Sobolev Spaces and Partial Differential Equations
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Weak solutions of elliptic boundary value problems - Part 2

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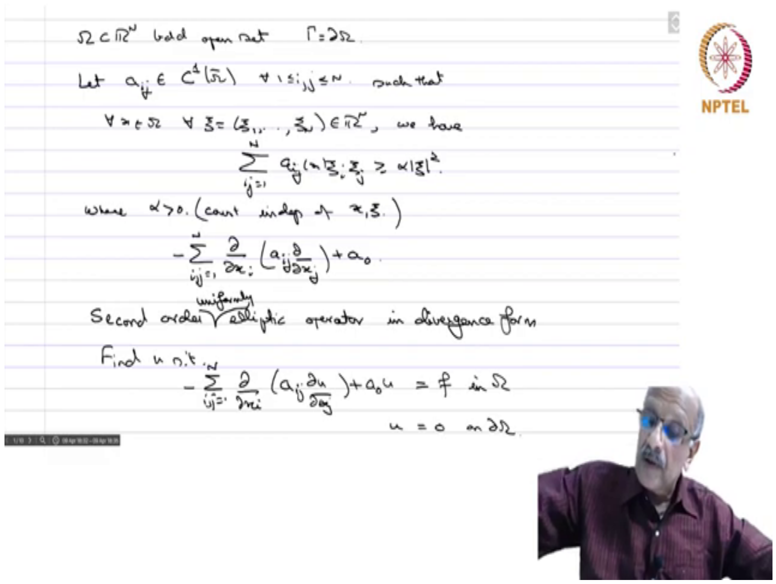

We now turn to more general second order elliptic operators. So, Ω as usual bounded open set, bounded domain as I also call it and $\Gamma = \partial\Omega$. So, let $a_{i,j} \in C^1(\overline{\Omega})$ for all $1 \le i, j \le N$. And such that for all $x \in \Omega$ and for all $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$ we have

$$\sum_{i,j=1}^{N} a_{i.j}(x)\xi_i\xi_j \geq \alpha|\xi_i|^2 ,$$

And where $\alpha > 0$ is a constant independent of x and ξ.

So, then you consider the following differential operator

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_i}\right) + a_0 u = f \quad in\,\Omega\ ,$$

$$u = 0 \quad on\,\Gamma\,.$$

So, this is called a second-order elliptic operator, uniformly elliptic in divergence form. The way you have written, usually, a second-order operator would have second derivatives, first derivatives, and lower order, zero-order term.

Here the second derivatives and first derivatives have been combined in a special way and therefore this is called the divergence form of the operator. So, and we now look for, find u such that so $-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j} \frac{\partial u}{\partial x_i}\right) + a_0 u = f \quad in\, \Omega \;, \;\; u = 0 \quad on\, \Gamma$.

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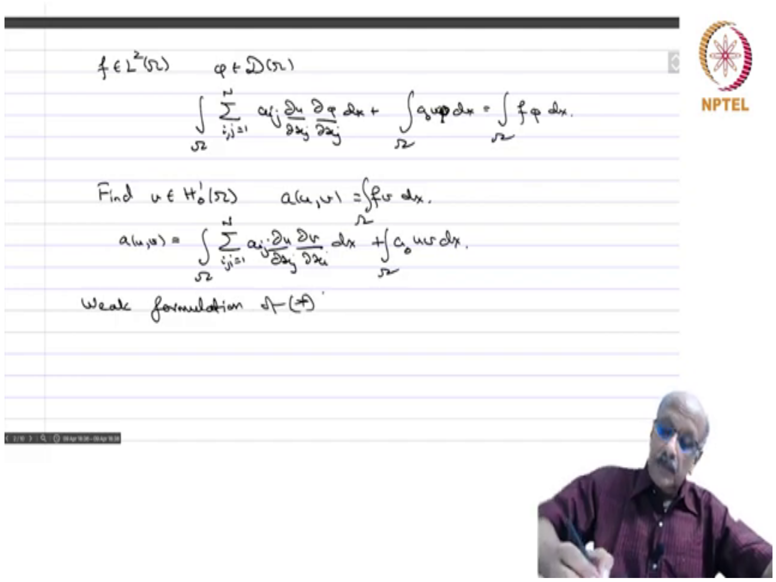

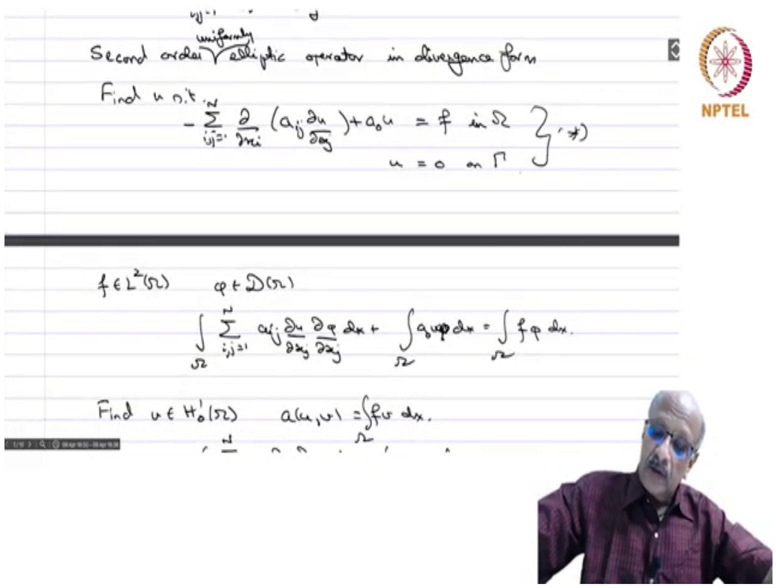

So, we assume now that $f \in L^2(\Omega)$ and then again so now if I multiply by phi so be $\varphi \in D(\Omega)$ multiply and integrate by parts and therefore this is where the d by d xj form will come into place. So, you will get

$$\int_{\Omega} \sum_{i,j=1}^{N} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Omega} a_0 \, uv dx = \int_{\Omega} fv \, dx \ , \ u, \ v \in H^1_0(\Omega).$$

So, if $u, v \in H^1_0(\Omega)$ belong to, so now everything is continuous as far as $H^1_0(\Omega)$ is concerned phi is dense there and therefore we get for all v, so you get that. So, find u in $H^1_0(\Omega)$, H^1 so that all these integrals make sense and 0 because it is in vanishes on the boundary such that you have

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{N} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega a_0 \, uv dx, \;\; u, v \in H^1_0(\Omega)$$

So, this is the, so we call this the weak formulation. So, the weak formulation you always have to mention three things what is the vector space, what is the bilinear form, what is the linear form. So, the linear form is $\int_\Omega fv \, dx$, $v \in H^1_0(\Omega)$, $f \in L^2$, bilinear form is given by this expression here $a(u, v)$ and the vector space of course is $H^1_0(\Omega)$, so that specifies. So, this is called the weak formulation of star, so star is the differential equation here with, together with the boundary conditions.

So, again one can check that if you have a weak formulation then it satisfies differential equation in the sense of distributions and of course the boundary condition is satisfied because we have put it in the space itself.

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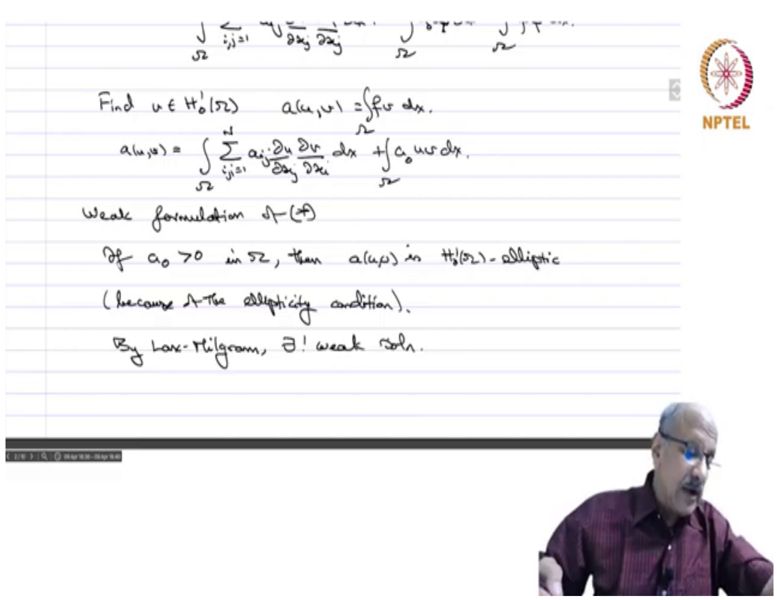

And so we now have if a naught is strictly positive in omega then $a(u, v)\ is\ H^1$ elliptic that is easy to see because of the condition of the ellipticity condition. What is the ellipticity condition? Recall, this is called the ellipticity condition. So, if you use this fact and for ξ_i and ξ_j you substitute $\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}$, then you will get that a u, u is greater than equal to alpha times integral mod u square, mod grad u square.

So, this is therefore, by Lax-Milgram there exists a unique weak solution and you can go back and forth as I said you can, if it is a classical solution then you can show it gives you a weak solution, if it is a weak solution which is sufficiently smooth then you get a classical solution.

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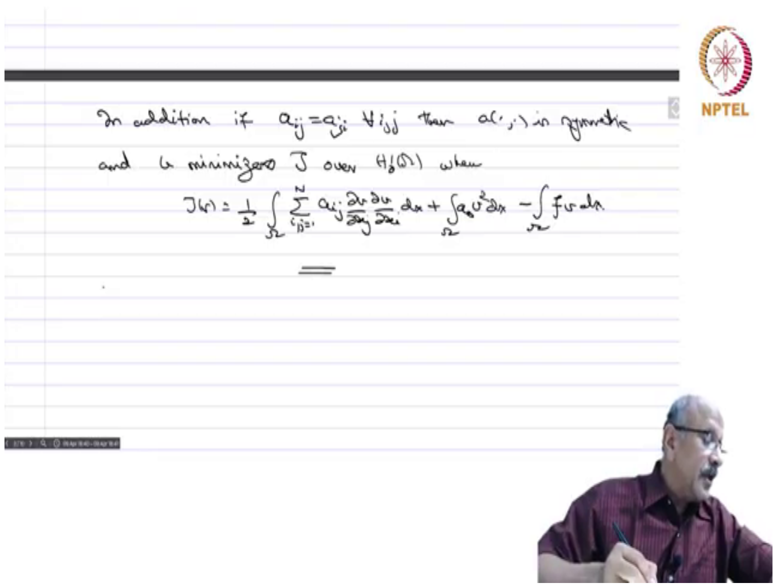

So, in addition from the abstract theorems we have proved in addition if a is symmetric that is $a_{i,j} = a_{j,i}, \ \forall i, j$, then a is symmetric and u minimizes J over $H^1_0(\Omega)$ where

$$J(v) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} a_{i,j} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx + \frac{1}{2}\int_\Omega a_0\, v^2 dx - \int_\Omega f v\, dx$$

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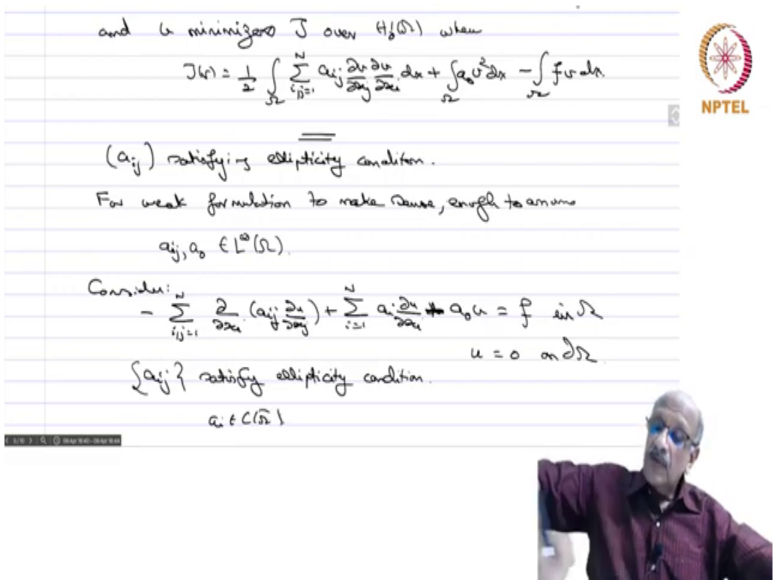

More generally we can consider the following so $a_{i,j}$ satisfies ellipticity condition. So, I also want to show for weak formulation to make sense enough to assume $a_{i,j}$, a naught are all in $L^{\infty}(\Omega)$ that is enough to make the integrals meaningful. So, you can always suppose a weak solution and then regularity theorem if you can prove using extra properties of the $a_{i,j}$, a naught and of the domain smoothness, etcetera. Then one can show that examine the question whether a weak solution is a classical solution or not.

So, weak solutions are what we will generally look for and numerical calculations will be attempting to approximate the weak solutions only. And therefore, the classical solution is a curiosity if you have it, it is fine, otherwise it still does not matter. So, now you consider, you have

$$- \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j} \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i} + a_0 u = f \quad in\,\Omega\ ,$$

$$u = 0 \quad on\,\Gamma\,.$$

So, again $a_{i,j}$ satisfy ellipticity and ai are all in c of omega bar so again we will see in a moment that we do not need so much, we will just see that everything is L^{∞}.

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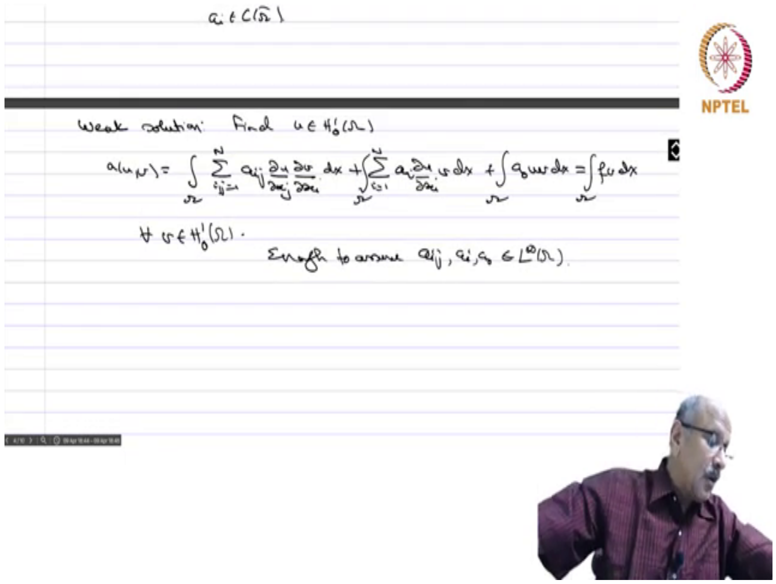

So, weak solution of this problem is find u in $H^1_0(\Omega)$a so that is the vector space now we have to specify that

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^{N} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Omega} \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i} + \int_{\Omega} a_0 \, uvdx, \;\; u, v \in H^1_0(\Omega)$$

So, once again you can check that this will satisfy the differential equation in the sense of distributions and we have imposed the boundary condition automatically and therefore this is what we call a weak solution of the problem. And in this case again enough to assume $a_{i,j}$, $a_{i,}$, a_i naught are all in L infinity omega then this makes sense and therefore we can talk of weak solutions of this differential operator.

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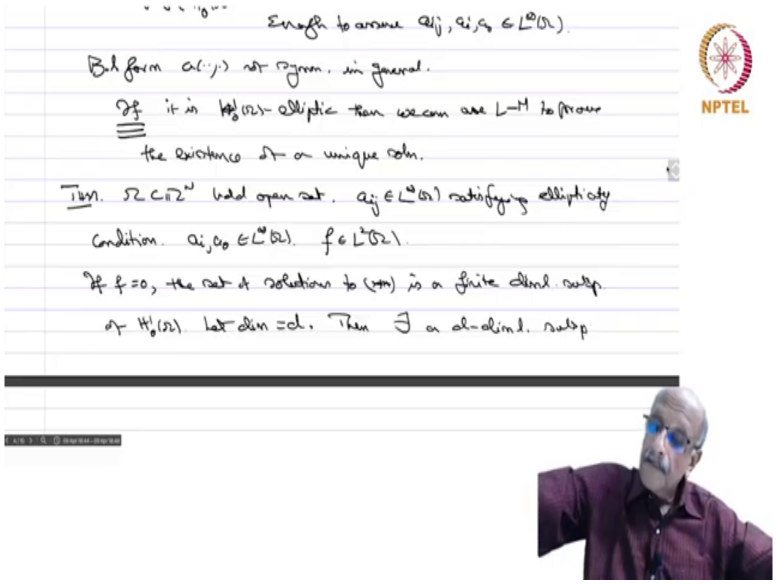

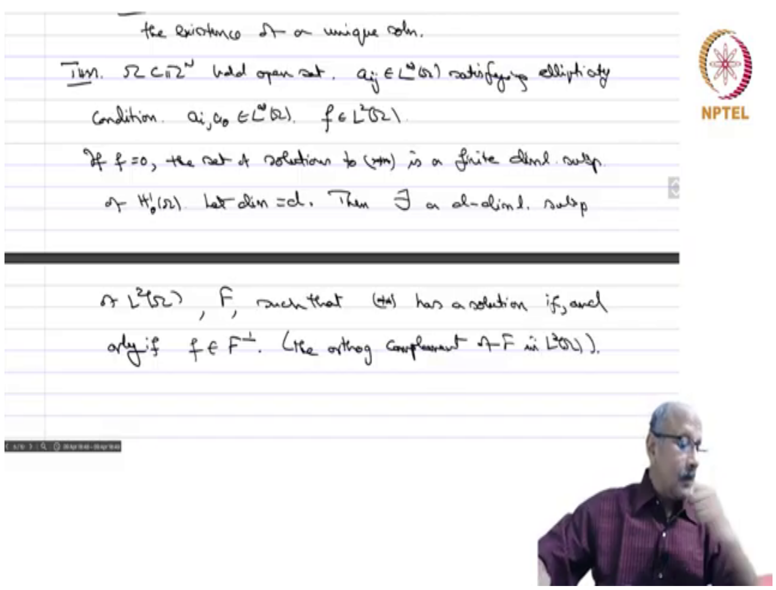

Now, bilinear form is not symmetric, not symmetric in general. If it is $H^1_0(\Omega)$ elliptic then we can use Lax-Milgram to prove the existence of a unique solution. But this may not happen. So, therefore, we have the following theorem. So,

Theorem,

$\Omega \subset \mathbb{R}^N$ bounded open set $a_{i,j}$ in $L^\infty(\Omega)$ satisfying ellipticity condition ai, a naught also in $L^\infty(\Omega)$ and f is in $L^2(\Omega)$.

If f equal to 0 the set of solutions to double star, so let us call this equation, this is the weak formulation is a finite dimensional subspace of $H^1_0(\Omega)$. So, let dimension equal to d. Then there exists a d dimensional subspace of $L^2(\Omega)$ capital F such that double star has a solution if and only if F belongs to the orthogonal complement in $L^2(F)$. So, this is the orthogonal complement of $F\ in\ L^2(\Omega)$.

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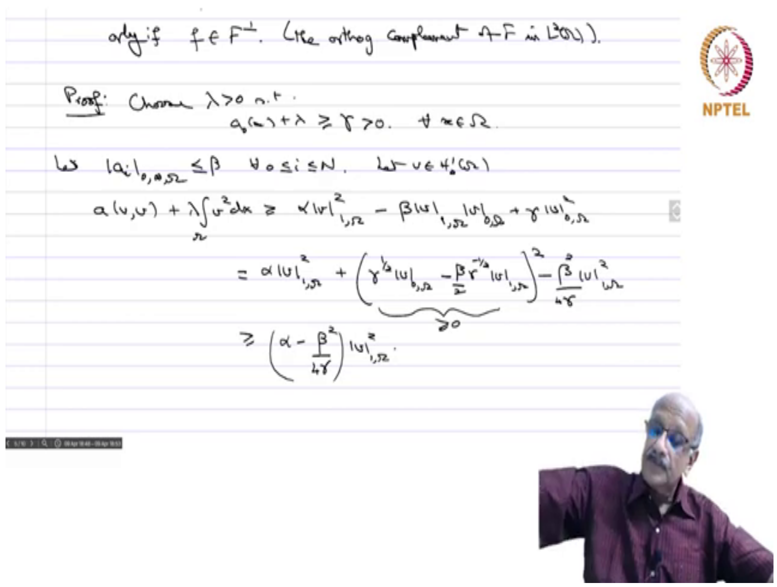

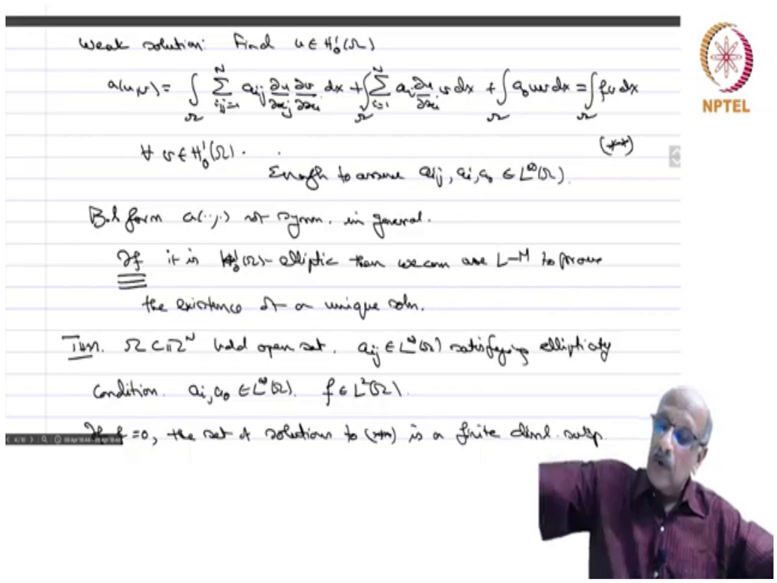

Proof:

So, now so we will prove this and you will see how nicely functional analysis comes to our help in solving these problems. So, choose lambda positive such that a naught of x plus lambda is greater than equal to gamma which is strictly bigger than 0. So, this can always be done because a naught is L infinity, so it is bounded so if I add a sufficiently large constant, then it will be always strictly positive and you can find a minimum value which is again strictly positive for all x in omega.

So, now let $|a_{0,i}| = 0,$ infinity omega less than equal to beta, for all 0 less than equal to i less than equal to N. Now, let v be arbitrary in $H^1_0(\Omega)$ then you look at a v, v plus lambda integral on omega v square dx. Now, because of the ellipticity a v, v. So, if you look at this expression here a v, v the first term will be $a_{i,j}$ du by d xj du by d xi sigma ij and that is greater than equal to alpha times mod u square d by d xi 1 omega.

So, du by d xi square. And therefore, you have this is greater than equal to alpha times mod u, mod v square on 1 omega. Then minus, so here you have a_i's are less than beta and therefore if you do the Cauchy-Schwarz inequality here you get mod u 1 omega mod v0 omega, so u equals v again. so if you and since you are going to get a lower bound you put a minus sign, this is minus beta mod v1 omega mod v0 omega.

Plus, then you have a naught v square, a naught v square plus lambda v square is also there and that is greater than equal to gamma. So, gamma mod v square 0 omega. And that is equal to alpha times mod v square 1 omega and then we complete the squares for these two terms and therefore you have gamma plus gamma power half mod v_0 omega minus beta by 2 gamma power minus half mod v1 omega whole square.

So, if you take the square term you have gamma mod v square 0 omega which is already there minus 2 beta and then the gamma will get cancelled, so and the 2 will also get cancelled beta times mod v_1 omega mod v_0 omega so we have this thing. And then we have a crossed, second term which is this one, so you, we have to subtract that minus beta by 4 gamma times mod v square 1 omega, beta square. So, this is therefore greater than or equal to so this square term is greater equal to 0 we ignore it, alpha minus beta square by 4 gamma times mod v square 1 omega.

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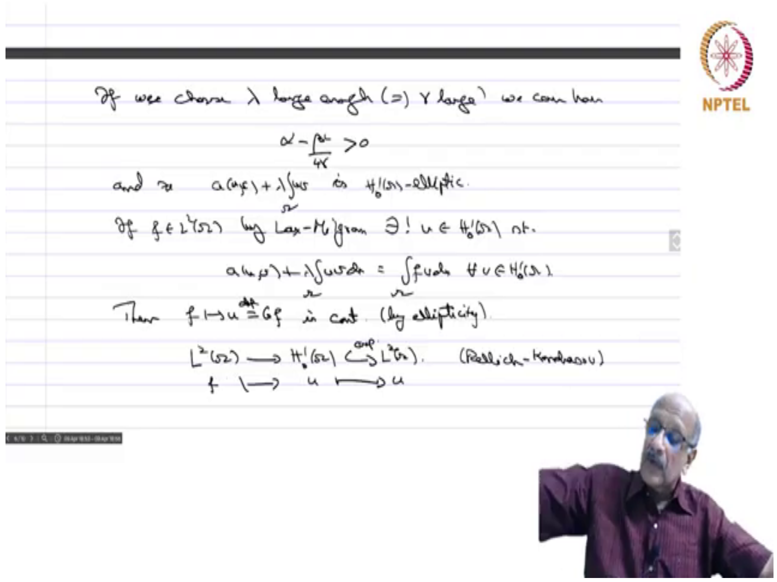

So, if we choose lambda large enough implies gamma large, we have, we can have alpha minus beta square by 4 gamma greater or equal to 0. And so a plus lambda times is $H^1_0(\Omega)$ elliptic that is $a(u, v)$ plus lambda times integral omega u v. This bilinear form is H^1 elliptic. So, if f belongs to $L^2(\Omega)$ by Lax-Milgram, there exists a unique u in $H^1_0(\Omega)$ such that a u, v plus lambda times integral uv on omega dx equals integral fv dx for every v in $H^1_0(\Omega)$.

Then by the ellipticity we know that f mapping to u which I am going to call g of f is continuous by ellipticity, we already know that if you have a u, v equals fv and then we have seen that f going to u is a continuous mapping g and it is 1 over the ellipticity constant is the norm of that operator.

Now, so you have from $L^2(\Omega)$ into $H^1_0(\Omega)$ you have f here going to u and now this will go back to u so this is inclusion in $L^2(\Omega)$. And we know this is, this inclusion is compact by Rellich-Kondrachov theorem $H^1_0(\Omega)$ into $L^2(\Omega)$ is elliptic because omega is a bounded open set and consequently composition of compact mappings is compact. And therefore, you have the

$$G: L^2(\Omega) \rightarrow L^2(\Omega),$$

is compact.

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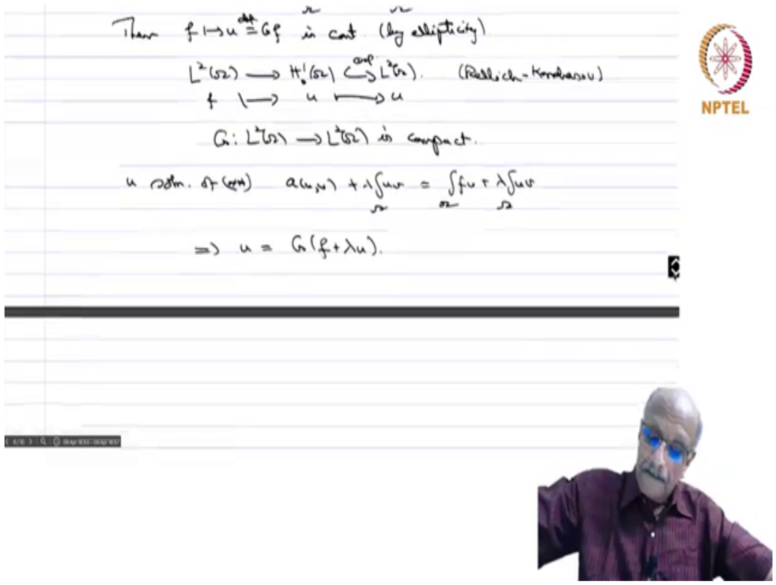

Now, if u solution of double star, what is double star, let us it is a weak formulation of the equation here, which is this expression here $a(u, v)$ is equal to this. Then we if you add a lambda uv to both sides then it, you get the this thing. Then you get

$$a(u, v) + \lambda \int_{\Omega} uv = \int_{\Omega} fv + \int_{\Omega} uv$$

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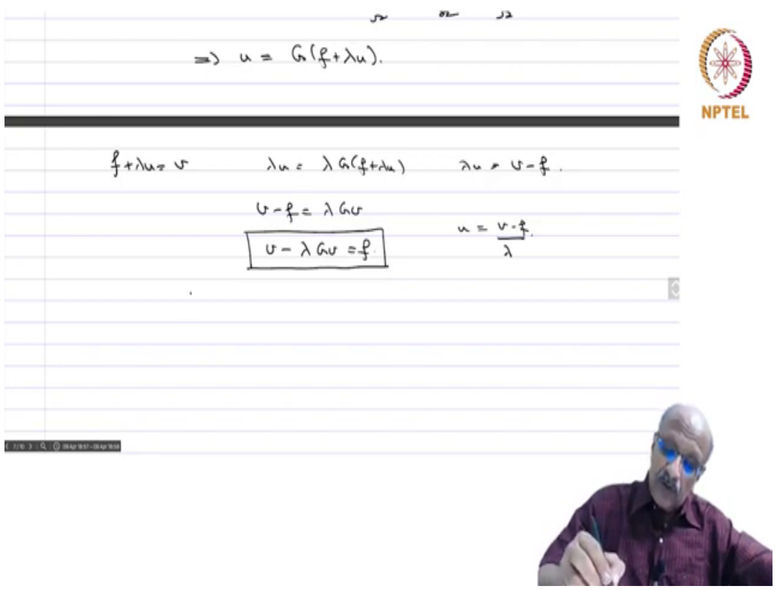

So, you let, put $f + \lambda u = v$, then you get $\lambda u = \lambda G(f + \lambda u)$ and λu is what, $\lambda u = v - f$. So, you substitute all these things here in this equation. So, you get $v - f = v - \lambda Gv = f$.

So, if you can solve for v you solve for u because $u = v - \frac{f}{\lambda}$. So, solving for u is same as solving for lambda, for v. But none, G is a compact operator and therefore its spectrum consists only of eigen values, non-zero spectrum.

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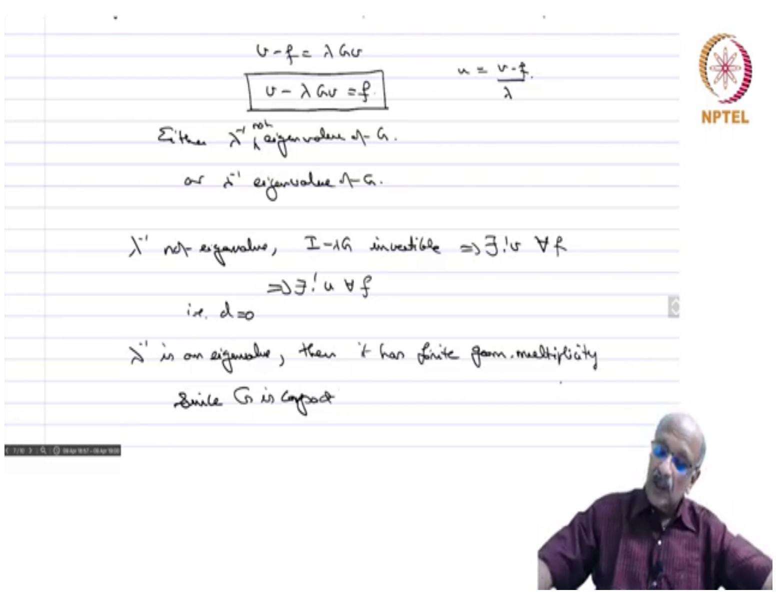

So, either lambda inverse eigen value of g are not an eigen value of G or lambda inverse is eigen value of G. So, if λ inverse is not eigen value, then $I - \lambda G$ is invertible implies there exists unique v for every f implies there exists a unique u for every f. That is d equal to 0, the set of all solutions for which f equal to 0 is the 0 solution only and therefore d equal to 0.

If lambda inverse is an eigen value, then it has finite geometric multiplicity. Since G is compact. So, if you have a compact space and you have a non-zero eigen value, then it is of course has to be finite, the null space that is the space of eigen vectors will be finite dimensional.

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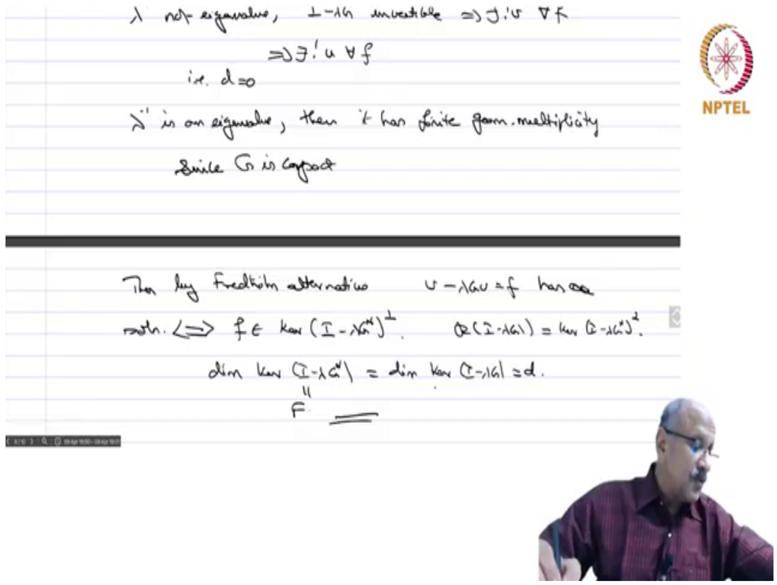

Then then by Fredholm alternative $v - \lambda Gu = f$ has a solution if and only if $f \in ker\,(I - \lambda G^{*})^{\perp}$ because you know range of $rang(I - \lambda G)$ is nothing but $ker\,(I - \lambda G^{*})^{\perp}$. So, this is the standard theorem, this is called the Fredholm alternative.

And therefore, you have an dimension of $dim\big(ker\,(I - \lambda G^{*})\big) = dim\big(ker\,(I - \lambda G\,)\big) = d$ is equal to dimension of which is equal to d. And therefore, you have the, this is our F and you have, so f, small f must be then capital F perp only then you have a solution. So, this proves that you have, this proves that theorem. So, we will continue with further examples later.