**Sobolev Spaces and Partial Differential Equations**
**Professor. S Kesavan**
**Department of Mathematics**
**Institute of Mathematical Sciences**
**Lecture 57**

# The Biharmonic operator

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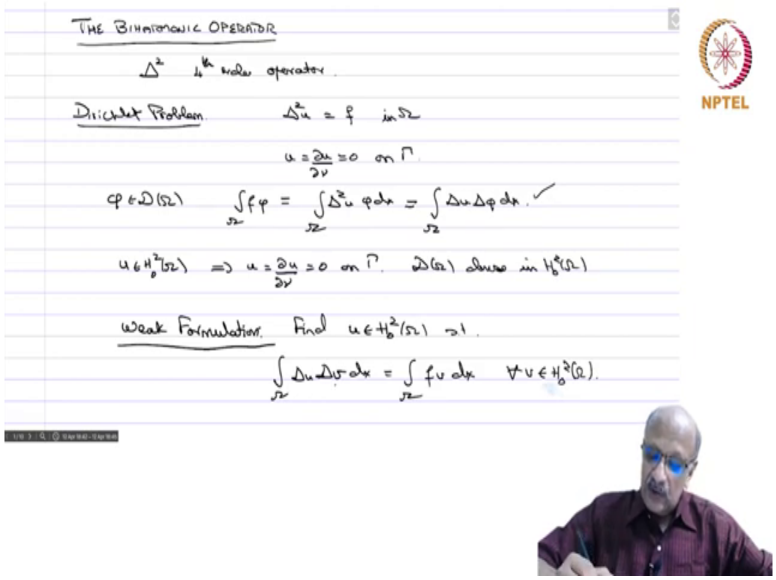


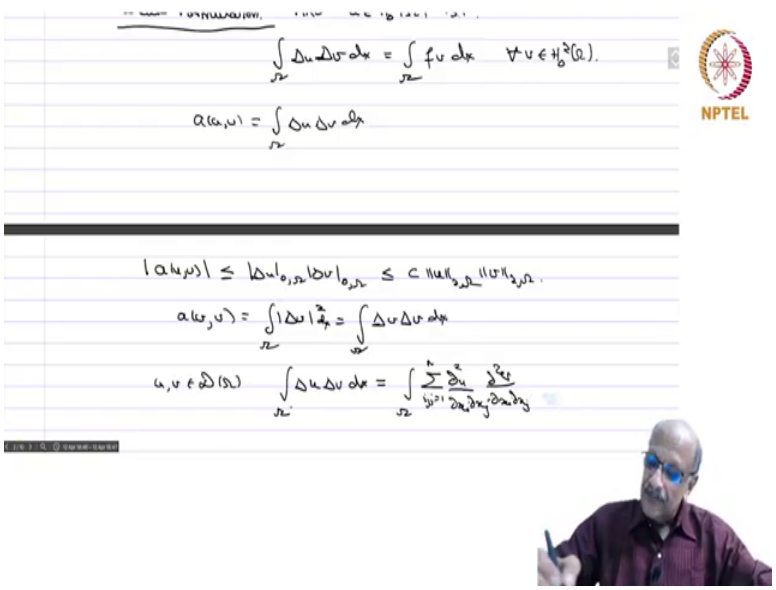


**Biharmonic operator:**

Up to now we were looking at examples of second order boundary value problems. So, today we look at a fourth order problem. So, the **Biharmonic operator**, so the Biharmonic operator is $\Delta^2$ square and therefore it is a fourth order operator, because delta, you apply $\Delta$ of $\Delta$ , $\Delta$ is a second order operator, you apply it once again. And therefore, you have a fourth order operator.

So, we are going to look for **Dirichlet problem.** So, you have

$$\Delta^2 u = f \quad in \ \ \Omega$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad on\, \Gamma.$$

So, here $\frac{\partial u}{\partial \nu}$ comes in the Dirichlet problem itself, because for the fourth order operator you need two boundary conditions. So, you might have had this experience in the differential equations also, when you are dealing with boundary were two point boundary value problems.

So, for second order operator, you needed one boundary condition, for the fourth order operator, you will need two boundary conditions to fix the solution uniquely. So, this is called the Dirichlet problem for the Biharmonic operator.

So, if $\varphi \in D(\Omega)$,then you have

$$\int_\Omega \Delta u \Delta\varphi \ dx = \int_\Omega f\varphi\, dx$$

So, now you can transfer the derivatives slowly and so by repeated application of green's theorem, because of the fact that u and $\frac{\partial u}{\partial \nu}$ are and f phi is in d omega. So, there is no problem, there are no boundary derivatives, or f phi, or a anything. So, you get Laplacian u times Laplacian phi dx.

So, this is the thing and now if you look at $u \in H^2_0(\Omega)$. So, this implies what? That $u = \frac{\partial u}{\partial \nu} = 0 \ \ on\, \Gamma$. And therefore, the boundary conditions are automatically satisfied. And you also know that $D(\Omega)$ dense in $H^2_0(\Omega)$. Now, the both sides of this equation here, are continuous with respect to the $H^2_0(\Omega)$, $H^2$ $(\Omega)$ norm and therefore we have the weak formulation, find $u \in H^2_0(\Omega)$, such that

$$\int_\Omega \Delta u \Delta v \; dx = \int_\Omega fv \, dx \; , \quad \forall v \in H^2_0(\Omega)$$

So, we have here the space is $H^2_0(\Omega)$, which automatically ensures that the boundary conditions are satisfied. And then the linear form is integral omega fv dx, which is continuous with respect to the $L^2$ norm. And therefore, in the $H^2$ norm and so on and so forth. And now and the bilinear form is integral delta u, delta v. So, this is of course obviously continuous. So,

$$a(u, v) = \int_\Omega \Delta u \Delta \varphi \; dx$$

And therefore, you have that

$$|a(u, v)| = |\Delta u|_{0,\Omega} |\Delta \varphi|_{0,\Omega} \leq C ||u||_{2,\Omega} ||v||_{2,\Omega}.$$

So, this now you have that

$$a(v, v) = \int_\Omega |\Delta v|^2 \; dx$$

So, now we have seen, I have given this probably as an assignment. So, if you have $u, \; v \in H^2_0(\Omega)$ and you look at $\int_\Omega \Delta u \Delta v \; dx$.

So, then you can so this is what? This is now slowly you can, you can transfer any derivative anywhere to either side, because everything is in $D(\Omega)$, there will be no boundary terms, and this will, then turn out to be

$$\int_\Omega \Delta u \Delta v \; dx = \int_\Omega \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_i \partial x_j} \; dx.$$

So, just transfer the derivatives one by one to from here to here, and then you can prove this. So, this is just a very you try it for instance with N=2. And then you can generalize it for linear.

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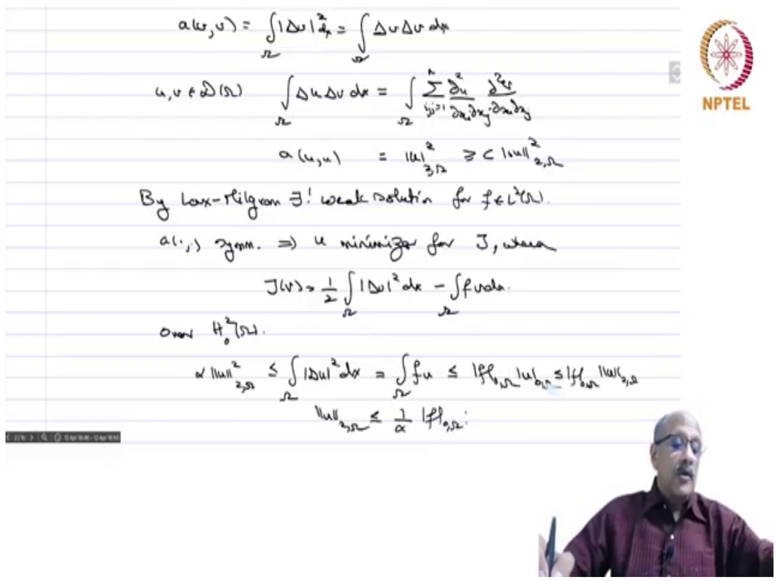


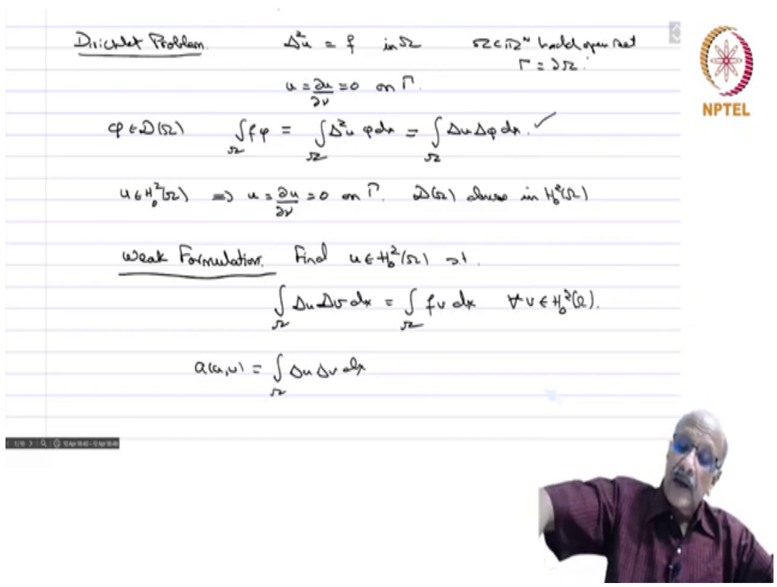


And therefore you have that this is equal to mod u square to, so delta a u, u is therefore equal to mod u square 2 omega and you know we have omega is a bounded open set. So, omega in $\mathbb{R}^N$, bounded open set and gamma equals $D(\Omega)$. And therefore you have Poincare inequality which tells you that this is less than or equal to mod u square is greater than equal to c times mod norm u square, 2 Ω, in fact mod u square 2 Ω itself is a norm. And therefore, you have the ellipticity.

Therefore, by **Lax-Milgram**, there exists a unique weak solution for $f \in L^2(\Omega)$ and a is symmetric implies u is minimizer for $J$, where

$$J(v) = \frac{1}{2}\int_{\Omega} |\Delta u|^2\, dx - \int_{\Omega} f v\ dx, \quad v \in H^2_0(\Omega).$$

So, over $H^2_0(\Omega)$ And further you also have continuity, because you have that

$$\alpha||u||^2_{0,\Omega} \leq \int_{\Omega} |\Delta u|^2\, dx = \int_{\Omega} fu\, dx \leq |f|_{0,\Omega}|u|_{0,\Omega} \leq |f|_{0,\Omega}||u||_{2,\Omega}.$$

So, by the standard you have

$$||u||_{2,\Omega} \leq 1/\alpha|f|_{0,\Omega}.$$

So, you have continuous dependence on the data. So, this is the thing.

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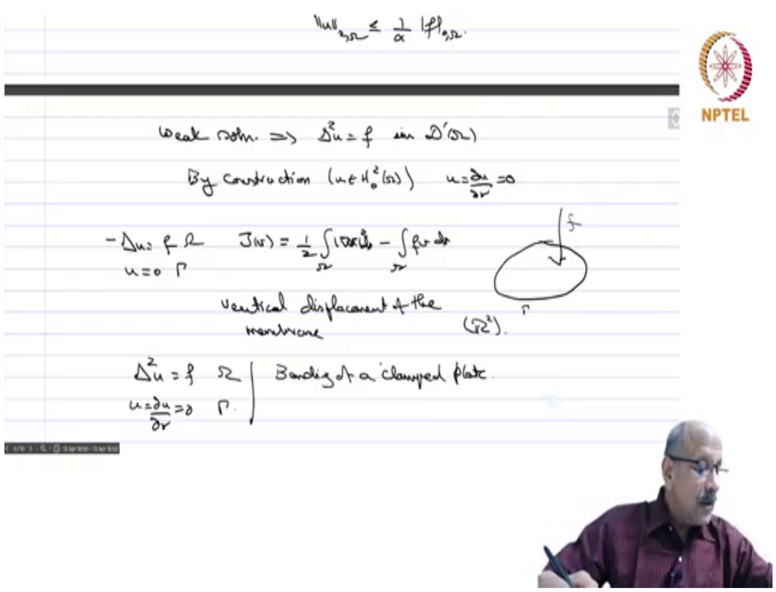


And as usual you can now convince yourself, that if you have a weak solution implies delta square u equals f in d prime of omega, and of course by construction, namely you have the u belongs to $H^2_0(\Omega)$. And therefore, you have u equals du by d nu equal to 0. So, then if you look at regularity theorems, then we can say that whether it is a classical solution, or not and of course a classical solution is the thing. So, now if you have the Laplace equation minus $-\ \Delta u = f$, which minimizes the with the associated and strain energy as it is called one half integral mod grad u square minus integral over omega f, f grad v square fv dx.

So, this you think of omega as a membrane, which is stretched over the thing and it is fixed to the boundary gamma and is acted upon by a vertical force, whose density is given by f. And then the Laplacian, so u gives you the vertical displacement of the membrane, I am talking of all $\mathbb{R}^2$.

Now, similarly if you have

$$\Delta\ u = f \quad in\ \ \Omega\, u = 0 \ \ on\, \Gamma,$$

so this is nothing but the bending of a clamped plate. So, you assume that you have very thin plate, which of course is a three-dimensional body. So, approximated by means of its middle surface. So, that will be a two-dimensional body and then you have a force, which is acted on this. And then clamping the plate along the boundary, means that you cannot even the not only that it does not move, but it does not have any lateral, rotational movements etcetera.

And therefore, that is called clamping. So, when you clamp a plate and then you act it upon by vertical force, then you have the bending vertical displacement is given by this equation. So, that is it. So, now we have of course given you a weak formulation, which is in $H^2_0(\Omega)$. Now, generally from a numerical analysis point of view, if you want to approximate solutions, especially using methods like the finite element method etcetera.

Then $H^2$ is a difficult space, because the finite element approximations are very cumbersome and very complicated, whereas it is much better if you work with $H^1$. So, we try to give you a mixed formulation, a different formulation which does not depend on the **Lax-Milgram** and therefore but it is a of a different kind. So, you increase the number of unknowns and then you see.

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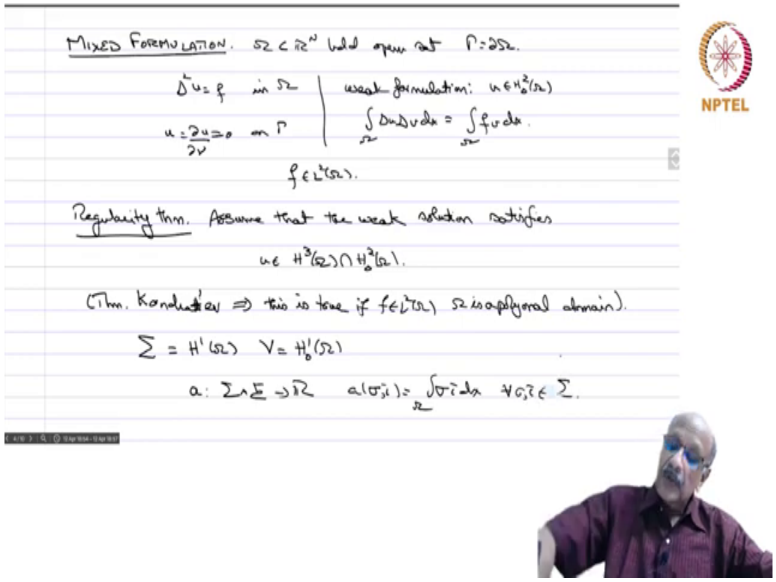


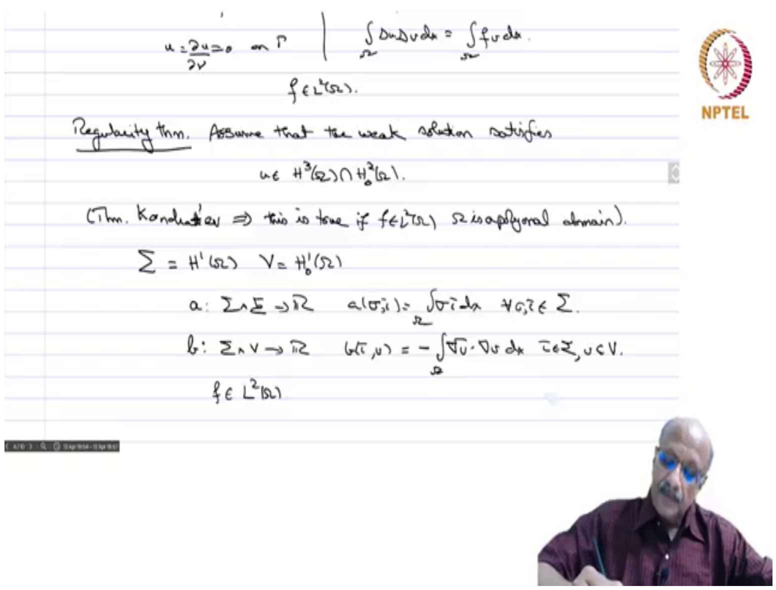


So, let us, so let us give you an example of a **mixed formulation**. So, u delta square u equal to, so $\Omega \subset \mathbb{R}^N$, bounded open set and $\Gamma = \partial\Omega$, this is

$$\Delta^2 u = f \quad in \quad \Omega$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad on\, \Gamma.$$

And then you have the weak formulation u in $H^2_0(\Omega)$, such that

$$\int_{\Omega} \Delta u \Delta v \ dx = \int_{\Omega} f v \, dx \ , \quad \forall v \in H^2_0(\Omega)$$

So, where $f \in L^2(\Omega)$.

So, now we so Regularity theorem, assume that the weak solution satisfies $u \in H^3(\Omega) \cap H^2_0(\Omega)$.

So, there is a theorem of Kondratieff which says that, this is true, if $f \in L^2(\Omega)$ and omega is of class is a polygonal domain. And obviously for much more also, if it is a smoother domain, it will, domain this is about the minimal hypothesis for this. So, this is a theorem of Kondratieff which says this. So, this is not an unreasonable hypothesis.

So, now I am going to say sigma is $H^1_0(\Omega)$ and v equals $H^1_0(\Omega)$. And we have a from sigma cross sigma to R. So, a of sigma tau equals integral sigma tau dx just the $L^2$ inner product, for all sigma tau in sigma. And then b is from sigma cross v into R, and b sigma v, of beta v, equals minus integral of omega grad tau, dot grad v dx. So, tau in sigma and v in v. So, and then you have of course that f is in $L^2(\Omega)$.

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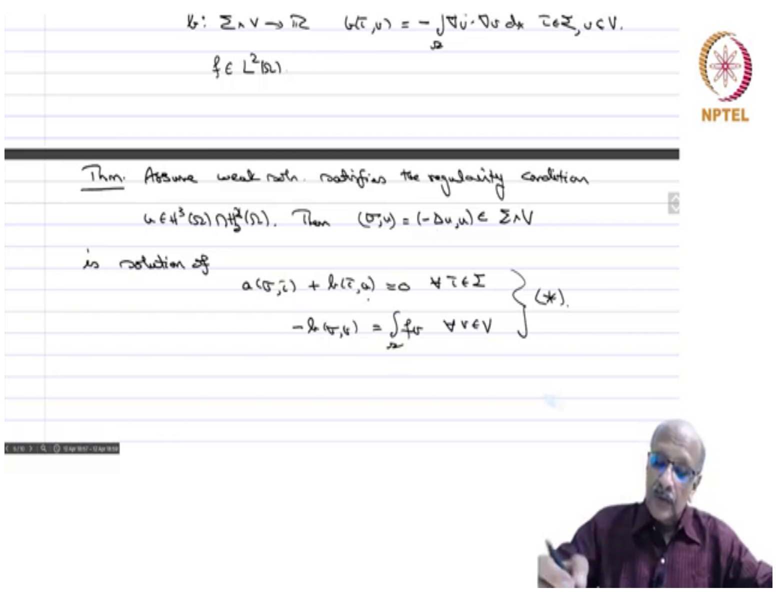

**Theorem:**

Assume weak solution satisfies, the regularity condition namely $u \in H^3(\Omega) \cap H^2_0(\Omega)$. Then

$(\sigma, u) = (-\Delta u, u) \in \sum \times V$.

So, this, so u is in $H^3$, so $\Delta u \in H^1$, u is in $H^2_0$. So, it is in $H^1_0$, so this belongs to $\sum \times V$, is solution of

$a(\sigma, \tau) + b(\tau, u) = 0, \ \forall \tau \in \sum,$

$-b(\sigma, v) = \int_\Omega fv\, dx, \quad \forall v \in V.$

So, you see we now have a system of equations with two unknowns. So, sigma is an unknown and v is an unknown. So, we have increased the number of unknowns, increase the size of the equation, but on the other hand we are working with simplest spaces namely $H^1(\Omega)$ and $H^1_0(\Omega)$, which for approximation purposes is the same.

Also, this very often, it is not that u, which is interesting if you are for instance interested in fluid mechanic problems, where the Biharmonic operator occurs naturally, in what is called the stream function vertex verticity formulation. Then we are interested in Laplacian u directly. So, instead of solving for u in $H^2_0(\Omega)$ and then differentiating it twice. So, you may you directly try to get an approximation of the Laplacian. So, that is why these mixed formulations, where you introduce a new unknown is sometimes useful.

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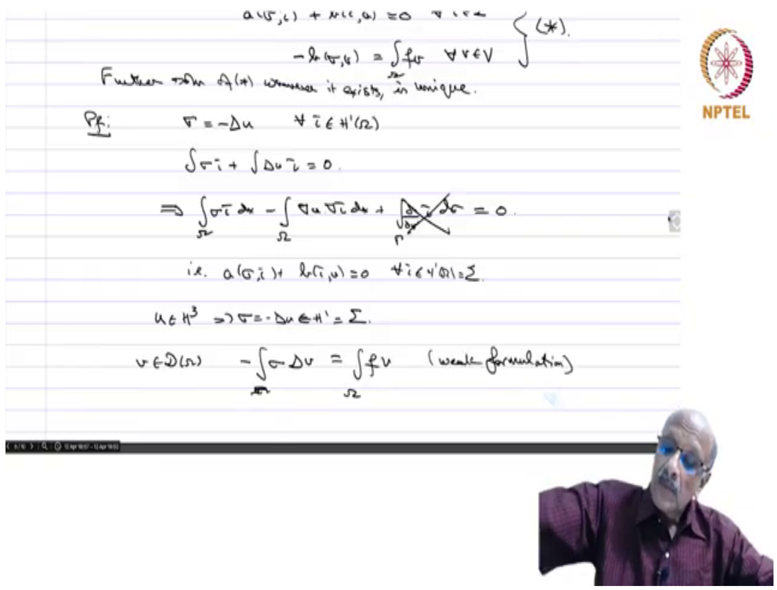


So, further solution of star, whenever it exists is unique.

**Proof,**

So we have $\sum = -\ \Delta u$ and therefore for every $\tau \in H^1\ (\Omega)$, which automatically in $L^2\ (\Omega)$, you have sigma tau plus integral Laplacian u tau equal to 0. And therefore, so this implies, that integral on omega 1 sigma tau dx plus or rather I am going to apply Green's theorem minus integral over u, you have that grad u, grad tau, dx, then plus integral du by d nu tau d sigma over gamma. But this term goes to 0, because du by d nu is equal to 0.

And therefore, you have this is equal to 0, that is a sigma tau plus b tau u equal to 0, for every $\tau \in H^1\ (\Omega)$, which is equal to sigma. So, $\tau \in H^3\ (\Omega)$ in place of course sigma equals minus Laplacian $\tau \in H^3\ (\Omega)$, which is capital sigma. So, that is what.

Now, you take v in d omega, so you have minus integral omega m, sigma delta v equals integral fv, because you know the delta u, delta v, the equals f v, that is sigma equals minus delta u and therefore you have this from the weak formulation. So, this is the weak formulation.

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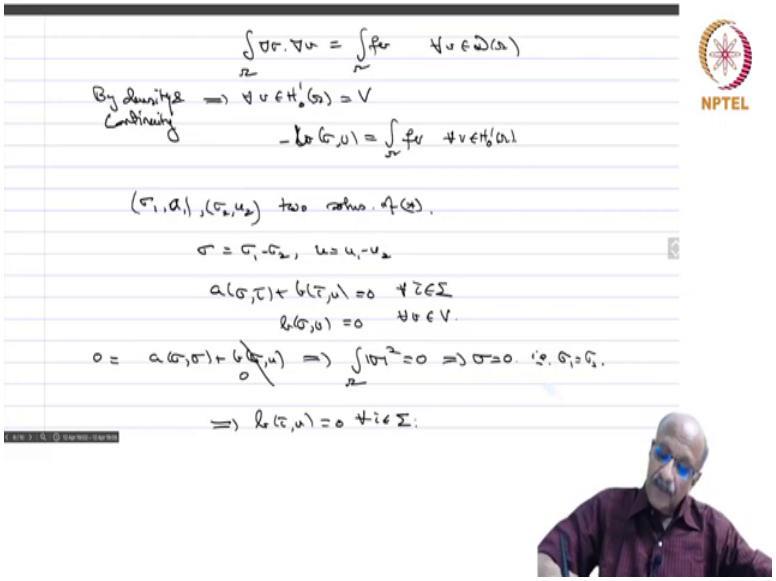

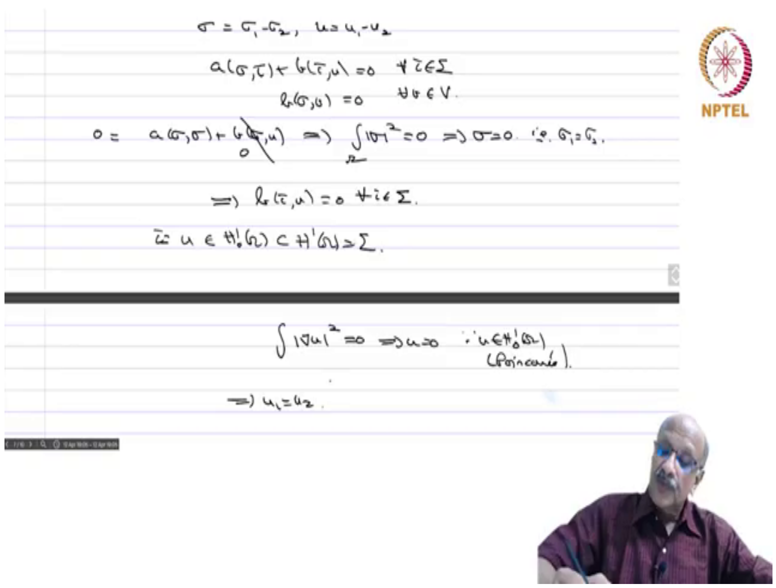

And now once again you approximate by means, once again you apply Green's theorem, you get integral grad sigma times grad v over omega, there will be no boundary term equals fv. And this is true for all v in d omega. Now, both sides of this equation are continuous in the H 1 0 norm and therefore this implies for all v in. So, by density and continuity for all $v \in H^1_0(\Omega)$ equal to v.

So, you have that minus of b sigma v equals integral omega fv, for every $v \in H^1_0(\Omega)$. So, you have that the this pair satisfies this system. So, now we want to show that the solution whenever it exists is unique. So, if sigma 1, see $u_1$, sigma 2 $u_2$ two solutions of star. Then what does this mean? This means that if sigma equals sigma 1 minus sigma 2, u equals $u_1 - u_2$. Then you have a sigma tau plus b tau u equal to 0, for all tau in sigma and then b sigma v equal to 0, for every b in v.

So, because you just subtracted the two equations the fv got cancelled and this is. So, if you now so if you put tau equal to sigma in the first equation you get a sigma, sigma plus b sigma u that is 0. But then this is already 0 by the second equation and this implies that $\int_\Omega |\sigma|^2\, dx = 0$, that implies that $\sigma = 0$, that is $\sigma_1 = \sigma_2$.

Now, $\sigma = 0$, so this now implies that b tau u equal to 0 for all tau in sigma and then you take tau equals u, because $u \in H^1_0(\Omega)$. So, it is in H 1 belongs to $u \in H^1(\Omega)$, which is of course contained in $H^1(\Omega)$ equal to sigma. So, if you do that, then you get integral mod grad u square equal to 0 and that implies implicit u equal to 0, since u equals H in $H^1_0(\Omega)$ and you have Poincare inequality.

And therefore, you have that $u_1 = u_2$ also. So, this proves the uniqueness of the theorem. So, this completely proves the theorem.

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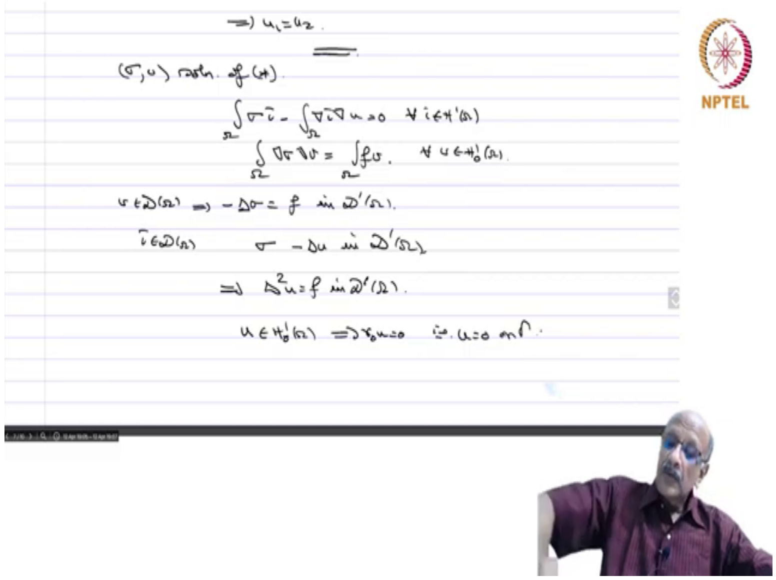


So, now let us look at this equation again, so suppose I have $(\sigma, u)$ solution of star and see how it we can recover these equations. So,

$$\int_\Omega \sigma\tau \ dx - \int_\Omega \nabla\tau \cdot \nabla u \ dx = 0, \ \ \tau \in H^1 \ (\Omega).$$

$$\int_\Omega \nabla\sigma \cdot \nabla v \ dx = \int_\Omega f v \ dx, \ \ \ v \in H^1{}_0(\Omega).$$

So, if you use $v \in D(\Omega)$, then this implies that

$$-\ \Delta\sigma = f \ \ in\, D^{'}.$$

And the first equation, if you use $\tau \in D(\Omega)$, then you get that

$$\sigma - \Delta u \in D^{'}(\Omega),$$

And therefore, from these two together you get $\Delta^2 u = f \in D^{'}(\Omega)$.

So, in the sense of distributions it satisfies the thing and you also have u is in $H^1{}_0(\Omega)$. So, in place $\gamma_0(u) = 0$, that is $u = 0 \ on \ \Gamma$. So, we now only have to recover the other boundary condition.

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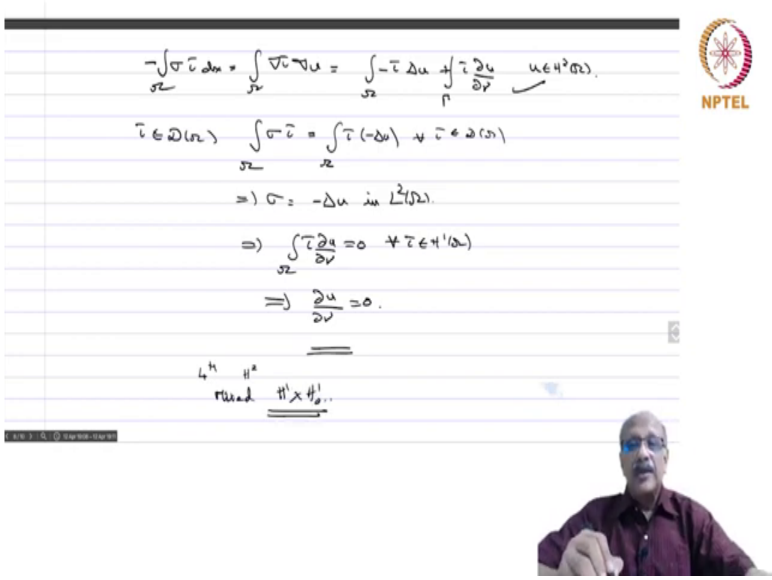

So, you have that, you have that

$$-\int_\Omega \sigma\tau\, dx = \int_\Omega \nabla\sigma \cdot \nabla u\, dx = \int_\Omega -\tau\Delta u\, dx + \int_\Omega \tau\frac{\partial u}{\partial \nu}, \quad u \in H^2(\Omega)$$

which is reasonable to assume, because $\tau\Delta u \in L^2$. So, you assume that $u \in H^2(\Omega)$.

And therefore, now if you take $\tau \in D(\Omega)$, you get that

$$-\int_\Omega \sigma\tau\, dx = \int_\Omega \tau(-\Delta u)\, dx, \ \forall\ \tau \in D(\Omega)$$

and therefore this is now completely a $L^2$ situation and therefore this implies that

$$\sigma = -\Delta u, \quad in\ L^2(\Omega).$$

And then going back to this equation here, this will imply that

$$\int_\Omega \tau\frac{\partial u}{\partial \nu} = 0, \ \forall\, u \in H^1(\Omega).$$

And then we know we have already seen this in the Neumann problem, this implies that

$$\int_{\Omega} \frac{\partial u}{\partial \nu} = 0, \ \forall\, u \in H^{1}(\Omega).$$

And therefore, you have u satisfies both the boundary conditions and this in satisfies the differential equation in the distribution sense. So, we can recover the original problem from this weak formulation. So, this is called a mixed formulation, because you have two kinds of unknowns, which are not only the primary unknown u, but you have introduced another unknown sigma, which is this thing and from so for a fourth order problem, generally we work with $H^{2}$.

But now in a mixed formulation, you only work with $H^{1} \times H^{1}_{0}$, which is a big improvement from the numerical analysis point of view as I already said. So, next time we will see a system of equations which occurs in elasticity.