Sobolev Spaces and Partial Differential Equations
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Lecture 58
The elasticity system

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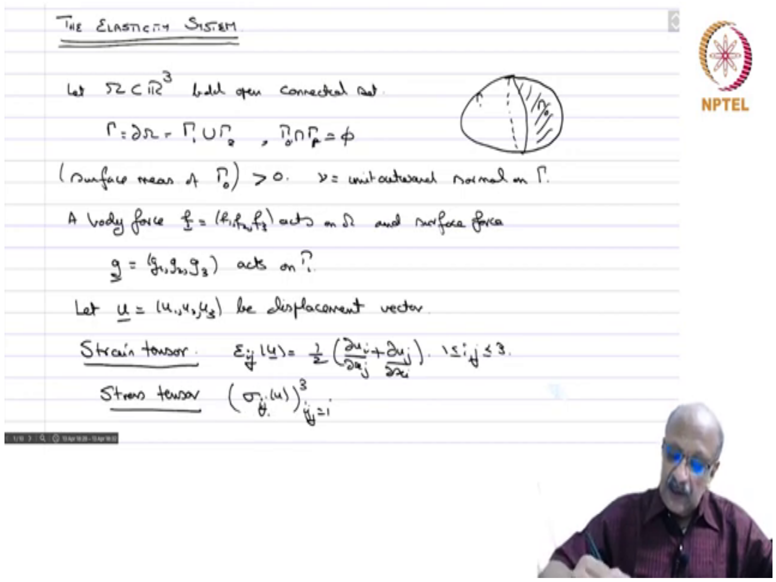

Elasticity system:

Now, our next example is the elasticity system. So, let $\Omega \subset \mathbb{R}^3$, so we are restricting ourselves to $\mathbb{R}^3$, We are bounded open and connected. This is the volume occupied by an elastic body. So, this is Ω, it is a volume occupied by an elastic body. So,

$$\Gamma = \partial\Omega = \Gamma_1 \cup \Gamma_2, \ \Gamma_1 \cap \Gamma_2 = \phi$$

so we have a portion here. So, this is a three dimensional body. So, this part we call as Γ_0 and this part of the boundary is Γ_1.

Now, surface measure of Γ_0 is strictly positive, that is assumed here, because we are going to impose a Dirichlet condition there. And then nu equals unit outward normal on Γ . A body force

$f = (f_1, f_2, f_3)$, acts on Ω and a surface force $g = (g_1, g_2, g_3)$, acts on Γ_1. So, there is a force, which acts on Γ_1 and then which could be something like a shear force, or something and then you have a force which x. So, we want to study the deformation of the bond.

So, let $u = (u_1, u_2, u_3)$, be the displacement vector. This is the one, which we are interested in. So, when you act this body on with these forces, we want to see how the body deforms, that means we have to compute u at every point of omega, then we know. Now, we have the

strain tensor, we have seen this already in the context of Khan's inequality. So, we have

$$\varepsilon_{i,j}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

, this is a symmetric tensor. So, $1 \leq i, j \leq 3$.

Now, the

stress tensor is $\{\sigma_{i,j}(u)\}_{i,j=1}^{3}$ and this is the one, which tells you how the body is going to deform under the action of the forces. So, the balance of forces will be written in terms of this.

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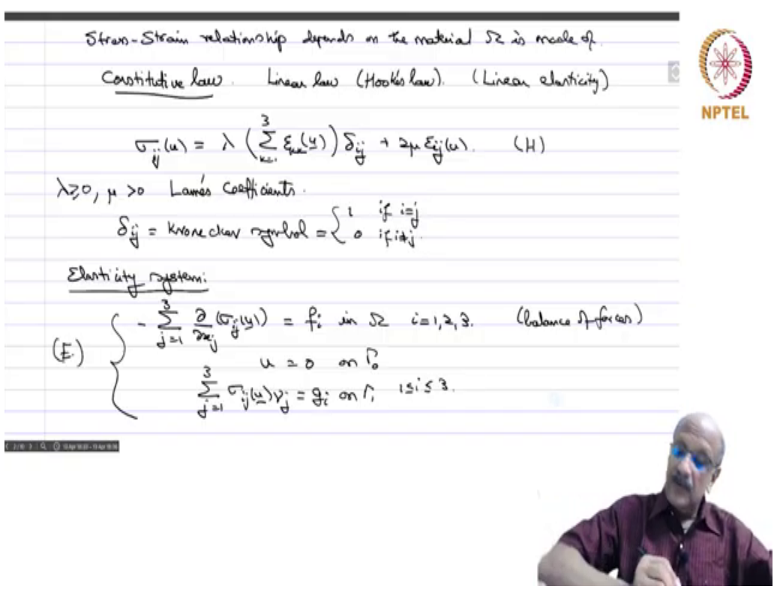

Now, we want to know, what is the material the body is made of, and how the stress and the strain are connected. So, the

stress- strain relationship depends on the material Ω is made of and it is called the constitutive law. In this case we will assume that it is a linear law, which we call **Hooke's law**. So, this is linear elasticity, so there is a linear relationship between the stress and the strain, which we will now write down.

So, we have

$$\sigma_{i,j}(u) = \lambda\left(\sum_{1}^{3} \varepsilon_{i,j}(u)\right)\delta_{i,j} + 2\mu\varepsilon_{i,j}(u).$$

$\lambda \geq 0,\ \mu > 0$ these are called Lama’s coefficients.

And $\delta_{i,j}$ is the Kronecker symbol. So, that is equal to 1, if i equals j and 0 if i is not equal to j.

So, this is the stress-strain relationship. So, lambda could be greater than or equal to 0 and mu strictly positive. These are called the lambda Lama’s coefficients. So, the

elasticity system is the following set of equations

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{i,j}(u)) = f_i \ in \ \ \Omega\ ,\ i = 1, 2, 3.$$

$$u = 0, \quad on\,\Gamma_0$$

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{i,j}(u))\nu_j = g_i \ in\ \Gamma_1, \quad i = 1, 2, 3$$

So, this is the elasticity system. So, *H* we will call *H* as **Hooke's law** and *E*, as the elasticity system.

So, we want to write down a weak formulation. So, we as we know from past experience, we are this is a second order equation and the system of equations. So, we are going to work on a product of $H^1(\Omega)$. And also we have to put the condition u equals 0 on comma naught, we have to impose it, because that is essential boundary condition, namely the Dirichlet boundary condition.

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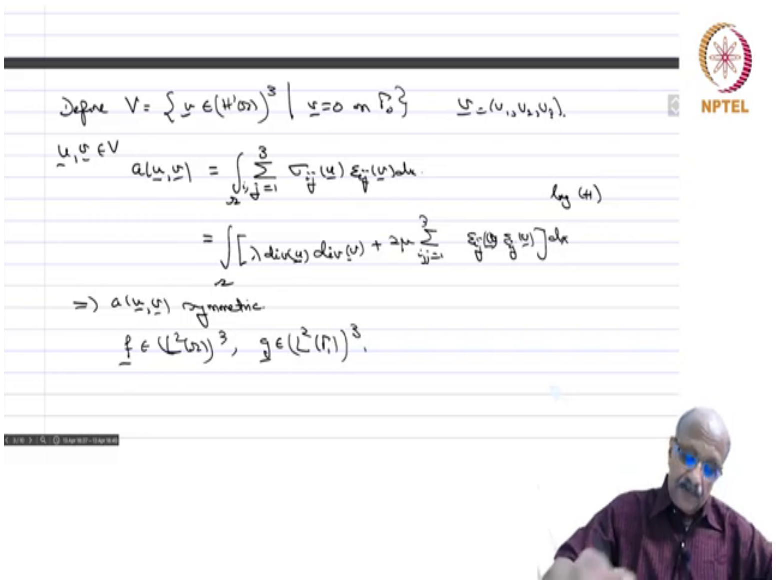

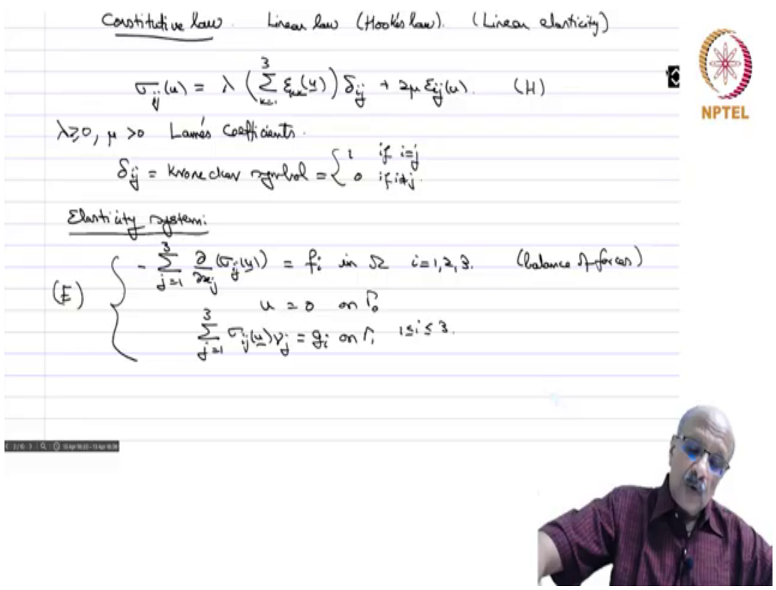

So, we define $V = \left\{ v\_ \in (H^1(\Omega))^3 \mid v\_ = 0 \ on \ \Gamma_0 \right\}$, So, we would be have three components $v\_ = (v_1, v_2, v_3)$, So, we define

$$a(u, v) = \int_{\Omega} \sum_{j=1}^{3} (\sigma_{i,j}(u))\varepsilon_{i,j}(v\_)\, dx$$

Now, we put in Hooke's law. So, by H, so what is Hooke's law, sigma ij if u is a lambda epsilon k, k of u δ_{ij} plus 2 mu ε_{ij} of u. So, 2 mu ε_{ij} of u to give you here i,j equal to 1 to 3. And therefore, the so the second term. So, you will get integral on omega.

So, plus 2 mu epsilon ij of u, ε_{ij} of v d x, and the first term will give you epsilon i only when i equals j you will have $\delta_{ij} = 1$. And therefore, so δ_{ij} is 1 only if i equals a. So, this term will not survive for any other ij and therefore, when i equals j x is on k, k is nothing but, so epsilon k, k is d uk by d xk plus d uk by dx a power half. So, half so that is equal to d uk by d xk sigma over 1 to 3 will give you just the divergence of u.

So, this will give you lambda divergence of u, times divergence of v plus 2 mu sigma over i,j equal to 1 to 3, of $\varepsilon_{ij}(v)$. So, this form implies that a, u, v, is symmetric, the first form does not tell you, but the second form tells you that it is a symmetric by linear form. So, f in $L^2(\Omega)$ power 3 and g belong to $(L^2(\Omega))^3$.

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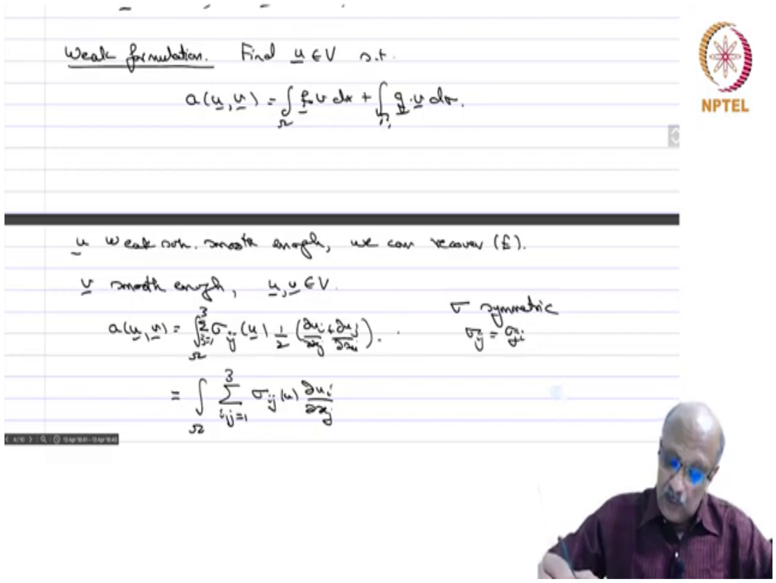

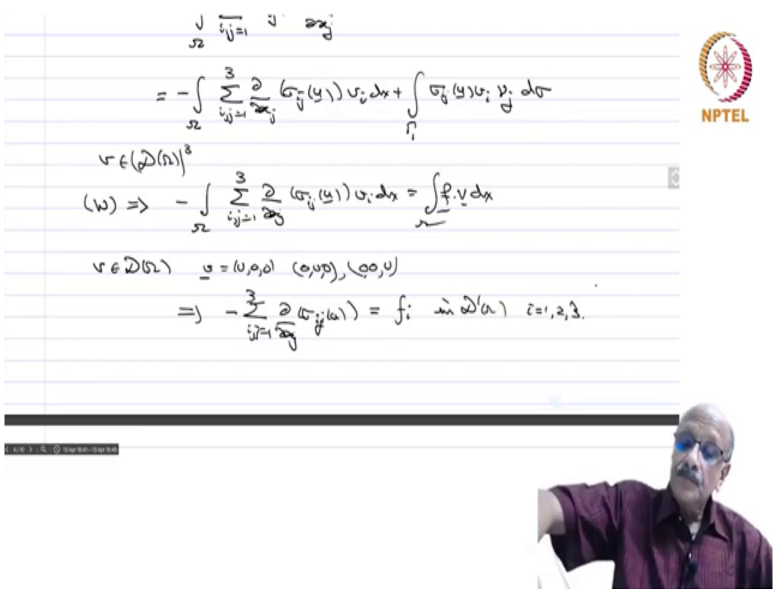

$v \in \mathcal{D}(\Omega)$ $\underline{v} = (v,0,0)$ $(0,v,0)$, $(0,0,v)$

$$\Rightarrow -\sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(u)) = f_i \text{ in } \mathcal{D}'(\Omega) \quad i=1,2,3.$$

$$\underline{u} = 0 \text{ on } \Gamma_0 \quad \because \underline{u} \in V.$$

$$\int_{\Gamma_1} \sigma_{ij}(\underline{u}) v_i \nu_j \, d\sigma = \int_{\Gamma_1} g_i v \, d\sigma \quad \forall \underline{v} \in V.$$

$$\Rightarrow \sigma_{ij}(\underline{u}) \nu_j = g_i \text{ on } \Gamma_1.$$

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Let $\underline{u} = (u_1, u_2, u_3)$ be displacement vector.

Strain tensor. $\varepsilon_{ij}(\underline{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. $1 \le i,j \le 3$.

Stress tensor $(\sigma_{ij}(u))_{i,j=1}^{3}$ (Symmetric tensor

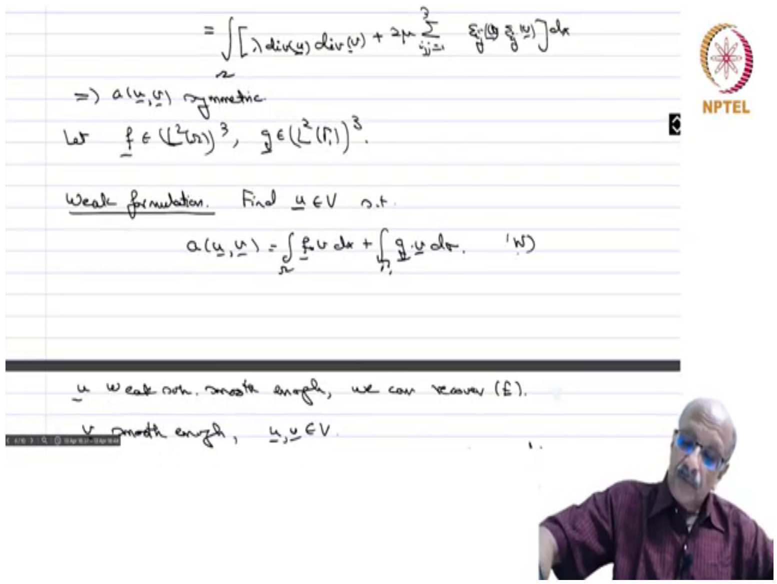
Stress-Strain relationship depends on the material Ω is made of.

Constitutive law. Linear law (Hooke's law). (Linear elasticity)

$$\sigma_{ij}(u) = \lambda \left(\sum_{k=1}^{3} \varepsilon_{kk}(\underline{u})\right)\delta_{ij} + 2\mu\, \varepsilon_{ij}(u). \quad (H)$$

$\lambda \ge 0$, $\mu > 0$ Lamé's coefficients.

δ_{ij} = Kronecker ... 1 if $i=j$



So, weak formulation find $u \in V$, so the trace is v such that a of u, v equals integral Ω f dot v d x plus integral over Ω 1 g dot v, $d\sigma$. So, this is the weak formulation, so you have the linear form, you have the bilinear form, which is symmetric and therefore, we want to know which is.

So, we want to know how we came to this? So, we want to justify this weak formulation. So, if u were a weak solution smooth enough, we can recover E how? So, if v is also smooth enough uv in v. So, you have a, u, v equals integral on Ω sigma ij of u epsilon ij of v is one half of dui by d xj plus d uj by d xi. But sigma is symmetric stress, tensor is always a symmetric tensor. So, I do not know if I said it here, this is symmetric tensor.

So, if you using that fact this is it is easy to check, that this is σ_{ij} equals 1 to 3 and that is equal to integral omega σ_{ij} equals 1 to 3 of sigma ij of u into dy by d xj. So, the half will get absorbed. So, sigma ij equals sigma ji. So, this there will be matching terms here, repeated twice each term will be repeated twice of this kind and so the half will go away and therefore, you will only get this dx.

And that if you use green's theorem, this will be minus integral on omega sigma ij equals 1 to 3 d by d xj of sigma ij of u vi, this is v here, dx and then the integral on the boundary on gamma naught v, naught v is 0. So, you will only have an integral on gamma 1, sigma ij of u and v i and because we turned over the j, derivative you have dj nu sigma.

So, now if you take v is in D omega cubed, then the boundary term will disappear and therefore, you will have. So, the weak solution call this W, W will imply that minus integral on omega sigma ij equals 1 to 3 d by d xj, sigma ij of u, vi d equals integral f dot v over omega dx, there also there will be no boundary integral term.

So, now you successively take $v \in \Omega$ and you take $v = (v, 0, 0)$ $or\ v = (0, v, 0)$ $or\ v = (0, 0, v)$ sub 1 by 1. So, you will get that minus sigma ij equals 1 to 3 d by d x_j of sigma ij u equals f i in $D'(\Omega)$ i equals 1, 2, 3, and then of course you have that u equal to 0 on gamma R, since u belongs to v. So, we only have to look at the other boundary condition. So, then of course you know that, this can be made if everything is smooth enough, this relation in L^2 as you can see from these equations here.

And therefore, going back to this equation here, you will get sigma integral on gamma 1, sigma ij of u vi mu j d sigma equal to 0 for all v in v, not equal to 0 equals integral on gamma 1 of gi dot v d sigma for every v in v and from this you can easily go back and get that sigma ij of u nu j is equal to gi on gamma. So, this is the so you cannot recover the original system by means of from the weak formulation.

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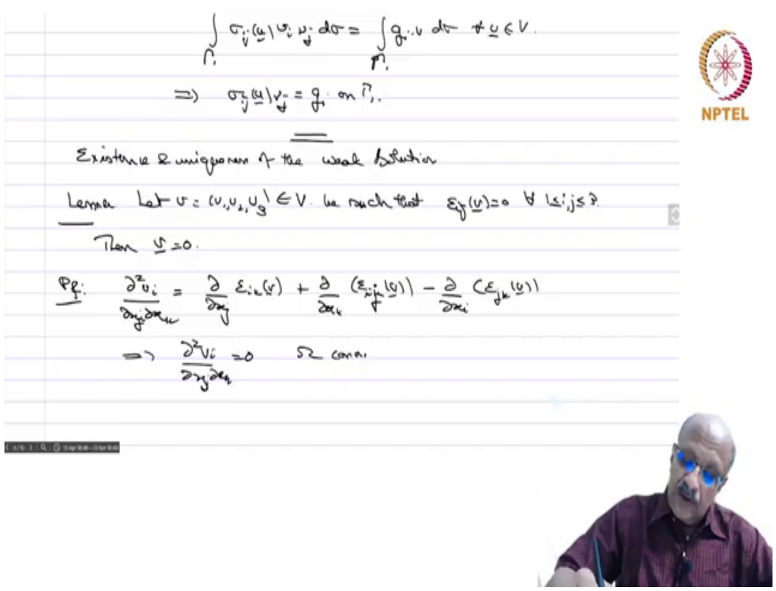

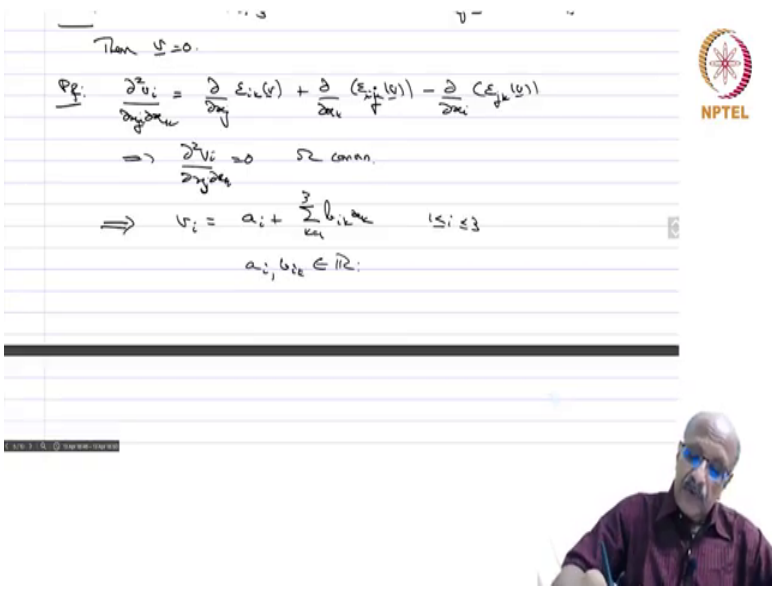

So, now we want to study the existence and uniqueness of the weak solution. So, we want to apply **Lax Milgram lemma**, so we have to show that it is elliptic. So, we now proceed to do that and for that we first need a preliminary

Lemma. Let $v = (v_1, v_2, v_3)$, belong to capital V be such that $\varepsilon_{ij} = 0$ for all $1 \leq i, j \leq 3$. Then $v = 0$.

Proof, we have already seen again in the proof of the **Khan's inequality** that

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}\left(\varepsilon_{ik}(v)\right) + \frac{\partial}{\partial x_k}\left(\varepsilon_{ij}(v)\right) - \frac{\partial}{\partial x_i}\left(\varepsilon_{jk}(v)\right)$$

. So, since all the epsilons are 0, this implies that $\frac{\partial}{\partial x_j}\left(\varepsilon_{ik}(v)\right) = 0,\ i, j, k$..

Now, Ω is connected, so all second derivatives vanish and therefore, this means that v is linear. So, $v_i = a_i + \sum_{k=1}^{3} b_{ik} x_k$.

since all the second derivative vanishes and Ω is connected. So, now so a_i, b_j are all real numbers.

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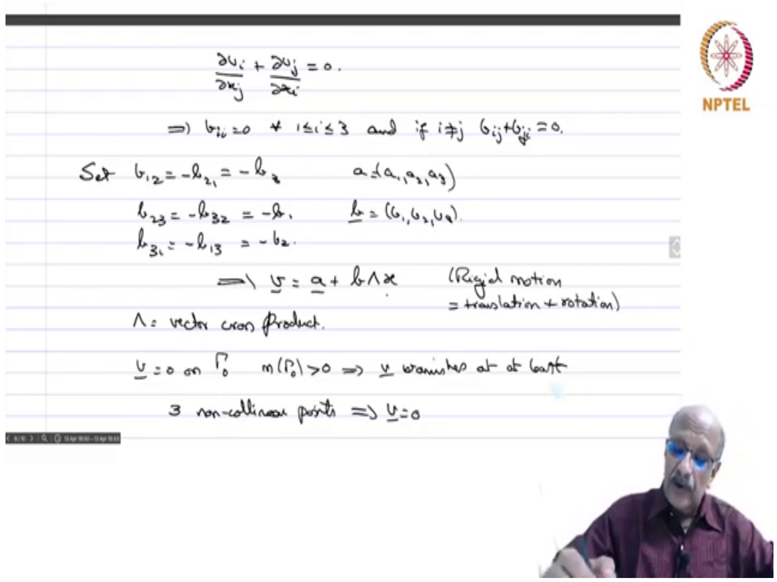

So, now you have that

$$\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = 0,$$

and therefore, you have that $b_{i,i} = 0$,, for all i and if i is not equal to j,

$$b_{i,j} + b_{ji} = 0, \ for\ i \neq j\ and\ b_{ii} = 0.$$

So, it is just algebra, you just plug it in here differentiate these expressions, you will get this.

So, now you put set $b_{12} =- b_{21} =- b_3 =- b_{32} =- b_1$ and

$b_{31} =- b_{13} =- b_2$.

And then $a = (a_1, a_2, a_3)$, and $b = (b_1, b_2, b_3)$, then you will get that

$$v = a + b \wedge x$$

where ∧ is the vector cross product. So, you just have to check out these calculations.

So, this so this means that, so what does this what do we actually prove? So, when you have the there is no strain in the thing that means, there is no deformation at all. So, we are just moving the body rigidly, it is just a rigid translation. So, rigid motion equals translation plus a rotation, that is exactly what each a is the translation part, b cross x vector product is the rotation part. And therefore, if you have a deformation in which there is no strain, that means the deformation tensor is 0, then it has to be a rigid body motion, that is all we have saw. So, v equals a plus b cross x.

And now v equal to 0 on gamma naught, a measure of gamma naught is strictly positive and this implies that v vanishes at, at least 3 non-collinear points and that will imply that v is equal to 0. So, if you put this equal to 0, then you will get you to put this equal 0 at 3 non-colonial points, then it will happen that v equal to 0. So, that is just some linear algebra, which you can check. So, now that we have this lemma,

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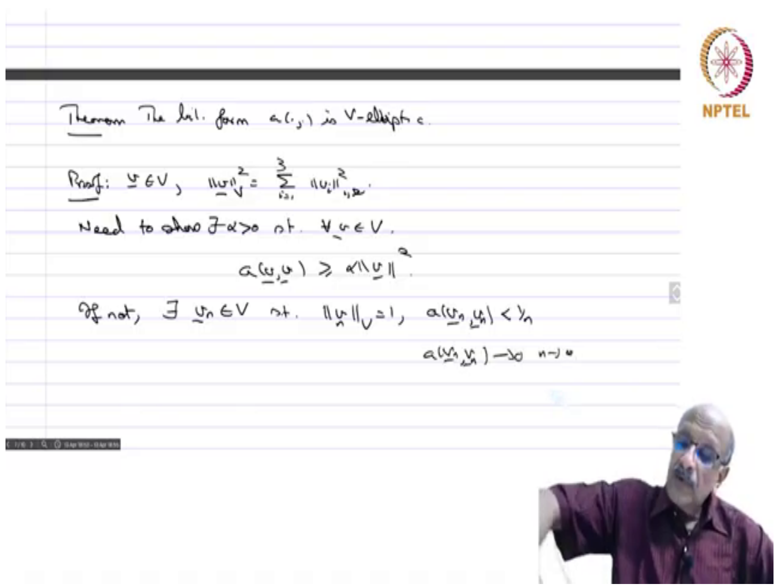

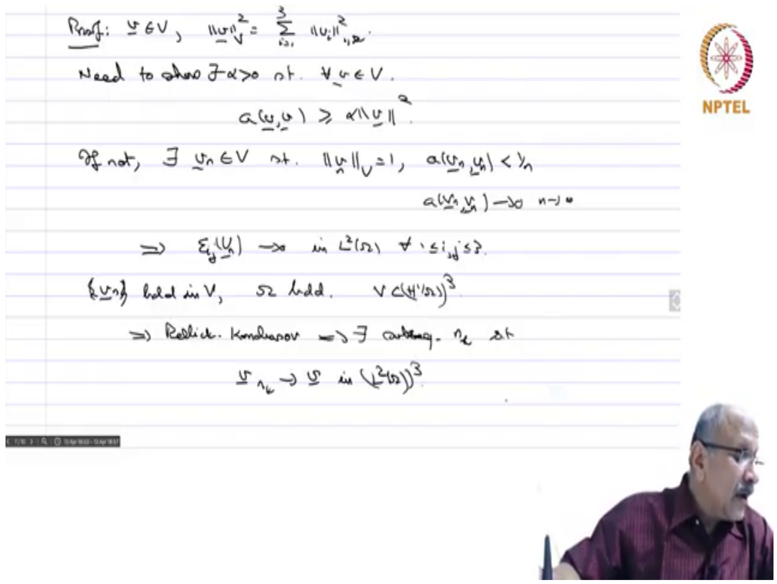

We can now state the following theorem.

Theorem:, The bilinear form $a(,)$ is V elliptic. So,

Proof, so for $v \in V,$ what do you have that

$$||v||_V^{\ 2} = \sum_{i=1}^{3} ||v_i||^2_{\ 1,\Omega.}$$

So, we need to show, so need to show there exists alpha positive, such that for all $v \in V,$, you have

$$a(v, v) \geq \alpha||v||^2_{\ V}$$

So, if not then for every however small an α you take, you will get a contradiction to this, that means there is a vector which is not supplying this one.

So, I can put take $\alpha = 1/n$ and I can also normalize v, we have done this kind of thing there exists $v_n \in V,$ such that $||v_n|| = 1$ and

$a(v_n, v_n) \rightarrow 0$ as n tends to infinity. But what is $a(v_n, v_n)$? If you look at $a(v\ , v\ )$, it is a sum of 2 squares one is divergence u v square plus $\varepsilon_{i,j}(v)$ square.

So, this means that, so this implies that $\varepsilon_{i,j}(v_n) \to 0$, in $L^2(\Omega)$ for all $1 \le i, j \le 3$. So, v_n is bounded in V, and Ω is bounded V is of course contained in $(H^1(\Omega))^3$. And therefore, **Rellich-Kondrachov** implies there exists a subsequence $\{v_{n_k}\}$, such tha tv_{n_k} converges to some $v \in (L^2(\Omega))^3$. Because it is bounded in $(H^1(\Omega))^3$. And therefore, it converges to some $v \in (L^2(\Omega))^3$.

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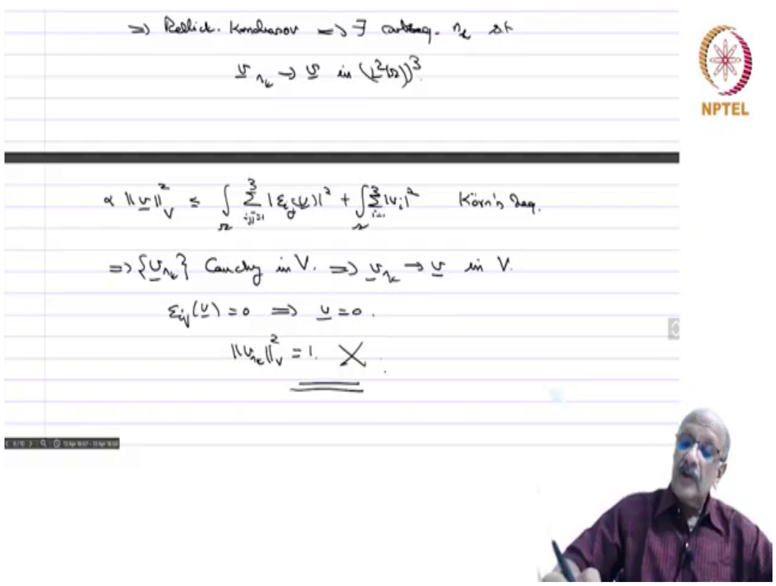

Now, what is we also have **Khan's inequality**. What is Khan's inequality

$$||v||^2_V \le \int_\Omega \sum_{i,j}^{3} |\varepsilon_{i,j}(v)|^2 + \sum_{i}^{3} |v_i|^2$$

This is Khan's inequality.

Now, $\varepsilon_{i,j}(v) \to 0$, and this is convergent in L^2. So, this means that v_{n_k} is Cauchy in V, and this implies that v_{n_k} converges to V, in fact in capital. So, since it already converges in L^2, so it will have to converge in V. But then $\varepsilon_{i,j}(v) = 0$, this implies that $v = 0$, but then $||v|| = 1$, and this is a contradiction. So, we have a contradiction and therefore, you know that this is elliptic.

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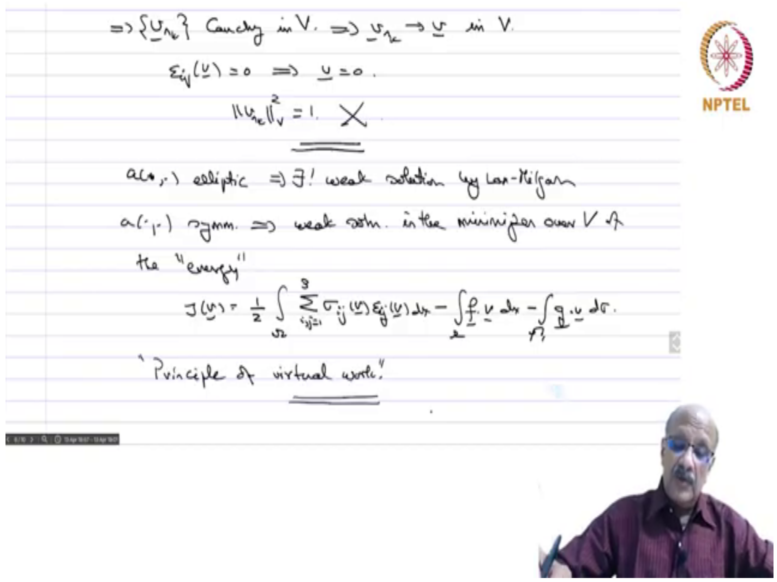

Therefore, a is elliptic, elliptic implies there exists a unique weak solution by **Lax Milgram** and a is symmetric, implies weak solution is the minimizer over V of the strain energy. of the energy

$$J(v) = \frac{1}{2}\int_{\Omega}\sum_{i,j}^{3} \sigma_{i,j}(v)\varepsilon_{i,j}(v)\,dx - \left(\int_{\Omega} f \cdot v\ dx + \int_{\Gamma_1} g \cdot v\ d\sigma(x)\right).$$

And this is the energy of the system.

And so for equilibrium you need that it has to be minimized and that is exactly called the principle of virtual work namely, the elasticity system, solution is nothing but the minimizer of the energy, so this thing. So, this is the existence- uniqueness and the weak formulation for the system of elasticity.