Sobolev Spaces and Partial Differential Equations
Professor. S Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Neumann problems

We were looking at the examples of second order elliptic operators, and we were looking at Dirichlet problems, that means the on the boundary the function is prescribed.

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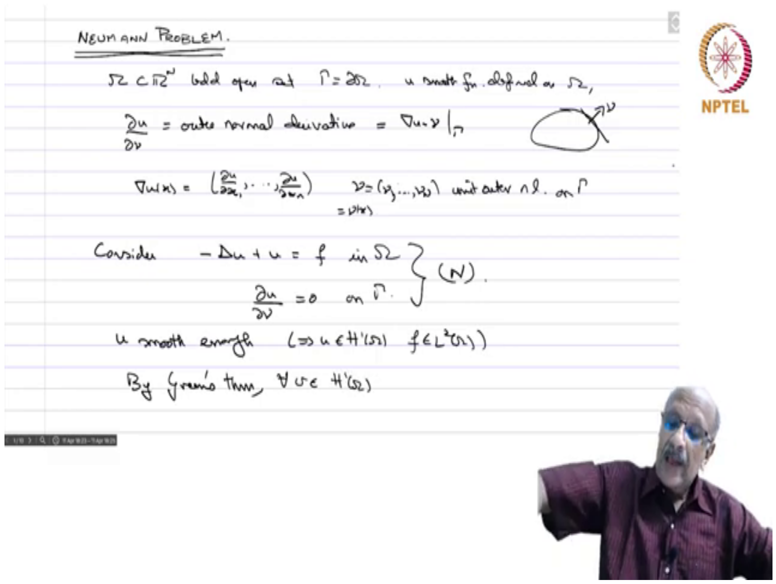

Neumann problem:

Now, we want to look at what is called the Neumann problem. So, $\Omega \subset \mathbb{R}^N$, bounded open set and $\Gamma = \partial\Omega$. So, if u is a smooth function defined on omega, then we denote by

$$\frac{\partial u}{\partial \gamma} = outer\ normal\ derivative = \nabla u \cdot \gamma|_{\Gamma}$$

$\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N})$ and $\gamma = (\gamma_1, ..., \gamma_N)$ outer normal to Γ

So, you have the domain, it will depend on x of course, so at each point you have a tangent and then you have the unit out of normal, which is like that. So, this is again dependent on x, nu equals nu of x. So, now we want to look at the following problem, so we consider

$$- \Delta u + u = f \quad in \ \Omega$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad on \ \Gamma$$

So, this is the Neumann problem, we will call it N.

So, if you have, if u is smooth enough and then so which will imply that $u \in H^1_0(\Omega)$ and $f \in L^2(\Omega)$. And so if you multiply by Green's theorem, for every $v \in H^1(\Omega)$, we have the following.

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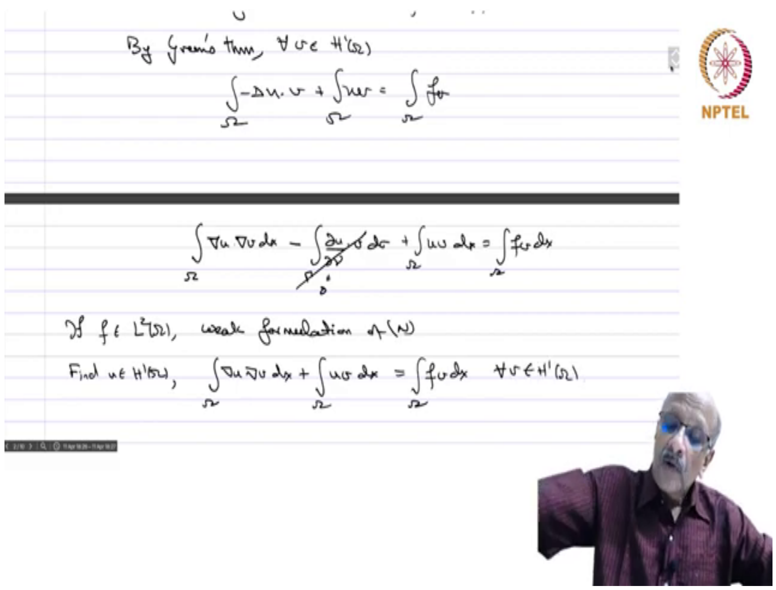

So,

$$\int_\Omega - \Delta u \cdot v + \int_\Omega uv = \int_\Omega fv.$$

So, if you integrate the first term by means of Green's formula, you will have

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial \gamma} \cdot v + \int_\Omega uv = \int_\Omega fv. \Rightarrow \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int_\Omega fv.$$

Since $\int_{\Gamma} \frac{\partial u}{\partial \gamma} \cdot v = 0.$

So, we have if $f \in L^2(\Omega)$, then weak formulation of N, this is nothing but integral omega find $u \in H^1 (\Omega)$, such that

$$\int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} uv = \int_{\Omega} fv, \quad \forall v \in H^1(\Omega).$$

This is called the **weak formulation** of the Neumann problem.

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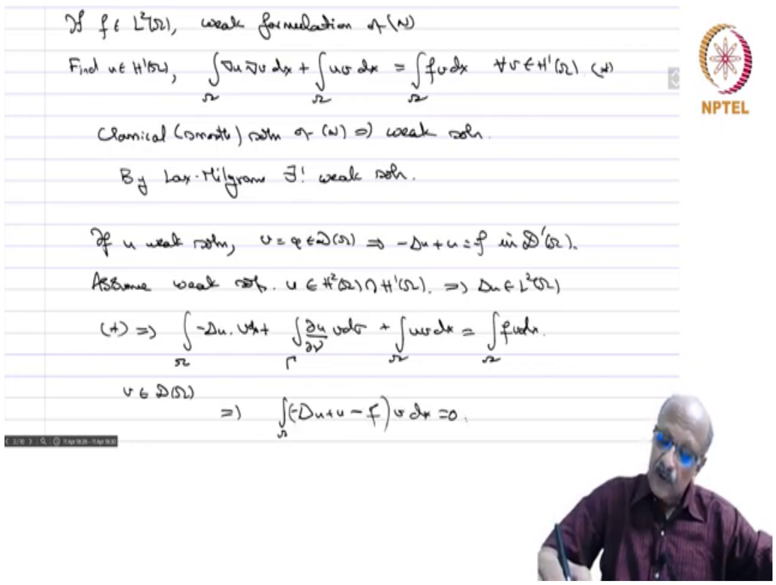

So, of course if you have classical, or a smooth solution of N implies weak solution, as we have just seen. So, weak formulation here, therefore the basic space is $H^1(\Omega)$ and then the linear form is integral omega fv dx and the bilinear form is integral grad u grad v plus integral u v, which is nothing but the inner product in $H^1(\Omega)$. And therefore, by **Lax-Milgram**, there exists a unique weak solution, that is immediate because you just have the inner product here, so that is definitely elliptic. So, we have nothing to prove.

So, on the other hand, if you, if u is a weak solution taking $\varphi \in D(\Omega) \Rightarrow -\ \Delta u + u = f\ in\ D(\Omega)$ in the sense of distributions. So, as distributions you will have that this is certainly true. So, now let us assume, weak solution $H^1(\Omega) \cap H^2(\Omega)$ So, let us assume that you have some additional smoothness, which again as I have repeatedly said is by means of a regularity theorem and generally for reasonable domains, this will always be true and therefore we will see this.

So, now we get back by going back retracing the integration by parts formula. So, star implies minus Laplacian u times v, because so here this means that Laplacian of u belongs to L^2, anyway it is in the sense of distributions and anyway you have this, minus Laplacian u, v integral omega plus integral $\frac{\partial u}{\partial \gamma}$, v d sigma on gamma plus integral on omega uv dx equals integral on omega fv dx.

So, now we take v in, so we take v and $D(\Omega)$, then of course this term will vanish. So, this will imply that minus integral omega, minus Laplacian u plus u minus f times v dx equal to 0.

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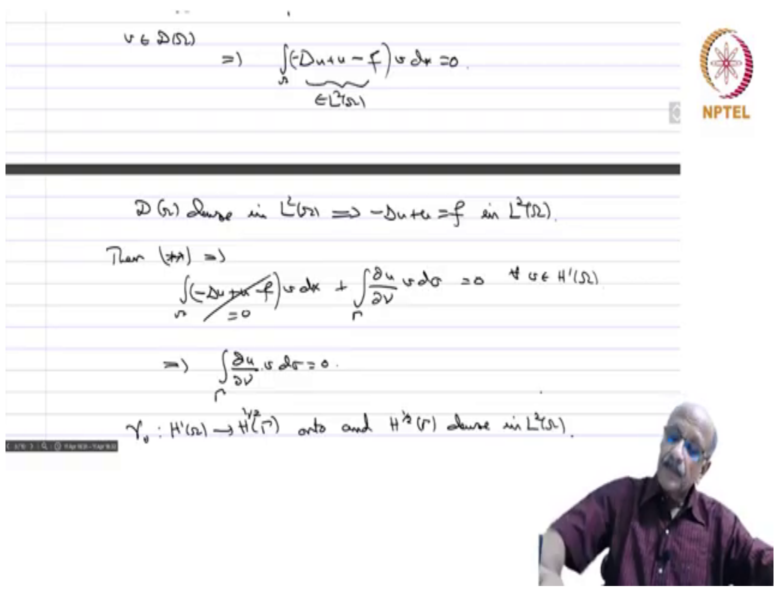

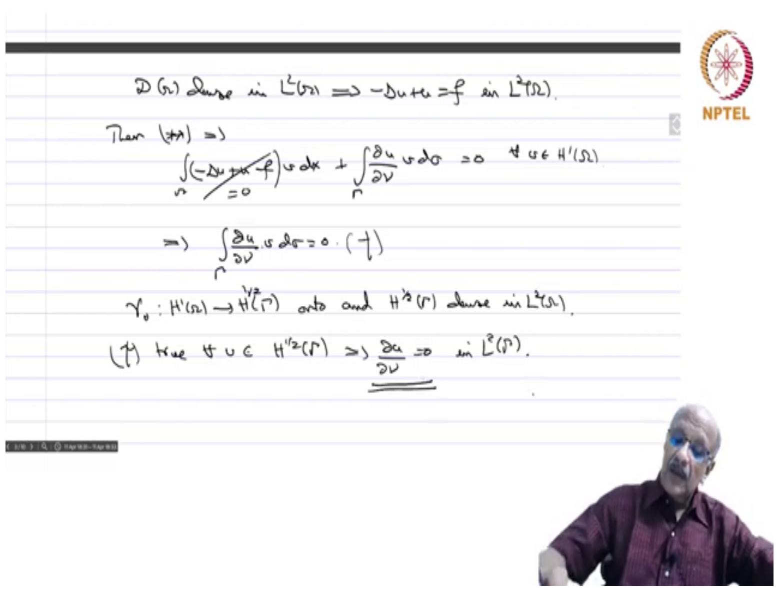

Now, we know that $-\ \Delta u, u, f$ are all in $L^{2}(\Omega)$ and you have $D(\Omega)$ dense in $L^{2}(\Omega)$. So, this implies that minus Laplacian u plus u equal to f in $L^{2}(\Omega)$, so as function of $L^{2}(\Omega)$, this is true.

So, now if you go back to star, so call this double star, then double star implies, that integral omega minus Laplacian u plus u minus f into v dx plus integral on gamma du by d nu, v d sigma equal to 0 for every $u \in H^1(\Omega)$.

But then minus Laplacian u plus u minus f is a L^2 function, which is 0. So, equal to 0 and therefore this implies that integral on gamma du by d nu times v d sigma equal to 0. Now, $H^1(\Omega)$, $\gamma_0: H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$,

$H^{1/2}(\Gamma)$ is dense in $L^2(\Gamma)$.

Therefore, this shows that, so call this dagger, so dagger true for all $v \in H^{1/2}(\Gamma)$, because of this ontoness of gamma naught. And therefore, you have this implies that du by d nu equal to 0 in $L^2(\Gamma)$. So, we show that, if u is in H^2, this satisfies the boundary condition.

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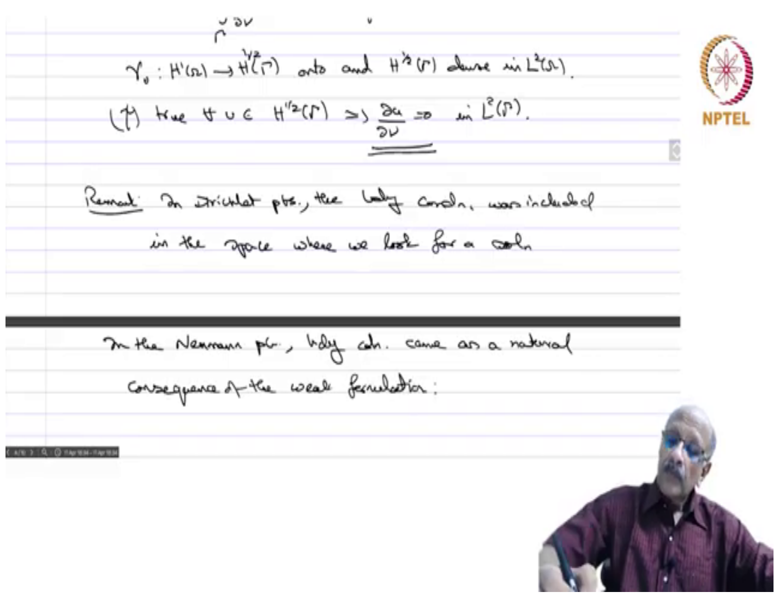

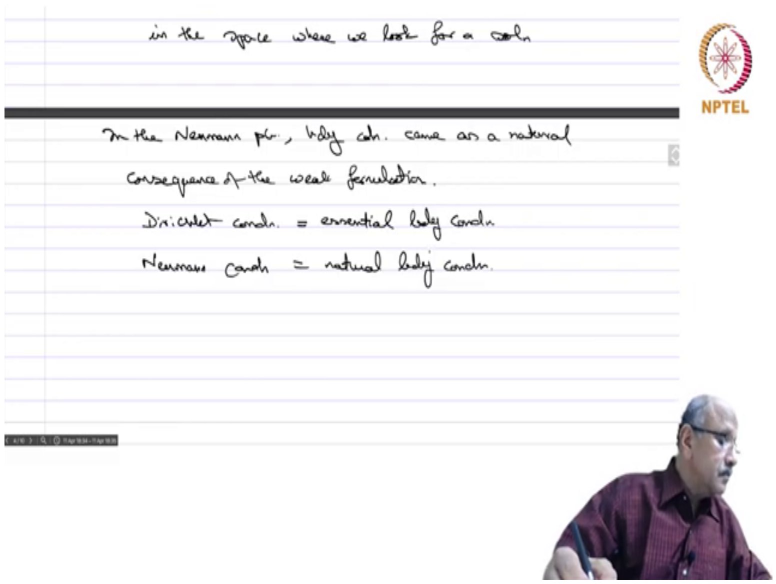

So, now we want to make a remark. So, there is a difference between the Dirichlet and the Neumann problems. In the Dirichlet problem, the boundary condition was included in the space concerned, space where we look for a solution. So, we worked in H^1_0, or we worked with the translate of H^1_0 by means of a lift of the boundary conditions. So, all functions where we are looking for a solution will automatically satisfy the boundary condition.

In the Neumann problem, boundary conditions came as a natural consequence of the weak formulation. So, this is why the Dirichlet condition is an essential boundary condition, and the Neumann condition is a natural boundary condition.

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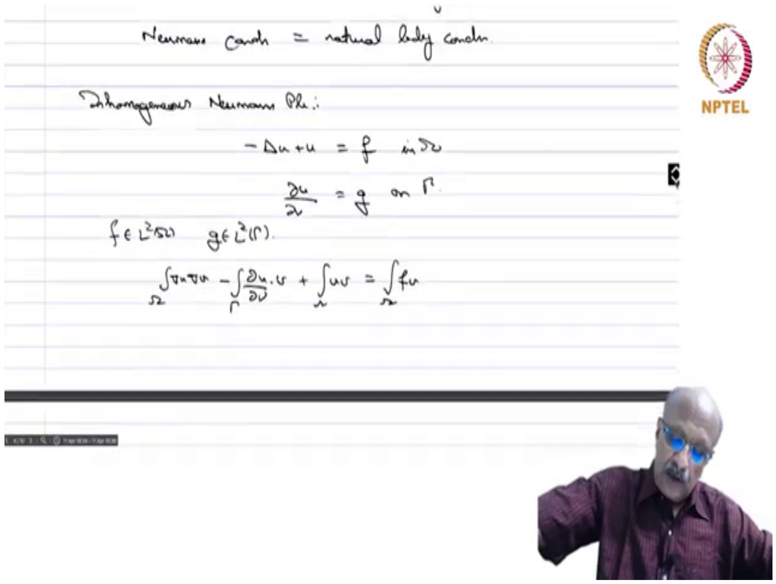

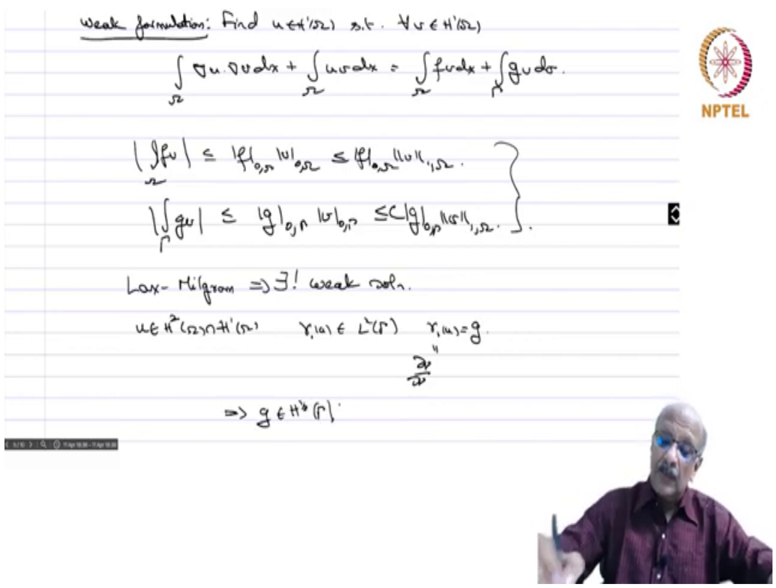

So, now let us look at the inhomogeneous problem,

$$-\ \Delta u\ +\ u\ =\ f \quad in\ \ \Omega$$

$$\frac{\partial u}{\partial \gamma} = g \quad on\ \ \Gamma.$$

So, now we assume that f is in $L^2(\Omega)$ and g is in $L^2(\Gamma)$, then again if you multiply by means of v and write it, so you will get integral grad u, grad v on omega minus integral on gamma du by d nu into v. And then plus integral on omega u v equals integral on omega f v.

So, if you now put in the condition du by d nu equal to g on gamma. Therefore, you will get the weak formulation find $u \in H^1(\Omega)$, such that for every $v \in H^1(\Omega)$, we have the

$$\int_\Omega \nabla u \cdot \nabla v dx - \int_\Gamma \frac{\partial u}{\partial \gamma} \cdot v\, dx + \int_\Omega uv dx = \int_\Omega fv\, dx \ \forall v \in H^1(\Omega).$$

Weak formulation: $\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx = \int_\Omega fv\, dx + \int_\Gamma gv\, dx, \quad \forall v \in H^1(\Omega)$

And the right hand side

mod fv is less than equal to mod f0 omega mod v0 omega, which is less than equal to mod f0 omega norm g, and norm v, 1 omega and integral on gamma gv is less than equal to mod g0 gamma, mod v0 gamma. But we know this is less than equal to, because v is trace operator. Therefore, mod g0 gamma C times norm v1 omega, we are just having. So, by the trace theorem, and therefore you have that the right hand side is continuously.

So, we have the basic space is $H^1(\Omega)$, the linear form is fv dx plus g v d sigma and then the bilinear form is grad u integral grad u, grad v plus integral uv. And that is again the inner product and therefore it is H^1 elliptic. And consequently **Lax-Milgram** implies, there exists a unique weak solution.

So, again if $u \in H^2(\Omega) \cap H^1(\Omega)$, then $\gamma_1(u) \in L^2(\Gamma)$ and you will have gamma 1 of u equals g, and this of course is nothing but $\frac{\partial u}{\partial \gamma}$, by the Green's formula you can just exactly as we said. And we also have that the weak solution satisfies the differential equation in the sense of distributions. And of course, if u is in $H^2(\Omega)$, $-\Delta u + u = f \in L^2$ functions. And therefore, you will get that $\gamma_1(u) = g$.

So, everything will come exactly as we have seen before and this will imply also that g belongs to $H^{1/2}(\Gamma)$ in this case. So, necessarily it will be unless it is in $H^{1/2}(\Gamma)$ a you cannot expect that u will be in $H^2(\Omega)$.

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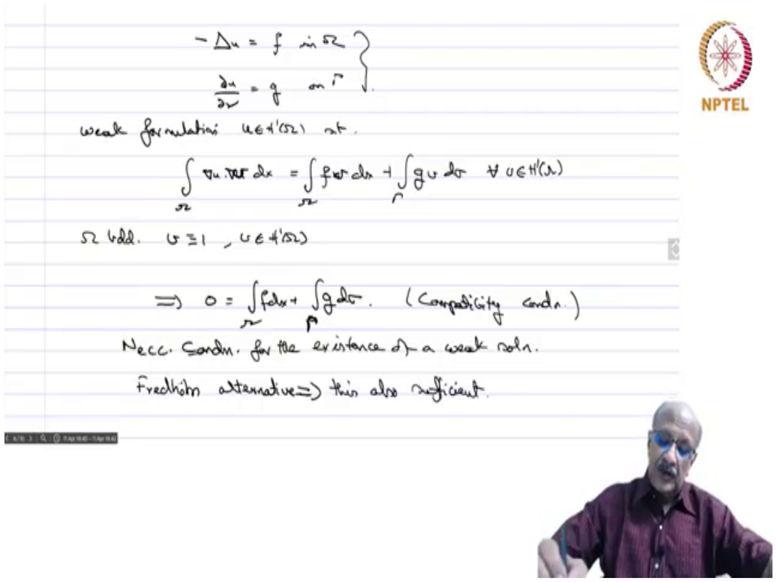

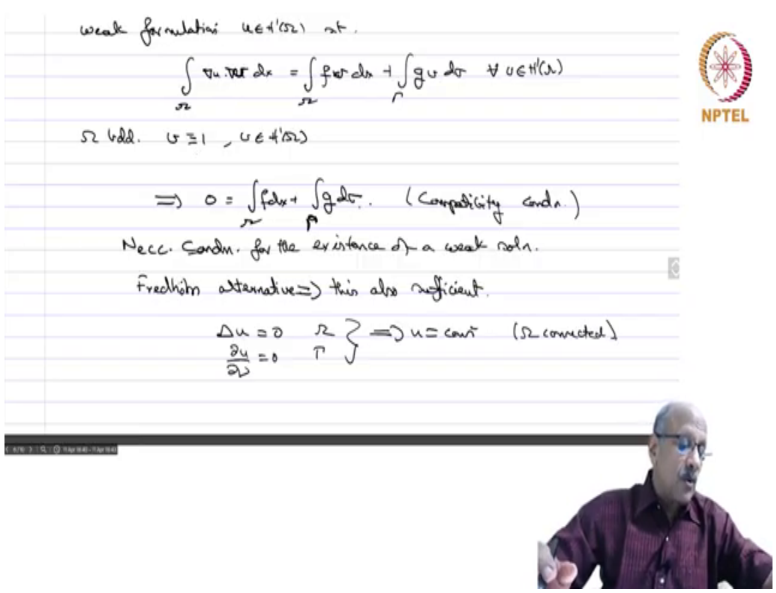

So, up to now there was a difference between the Dirichlet and Neumann problems, we had other than the imposition of the boundary condition, namely that in the Dirichlet case, we looked at $-\Delta u + u = f \quad in\ \Omega$

$$u = g \quad on\ \Gamma.$$

And here I always seem to put $-\ \Delta u\ +\ u\ =\ f$, because I wanted ellipticity in the $H^1(\Omega)$ norm. And since we do not have Poincare inequality for $H^1(\Omega)$ in a bounded domain, we had to put this extra term.

So, now what happens if I do not have that extra term. So, if I have $-\ \Delta u\ +\ u\ =\ f$ in Ω and u, $\frac{\partial u}{\partial \gamma} = g\ \ on\ \Gamma$, then the weak formulation will be $u \in H^1(\Omega)$, such that again you multiply integrate by parts and write everything will be grad u dot grad nu, dx equals integral of omega fv grad nu grad v. So, the fv dx plus integral on gamma gv d sigma for every $v \in H^1(\Omega)$.

So, now if I take omega is bounded, so v identically 1, $v \in H^1(\Omega)$, then if I substitute this will imply that 0 equals integral of f dx plus integral j on gamma g d sigma. So, this is a necessary condition, for the existence of a solution. So, there is a compatibility condition is called compatibility condition.

Now, if you write it out in the functional analytic framework in using the operator capital G, which maps the function to the f data to the solution and so on, then there is we can, we can get a compact operator by relic's theorem, as we have done before in the general second order elliptic Dirichlet case. And one can prove by Fredholm alternative implies that, this is also sufficient. So, this is a necessary and sufficient condition for the existence of a solution, which comes from the Fredholm alternative.

Now, this is because if you look at the Laplacian u equal to 0, in Ω and $\frac{\partial u}{\partial \gamma} = 0\ on\,\Gamma$, the homogeneous problem, then this will mean that u is a constant, one can prove if omega is connected of course. And what is the Fredholm alternative, if you want to have a solution for the inhomogeneous equation, we have already seen you can show that, this map is the map with g star, which is associated with this will be the same as g namely it is sulphate joint and therefore it should be orthogonal to the kernel of this kernel.

And that means there exists a solution if only if f and g, belong, are orthogonal to the function v equal to 1. And that is precisely this condition, which we have put here. So, by the Fredholm alternative one can show that this is in fact the solution. So, so the general you do not have a solution for this problem unless you have that, that is this compatibility condition is satisfied.

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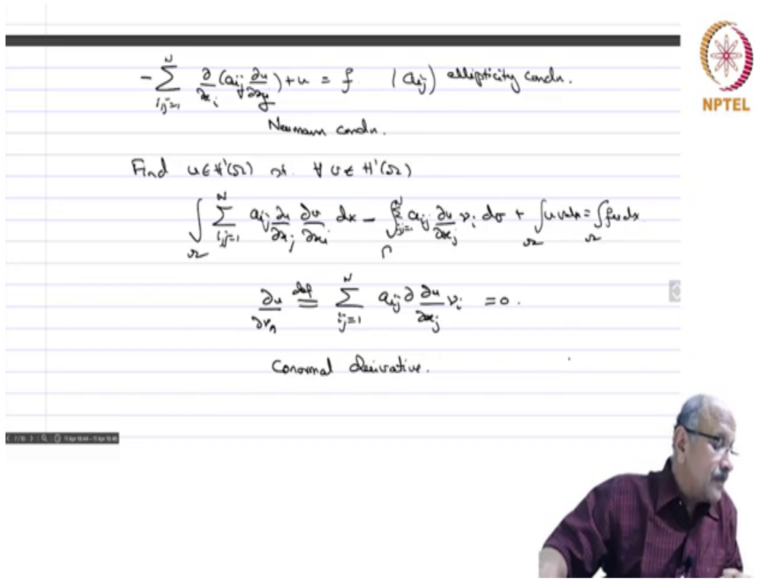

Now, we can also study the **Neumann problem**, for the general second order elliptic operator. So, $-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right)dx + u = f$ for instance $(a_{i,j})$ satisfying uniform ellipticity condition, and you can so if you now write the corresponding weak formulation.

So, plus Neumann condition, so then what will be the condition find $u \in H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$, you have

$$\int_{\Omega}\sum_{i,j=1}^{N} a_{i,j}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx - \int_{\Omega}\sum_{i,j=1}^{N} a_{i,j}\frac{\partial v}{\partial x_j}\gamma_i d\sigma + \int_{\Omega} uv\,dx = \int_{\Omega} fv\ dx.$$

So, the correct natural boundary condition associated to this operator is

$$\frac{\partial u}{\partial \gamma_n} = \sum_{i,j=1}^{N} a_{i,j}\frac{\partial v}{\partial x_j}\gamma_i = 0.$$

And this is called the co normal to the differential operator, which is given above. So, this is about the Neumann problem.

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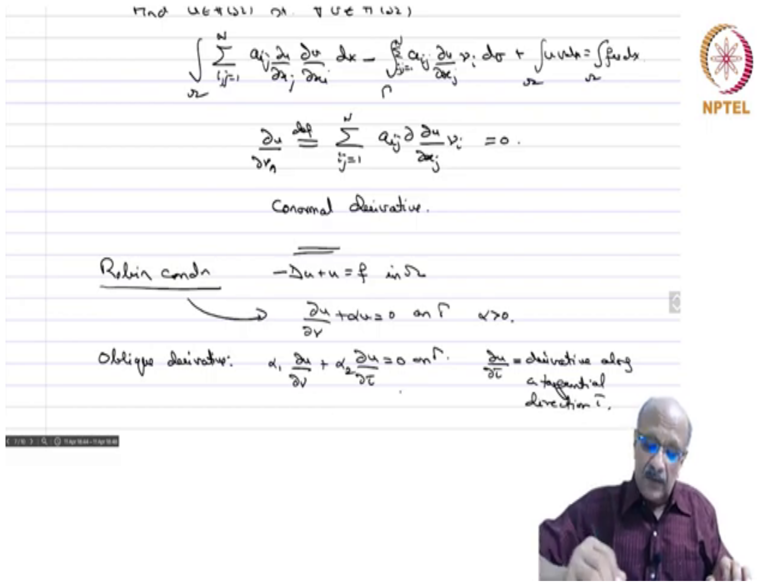

Now, we can look at various other kinds of boundary value problems, we will see some of them in the exercises and so on. So, you can have a Robin boundary condition, which is a combination of the Dirichlet and the boundary Neumann condition, which is equal to so. So, you get $-\Delta u + u = f$ in Ω. So, what is the Robin condition? This will be u, $\frac{\partial u}{\partial \gamma} + \alpha u = 0,\ \alpha > 0$, So, this is called the Robin condition.

Now, you can also have the oblique derivative, you have namely

$\alpha_1 \frac{\partial u}{\partial \gamma} + \alpha_2 \frac{\partial u}{\partial \tau},\ on\, \Gamma,$ So, $\frac{\partial u}{\partial \tau} =$ will be derivative along a tangential direction in, in two dimensions, you have only one, one tangential direction in higher dimensions you will have more than one. And therefore, you if tau is one of the unit tangent vectors at a point. So, in that direction you can have. So, this is called the oblique derivative problem.

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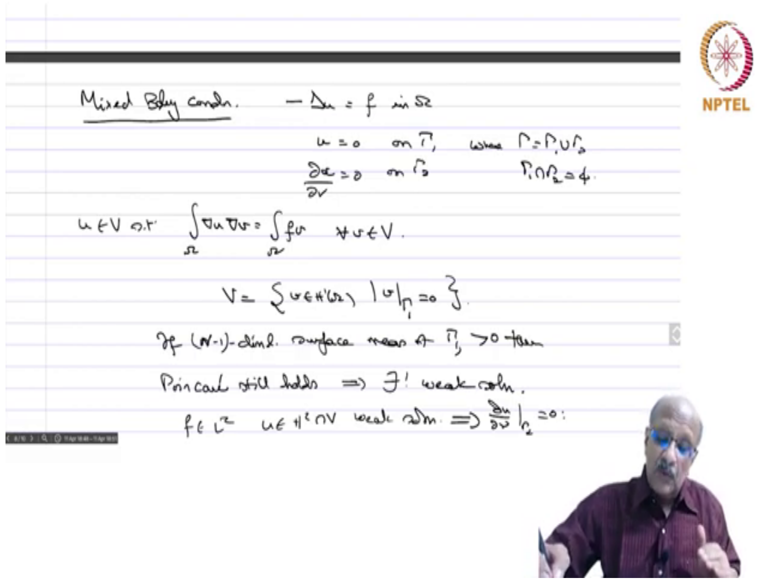

You can also have mixed problem,

mixed boundary conditions. So, for instance you have

$$-\Delta u + u = f \quad in\ \Omega$$

$$u = 0 \quad on\ \Gamma_1$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad on\ \Gamma_2$$

$$\Gamma = \Gamma_1 \cup \Gamma_2,\ \ \Gamma_1 \cap \Gamma_2 = \phi.$$

So, you have these things. And now in this case the weak formulation would be integral on omega

$$u \in V\ \ such\ that\ \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} fv\, dx\ \ \forall v \in V.$$

$$V = \{v \in H^1(\Omega)\ :\ v|_{\Gamma_1} = 0\}$$

So, you take the trace and then its restriction to Γ_1 should be equal to 0. If the N-1dimensional surface measure of Γ_1 is strictly positive, then Poincare still holds, that is because you have essentially what is this, if the first gradient is 0, then the function is a constant. But if it is 0 on a positive part of the boundary with positive surface measure, then the constant function has to be identically 0.

So, that is an idea, so Poincare still implies there exists a unique weak solution. And then if you can go and check that this condition we have imposed on the boundary. So, the Dirichlet condition, which is essential condition and if you work backwards with this bilinear form and write down explicitly the Green's formula and so on, then you should be able to show that a smooth solution say H^2, if $f \in L^2$ and so $f \in L^2$ and $u \in H^2$, intersection V, then this will imply and weak solution, this will of course imply that $\frac{\partial u}{\partial \gamma}|_{\Gamma_2} = 0$. So, this you can just check, it is very easy. So, now we will next, the next item on the agenda is to go and still look at some other kinds of problems. So, we have been looking at second order problems with various boundary conditions. So, the next thing we will do is, to look at the Dirichlet problem for the biharmonic operator, which is a fourth order differential operator.