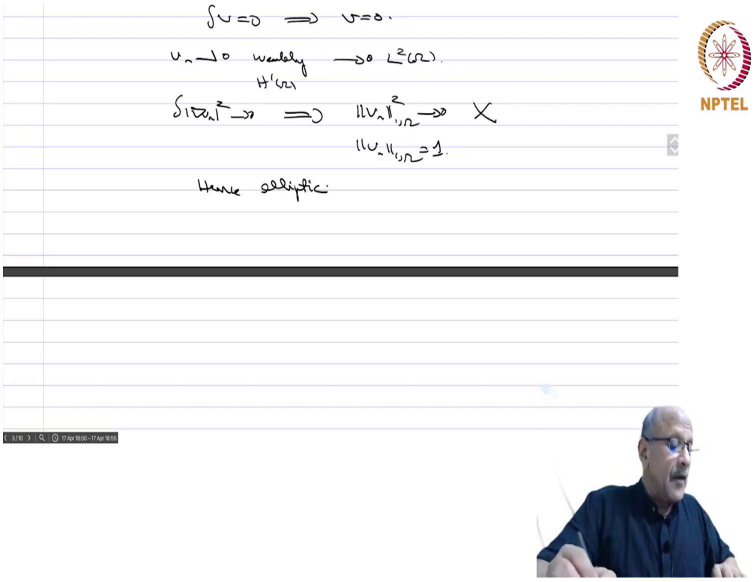

So, $v_n \to v_0$ *weakly* $v_n \to 0$, $H^1(\Omega)$ *and* $L^2(\Omega)$, but we also know that integral grad v_n square goes to 0 and therefore, this implies that norm v_n square 1 omega also goes to 0 but that is a contradiction because norm v_n 1 omega equal to 1.This is therefore it is elliptic, hence elliptic.

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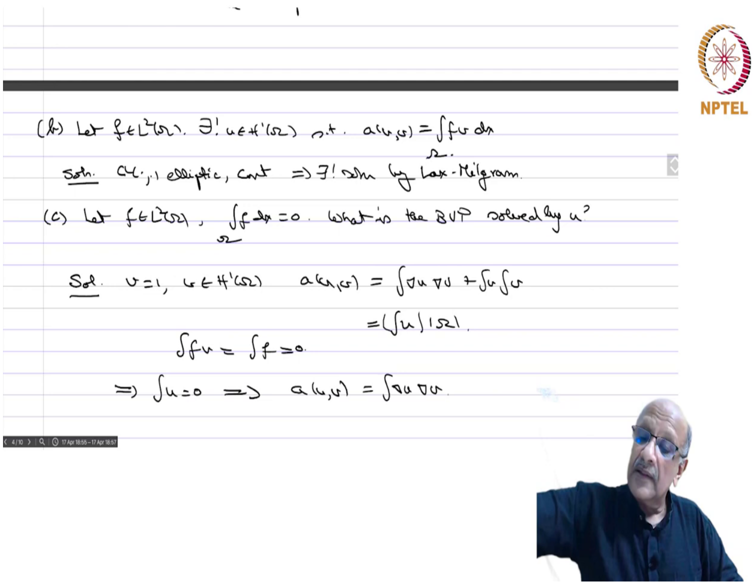

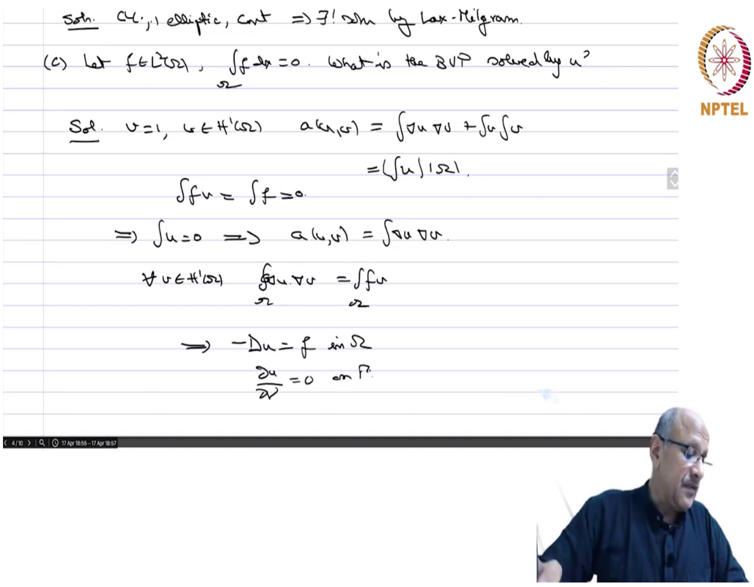

(b) let $f \in L^2(\Omega)$ then there exists a unique $u \in H^1(\Omega)$ such that

$a(u, v) = \int_\Omega fuv\, dx$. Solution, a is elliptic obviously continuous implies there exists unique solution by Lax-Milgram. C, let $f \in L^2(\Omega)$, $\int_\Omega f\, dx = 0$, what is the bvp solved by you?

Solution, so if you take v equal to 1, then $v \in H^1(\Omega)$ then,

$a(u, v) = \int_\Omega \nabla u. \nabla v + \int_\Omega u \int_\Omega v$ and that is equal to integral u times mod omega of course, integral u times mod omega. $\int_\Omega fv\, dx = \int_\Omega f = 0$ and therefore, this implies that $\int_\Omega u = 0$ and this implies therefore that $a(u, v) = \int_\Omega \nabla u. \nabla v$.

Therefore, for every $v \in H^1(\Omega)$ integral grad u grad v equals integral fv omega and this implies that minus laplacian u equals f in omega and du by d nu equal to 0 on gamma, this we know we have seen already. This is another way of looking at the problem Neumann problem without the plus u term.

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(3) (Obstacle Problem) $\Omega \subset \mathbb{R}^n$ bdd, conn. open set $\Gamma = \partial\Omega$

$\chi \in H^2(\Omega)$, $\chi \leq 0$ on Γ.

$K = \{ v \in H_0^1(\Omega) \mid v \geq \chi \text{ a.e. in } \Omega \}$.

$f \in L^2(\Omega)$. Show that $\exists!\, u \in K$ s.t. $J(u) = \min_{v \in K} J(v)$

where $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v \, dx$.

Soln. $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ cont, $H_0^1(\Omega)$-elliptic, symm.

$v_n \in K$, $v_n \to v$ in $H_0^1(\Omega)$. $\Rightarrow v_n \to v$ in $L^2(\Omega) \Rightarrow v_{n_k} \to v$ pt.wise a.e.

$v_{n_k} \geq \chi \Rightarrow v \geq \chi$ a.e. $\Rightarrow v \in K$.

K closed. Clearly K is convex.

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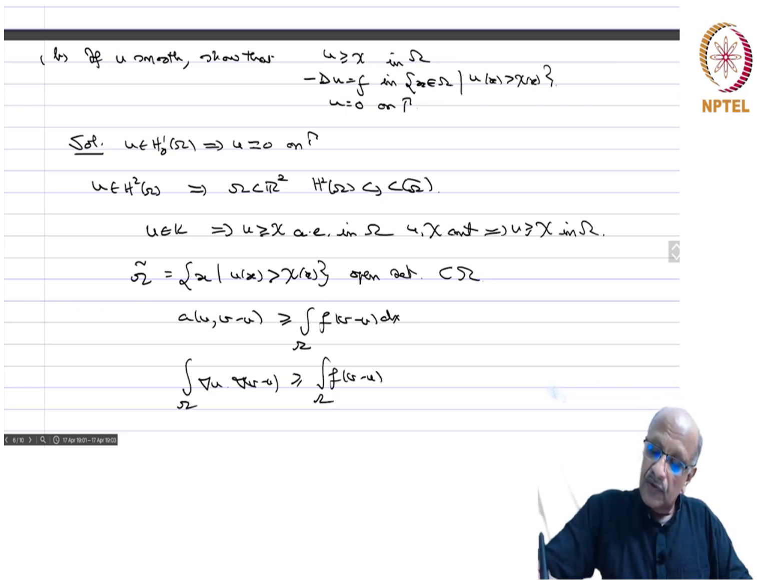
K closed. Clearly K is convex.

$\exists!\, u$. min J.

(b) If u smooth, show that $u \geq \chi$ in Ω

$-\Delta u = f$ in $\{x \in \Omega \mid u(x) > \chi(x)\}$.

$u = 0$ on Γ.



Third problem is called the

(3): Obstacle problem. $\Omega \subset \mathbb{R}^2$ bounded connected open set $\Gamma = \partial\Omega$. $\chi \in H^2(\Omega)$, $\chi \leq 0$ on Ω,

$$K = \left\{ v \in H^1_0(\Omega) |\ v \geq \chi\, on\, a.e\ \ in\, \Omega \right\}$$

$f \in L^2(\Omega)$. Show that there exists a unique $u \in K$ such that

$$J(u) = Min_{v \in K} J(v)$$

where $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v\, dx$

Solution,

$$a(u, v) = \int_\Omega \nabla u . \nabla v \ \ dx$$

is continuous $H^1_0(\Omega)$ elliptic, symmetric. So, let $v_n \in K$, $v_n \to v\ in\ H^1_0(\Omega)$, that implies v_n converges to $v \in H^2\ (\Omega)$ and therefore for the subsequence v_{n_k} converges to v pointwise almost everywhere. So,

$v_{n_k} \geq \chi \Rightarrow v \geq \chi$ almost everywhere and this implies that $v \in K$. Therefore, K is closed, clearly K is convex and therefore, there exists unique u minimizing J.

B, if u smooth show that $u \geq \chi\, in\ \Omega$,

$- \Delta u = f$

in set of all x and omega $u \geq \chi\, in\ \Omega$, $u_n > \chi\, in\ \Omega$ and $u = 0\ on\ \Gamma$. So solution, $u \in H^1(\Omega)$ implies $u = 0\, on\, \Gamma$, $u \in H^1\ (\Omega)$ assume, then omega is in $\mathbb{R}^2$. So, $H^2\ (\Omega)$ is contained in c of omega bar by the Sobolev theorem.

And therefore, you have that $u \in K$, so this implies $u \geq \chi\, in\ \Omega$ almost everywhere in omega but then u kai continuous implies $u = \chi\, in\ \Omega$. Then

$\tilde{\Omega} = \{x:\ u(x) > \chi(x)\}$

is therefore an open set contained in omega. You have

$$a(u, v - u) \geq \int_{\Omega} f(v - u)\ dx$$

by the characterization of the minimizer and therefore, you have that

$$\int_{\Omega} \nabla u. \nabla(v - u) \geq \int_{\Omega} f(v - u)\, dx$$

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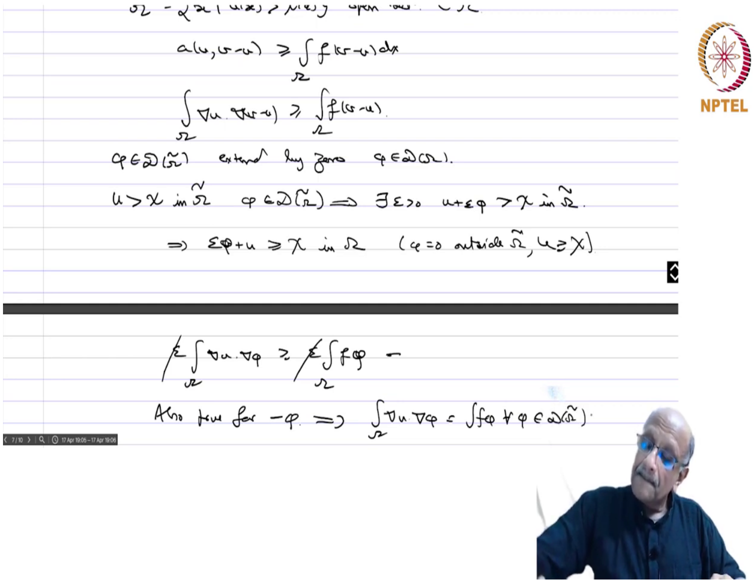

Let take $\varphi \in D(\tilde{\Omega})$, extend by 0 then $\varphi \in D(\tilde{\Omega})$ as well. Now $u \geq \chi$ in $\tilde{\Omega}$ and then $\varphi \in D(\tilde{\Omega})$, $\varphi \in D(\tilde{\Omega})$. So that means its support is inside a compact set and this implies, by the way I did not… I have sorry, fine. So, this implies there exists an $\varepsilon > 0$ such that

$u + \epsilon\varphi \geq \chi \; in \, \tilde{\Omega}$, because on the support which is a compact set you can certainly find an epsilon because $u - \chi$ attains, a minimum there a positive minimum and it is positive everywhere in that support and it attains a positive minimum.

Then outside $\varphi = 0$, so $u \geq \chi$ we know that and therefore this is being $\tilde{\Omega}$ and

$\varepsilon v\varphi + u \geq \chi \; in \, \Omega$

as well because $\varphi = 0$ and $u \geq \chi$. If you substitute that in the variational characterization then you get

$$\varepsilon \int_\Omega \nabla u.\nabla\varphi \, dx \geq \int_\Omega \varepsilon f \varphi$$

, so epsilon gets cancelled. And, now also true for $-\varphi$ and therefore this implies that

$$\int_\Omega \nabla u.\nabla\varphi \, dx = \int_\Omega f\varphi, \quad \varphi \in D(\tilde{\Omega}).$$

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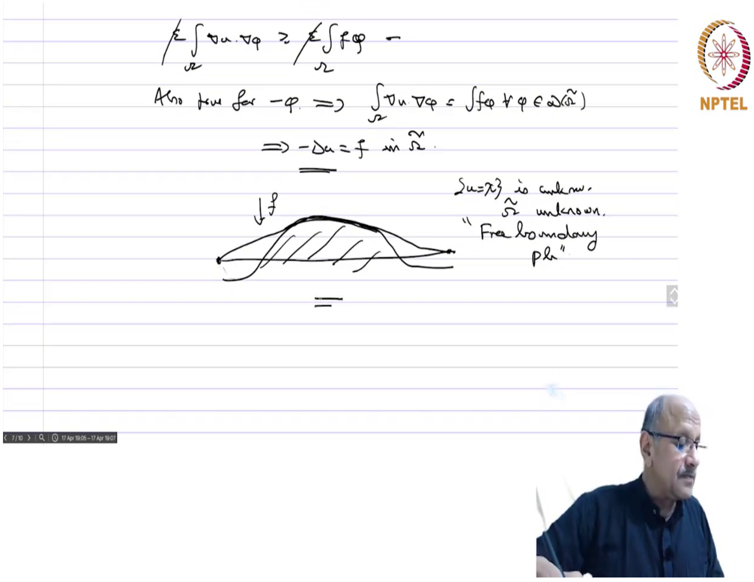

Implies $-\Delta u = f$ in $\tilde{\Omega}$. This way it is called the **Obstacle problem**, you have omega here kai is some function which is less than equal to 0 on the boundary and then you are looking for u which is above kai and then its its acted upon but think of a membrane which is stretched over a solid obstacle here and then is acted upon by a vertical force which is f and if you have the displacement.

So, there is a portion where it will come in contact with the obstacle and therefore there it will be equal to k elsewhere, it will satisfy the usual Laplace equation and this is called the obstacle problem. And then the the $u = \chi$ is unknown, so omega tilda itself is unknown. It is part of the unknown in the problem, so this is called a free boundary problem. It is an example of the application of the variational inequality.