**Sobolev Spaces and Partial Differential Equations**
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**Lecture 9**
**Weak Solutions of Elliptic Boundary Value Problems - Part 1**

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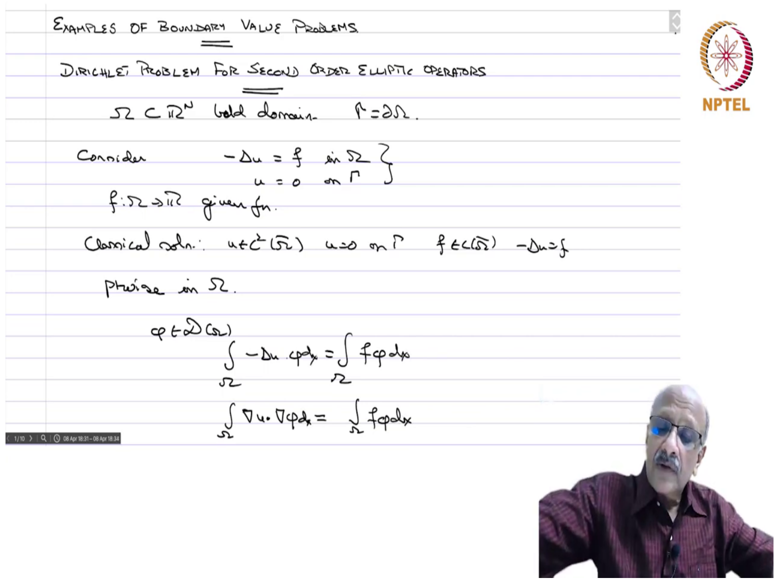


We were looking at a lot of abstract variational problems. Now we will see several examples of applications of the general result, especially the Lax-Milgram lemma. So, examples of boundary value problems.

So, we first look at the Dirichlet problem for second-order elliptic equation, operators. So, throughout, whether I say $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\Gamma = \partial\Omega$. So, this is the thing.

So, we consider the problem

$$-\Delta u = f \ \ in\, \Omega$$

$$u = 0 \ \ on\, \Gamma$$

So, this is a problem, $\Delta$ is a Laplacian and f is some given data. So, in the domain it should satisfy this differential equation, and on the boundary, it should vanish. So, $f: \Omega \rightarrow \mathbb{R}^N$ given function.

So classical solution means $u \in C^2(\overline{\Omega})$, $u = 0\ on\, \Gamma$, and (omega) $f \in C\ (\overline{\Omega})$, and $-\ \Delta u = f\ \ in\, \Omega$ point wise in omega. So, this is what we would look at, a classical solution.

So, if u is a classical solution let us multiply this by a C infinity function with compact support. So, let $\varphi \in D(\Omega)$. So, you have

$$-\int_{\Omega}(\Delta u)\varphi\, dx \ = \int_{\Omega} f\varphi\, dx$$

So now let us use Green's theorem. So, you get

$$\int_{\Omega} \nabla u \nabla\varphi\, dx \ = \int_{\Omega} f\varphi\, dx$$

Then there is no boundary term because $\varphi \in D(\Omega)$ and therefore there will be no boundary term equals $\int_{\Omega} f\varphi\, dx$. So, this is what we get when we satisfy this.

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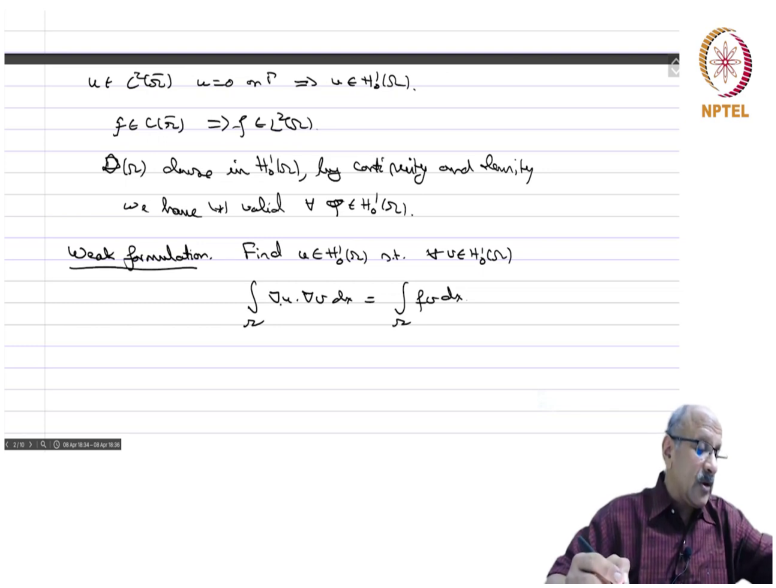

So then, so, so $u \in C^2(\overline{\Omega})$, f is in, so therefore the, implies that, and $u = 0 \; on \, \Gamma$, and therefore this implies that u belongs to $H^1_0(\Omega)$. And then $f \in C \; (\overline{\Omega})$ . And therefore, this implies that $f \in L^2(\Omega)$.

So now if you look at star, by, since $D(\Omega)$ dense in $H^1_0(\Omega)$, by continuity and density, we have star valid for all v, $\varphi \in H^1_0(\Omega)$, not just, because of the density. This we call as the weak formulation. So, we have weak formulation. Find $u \in H^1_0(\Omega)$ such that for every $v \in H^1_0(\Omega)$ we have

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx \text{ for every } v \in H^1_0(\Omega).$$

So, there is nothing about the second derivative in, even though the given differential equation is a, is called the is a second order differential equation, namely, the its, for the Laplace operator, this equation does not involve anything about second derivatives.

We are looking at a function on which the vector space involves only function and its first distribution derivatives. And the formulation here also, the equation which we have written for every $v \in H^1_0(\Omega)$ does not involve any second derivatives at all. So, that is why we call this a weak formulation. So, we (as) we say that this is a solution, a weak solution of the original equation.

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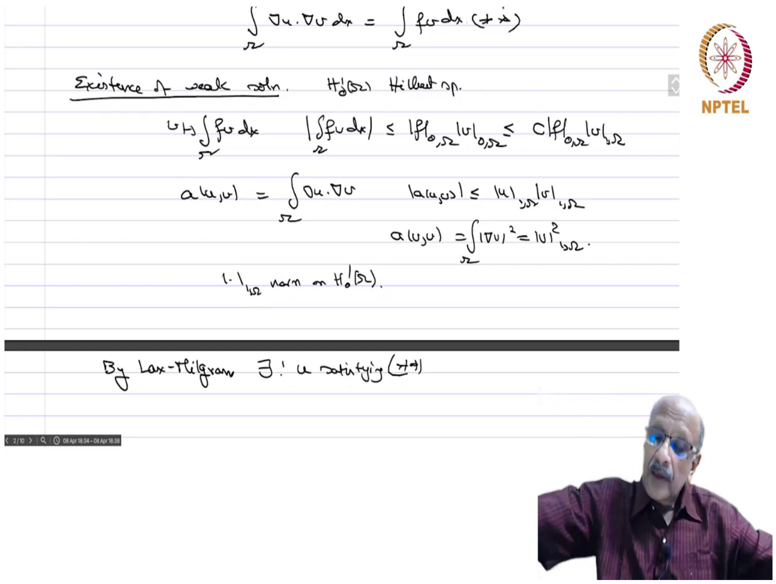


So let us, first let us dispose of the question of existence. So, existence of weak solution.

So, $H^1_0(\Omega)$ is a Hilbert space. f v going to $\int_\Omega fv\,dx$. So,

$$\left|\int_\Omega fv\,dx\right| \leq ||f||_{0,\Omega}||v||_{0,\Omega},$$

and by Poincare's inequality this is

$$\leq C\,||f||_{0,\Omega}||v||_{1,\Omega}$$

And therefore, this is a continuous linear functional on this thing.

And you have that a of u v equals integral grad u grad v on omega, so mod a u v is less than equal to mod u 1 omega mod v 1 omega and you have that a v, v equals integral mod grad v square and that is of course mod v square 1 omega, which is a norm on, so mod 1 omega norm, on $H^1_0(\Omega)$, because omega is bounded and we have Poincare's inequality.

So, we have the solution. So, by **Lax-Milgram**, so we have a continuous, symmetric and elliptic, bilinear form, and a linear functional on the other side, therefore by **Lax-Milgram** there exists a unique u satisfying double star. So, there exists a unique solution.

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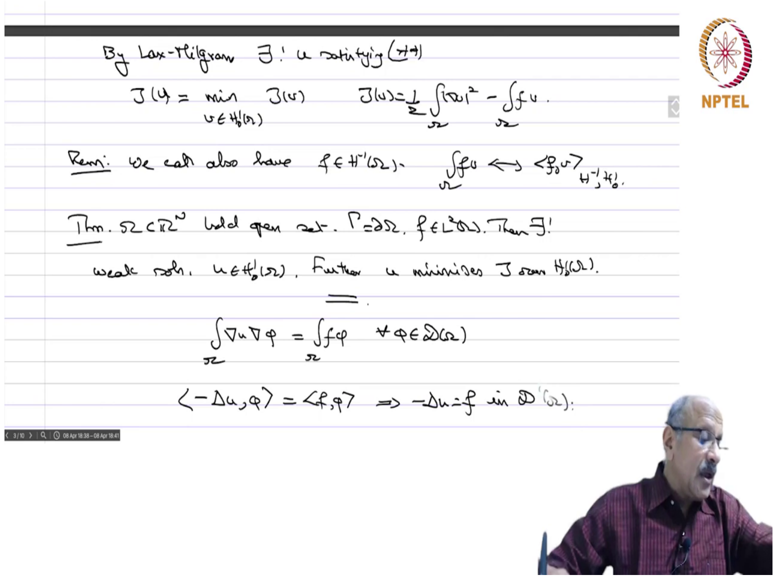


Further, because by the symmetry of the bilinear form we have that u equals,

$J(u)$ equals minimum over all v in $H^1_0(\Omega)$, of $J(v)$ where

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_\Omega f v\, dx.$$

So, this is the thing.

Now

**Remark.** We can also have $f \in H^{-1}(\Omega)$. And then instead of integral f v omega this will be replaced by the duality bracket f in $H^{-1}$ $(\Omega)$. So, the ins, that is the only change, otherwise this will have a solution. So, we have the following, we have proved the following theorem.

**Theorem:**

$\Omega$ bounded open set $\Gamma = \partial\Omega$, f in $L^2(\Omega)$, then there exists a unique weak solution $u \in H^1_0(\Omega)$. Further, u minimizes J over $H^1_0(\Omega)$. So, this is the theorem. So, now let us see how the weak solution is connected to the original equation.

**Proof:**

So, suppose we have, so we have

$$\int_{\Omega} \nabla u \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \ \forall \varphi \in D(\Omega),$$

for instance. But this tells you that minus Laplacian u acting on phi equals f phi as distributions and therefore this implies that minus Laplacian u equals f in $D'(\Omega)$. So, the weak solution is connected to the original differential equation. Namely, it satisfies the same, (eq) differential equation in the sense of distributions.

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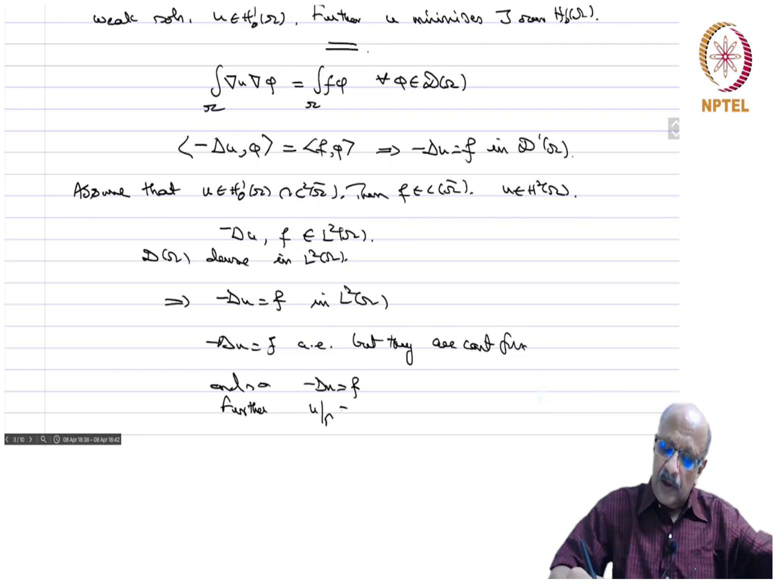


Now, so assume now, u belongs to $H^1_0(\Omega)$ intersection $C^2(\overline{\Omega})$. Then f will belong to c of omega bar and minus Laplacian u f belong to L2 of omega because it is in H u, is in $H^2$, and also $u \in H^2(\Omega)$. And therefore, you have, and they are equal as functions and $D(\Omega)$ is dense in $L^2(\Omega)$. This implies that minus Laplacian u equals f in $L^2(\Omega)$. That means minus Laplacian u equals f almost everywhere but they are continuous functions, and so minus Laplacian u equals f. Further, u restricted to gamma equal to 0.

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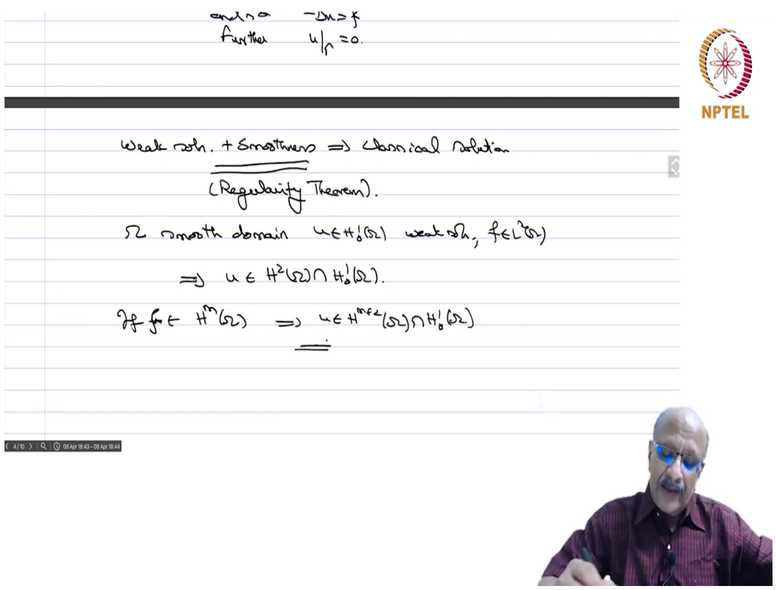


So therefore, you have that, it is a classical solution. Therefore, weak solution plus smoothness implies classical solution. So, the question is, when do you have the smoothness. So, this is guaranteed by means of what is called a regularity theorem. So, we have to prove it each time. So, for given any problem you have a weak solution, weak formulation, you have weak solution, then by some other technique you have to show that it is a regularity theorem.

Now for the record, we will say, for instance, that if omega is a smooth domain, how smooth I am not specifying, let us say reasonably smooth, then, and u is in $H^1_0(\Omega)$ weak solution, f in $L^2(\Omega)$, this will imply that u belongs to $H^2(\Omega) \cap H^1_0(\Omega)$. If u belongs to $f \in H^m(\Omega)$, this will imply that u belongs to $H^{m+2}(\Omega) \cap H^1_0(\Omega)$. These are examples of regularity theorems for the Laplace operator with reasonably smooth domains.

And therefore, if you, m is large enough then of course with the Sobolev embedding theorems you can deduce the u is differentiable, so, as many times as you want, and then you can deduce that it is a classical solution. So, this is how we go. So, the regularity theorem needs to be proved. It is, it is not an obvious thing. And that is a different story.

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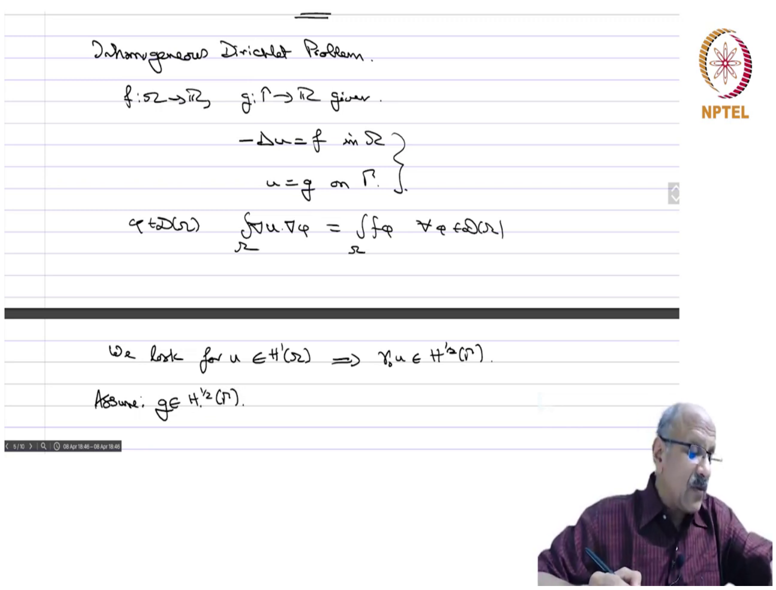


**Inhomogeneous Dirichlet problem:**

So now let us look at the inhomogeneous Dirichlet problem. Dirichlet problem means you are prescribing the value of the function on the boundary. This is called a Dirichlet problem. And therefore, homogeneous means u equal to 0 on the boundary, inhomogeneous means you have some other function.

So, $f: \Omega \rightarrow \mathbb{R}$, $g: \Omega \rightarrow \mathbb{R}$ given, and we want to look at the problem

$$-\Delta u = f \ \ in\, \Omega$$

$$u = g \ \ on\, \Gamma.$$

So, this is called a inhomogeneous Dirichlet problem.

So as usual if you multiply by $\varphi \in D(\Omega)$, you get

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \text{ for every } \varphi \in D(\Omega),$$

because again we used integration by parts and there was no boundary term because phi is 0 on the boundary. So, we are trying to look for, so we look for u in $H^1_0(\Omega)$ so that these integrals make sense.

And this will imply of course that gamma naught u will have to belong to $H^{1/2}(\Gamma)$. That is the Trace theorem which we have proved. So, therefore, we assume, so assume g belongs to $H^{1/2}(\Gamma)$. Only then we can try to make sense of the weak solution.

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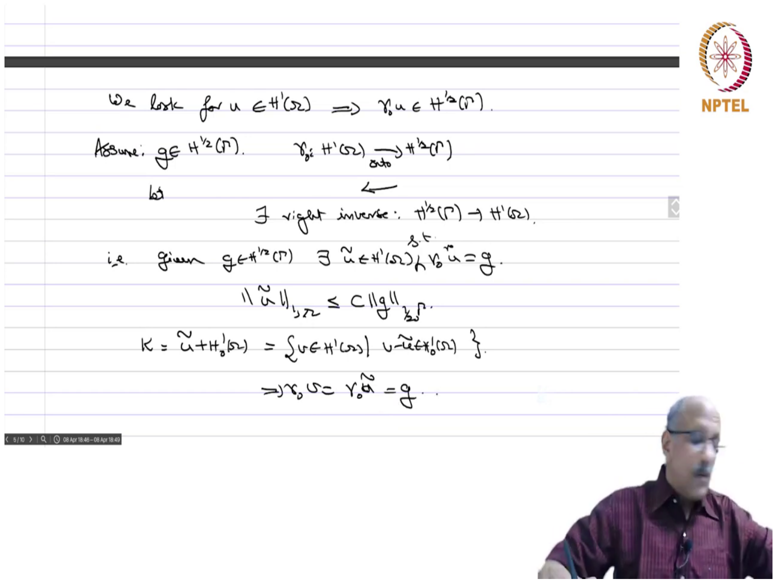


So now let, so you have, what you have, from $H^1(\Omega)$ gamma naught to $H^{1/2}(\Gamma)$, this is a continuous and onto. So, when you have onto from a Hilbert space then always you know that there exists a right inverse. So, there exists a right inverse from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$.

That is, given g in $H^{1/2}(\Gamma)$ there exists a $\tilde{u} \in H^1(\Omega)$, and that is right inverse that means u tilde, gamma naught of u tilde equals g such that, and further, you can say norm of u tilde in $H^1(\Omega)$ will be less than equal to some C times norm g in $H^{1/2}(\Gamma)$. So, this is the lifting you can always have because you have, we are in Hilbert spaces and we have right inverse.

So now you define $K = \tilde{u} + H^1_0(\Omega)$. So, this is set of all v in $H^1(\Omega)$ such that v equal to g on gamma, or in other words, if you do not, so v minus u tilde is in $H^1_0(\Omega)$. And this will imply that v, gamma naught v equal to gamma naught g u tilde equal to g. So, v will be equal to g on the boundary.

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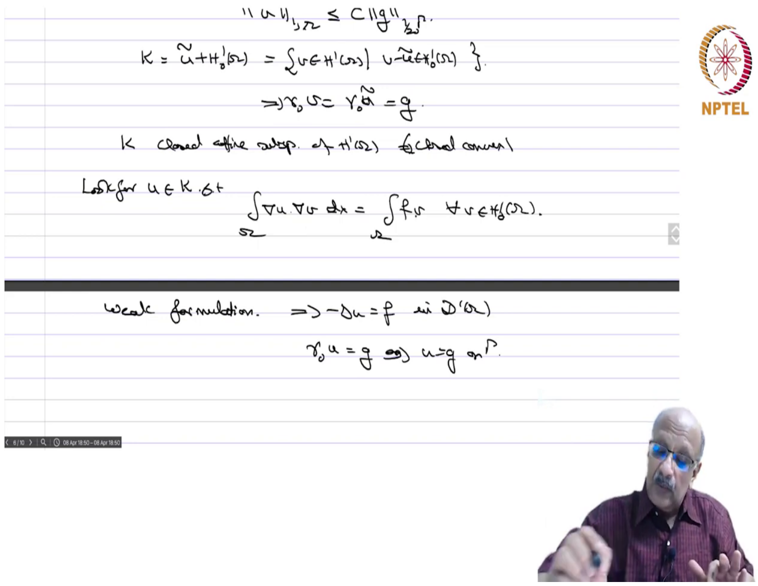


Then $K$ is a closed affines subspace of $H^{1}(\Omega)$. So, it is closed convex, so implies closed convex. And we want u belong, we are, look for u belonging to $K$. And we have that integral grad u grad v d x equals integral f v for every v in $H^{1}{}_{0}(\Omega)$. So, we call this the weak formulation.

So as usual, so we arrived at it starting with the solution of the equation, and we can again check this obviously means, so if you have a weak solution this obviously means that minus Laplacian u equals f in $D^{'}(\Omega)$ as before, and since we have u gamma naught u equal to g on, implies u equal to g on gamma. So, we have that the weak solution satisfies the differential equation in the sense of distributions, and of course it satisfies the boundary condition also. And therefore, this is a called a weak formulation of the equation.

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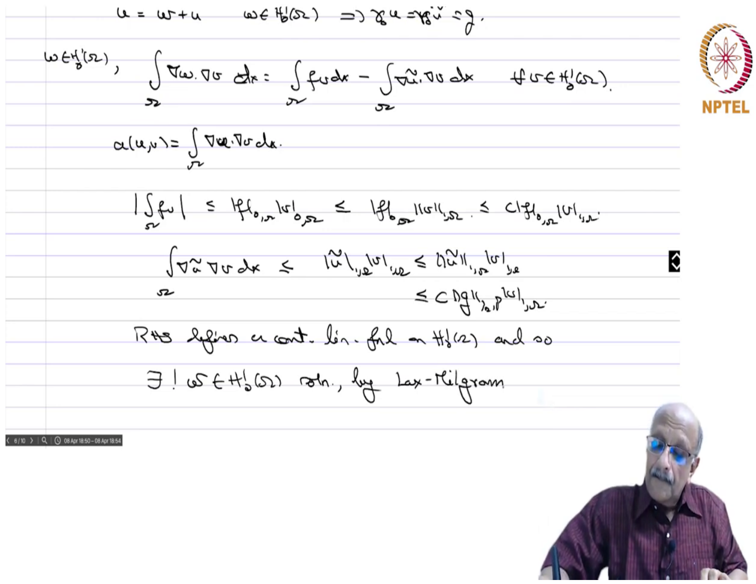

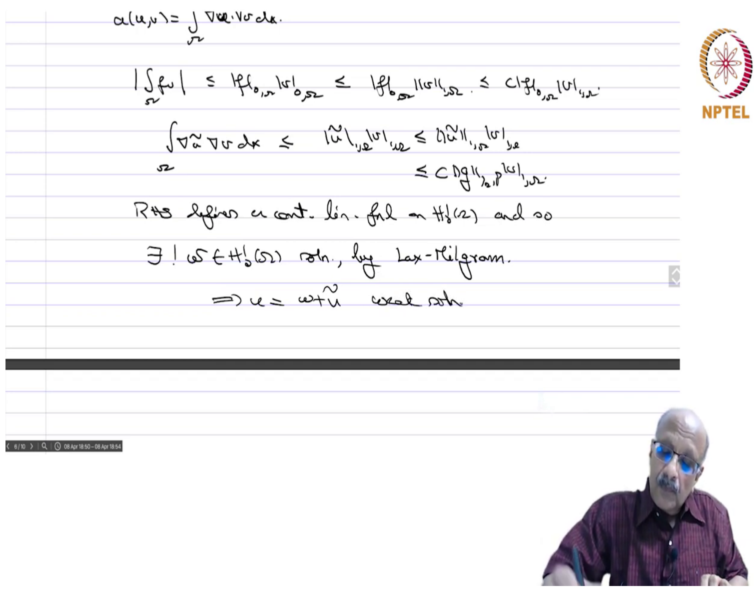

Now, what about the existence of the solution for this weak solution? So, for that we look at u, we write it as w plus u tilde where w belongs to $H^1(\Omega)$. So, this will imply of course that gamma naught of u equals gamma naught of u tilde which is equal to g.

And then if you substitute it in the formulation above, so that will give you grad w, so w in $H^1_0(\Omega)$ such that

$$\int_\Omega \nabla w \nabla v \, dx = \int_\Omega f v \, dx - \int_\Omega \nabla \tilde{u} . \nabla v \, dx \text{ for every } \varphi \in H^1_0(\Omega).$$

So now you have completely a problem in $H^1_0(\Omega)$. So, we have, which is Hilbert space, and of course a u v is equal to integral on omega grad w grad v, which we know, grad u grad v which we know is an elliptic, symmetric, continuous, bilinear form. It is in fact just the inner product in $H^1_0(\Omega)$ by the Poincare inequality. So, now what about the right hand side?

So,

$$\left|\int_\Omega fv\,dx\right| \leq ||f||_{0,\Omega}||v||_{0,\Omega} \leq C||f||_{0,\Omega}||v||_{1,\Omega}.$$

So, that is, that is fine, and then you have about

$$\int_\Omega \nabla \tilde{u} \cdot \nabla v\,dx \leq ||\tilde{u}||_{1,\Omega}\,||v||_{1,\Omega} \leq C||\tilde{u}||_{1,\Omega}\,|v|_{1,\Omega}.$$

So, this is less than C times mod f 0 omega mod v 1 omega. And

$$||\tilde{u}|| \leq C||g||_{1/2,\Gamma}$$

norm u is less than equal to some C times norm g half gamma mod v 1 omega. So, therefore, RHS defines a continuous linear functional on H1 0 of omega, and so there exists a unique w in H1 0 of omega solution by Lax-Milgram. And of course, so this implies that u can be written as w plus u tilde solution, weak solution of the inhomogeneous problem. So, that is fine. Now what about uniqueness?

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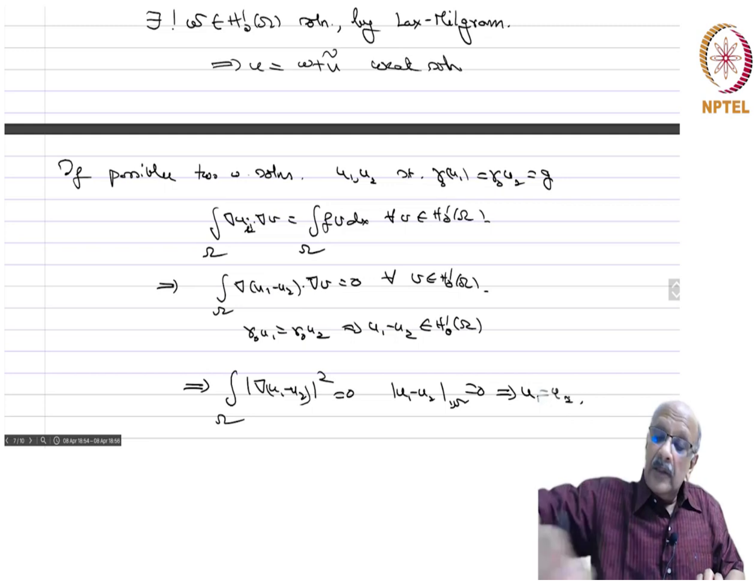


Now given u tilde, w is unique. That is all we know. But we want to know if there is a unique solution. Suppose you have two unique, two solutions, two (pos) if possible two solutions, two weak solutions. So, that means $u_1, u_2$ such that gamma naught $u_1$ equals gamma naught $u_2$ equals g. And

$$\int_{\Omega} \nabla(u_1 - u_2) \cdot \nabla v \, dx = 0, \quad \forall v \in H^1_0(\Omega)$$

So, this implies that integral on omega grad $u_1$ minus $u_2$ grad v equal to 0 for every $v \in H^1_0(\Omega)$. And since gamma naught $u_1$ equals gamma naught $u_2$, we have gamma naught $u_1$ equals gamma naught $u_2$ implies $u_1$ minus $u_2$ itself is in $H^1_0(\Omega)$.

And therefore, this implies, so if you put $v = u_1 - u_2$ as a test function, you get mod grad $u_1$ minus $u_2$ square integral on omega equal to 0. That means

$$|u_1 - u_2|_{1,\Omega} = 0$$

and by Poincare's inequality this implies that $u_1 = u_2$. So, the solution is unique. Now what about the continuous dependence on the data?

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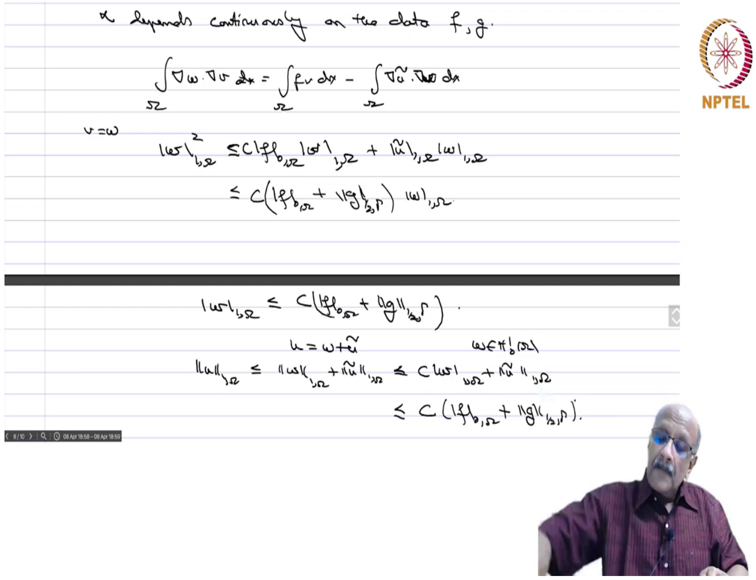


So, u depends continuously on the data. So, we have that

$$\int_{\Omega} \nabla w \cdot \nabla v \, dx = \int_{\Omega} f v \, dx - \int_{\Omega} \nabla w \cdot \nabla v \, dx \cdot \quad \forall v \in H^{1}_{0}(\Omega).$$

Now put $v = w$.

So, you get

$$|w|^{2}_{1,\Omega} \leq |f|_{0,\Omega} |w|_{0,\Omega} + |\tilde{u}|_{1,\Omega} |w|_{1,\Omega} \leq (C|f|_{0,\Omega} + ||u||_{1,\Omega}) |w|_{1,\Omega}.$$

And

$$||\tilde{u}||_{1,\Omega} \leq C \; ||g||_{1/2,\Gamma}.$$

So, this shows that it is continuously dependent on the data.

$$|w|_{1,\Omega} \leq C|f|_{0,\Omega} + |\tilde{u}|_{1,\Omega} |w|_{1,\Omega} \leq C|f|_{0,\Omega} + C\, ||g||_{1/2,\Gamma}.$$

$u = w + \tilde{u}$

So, and you have that

$$||u||_{1,\Omega} \leq C(|f|_{0,\Omega} + \; ||g||_{1/2,\Gamma})$$

by the Poincare inequality because $\forall w \in H^{1}_{0}(\Omega)$.

So, we use Poincare inequality 1 again, $||\tilde{u}||_{1,\Omega} \leq C \; ||g||_{1/2,\Gamma}$, that is right inverse we took, and w we have just calculated, so the whole thing is, like, by some constant f 0 omega plus norm g half gamma. So, we have, there exists a unique solution and it is continuously dependent on data.

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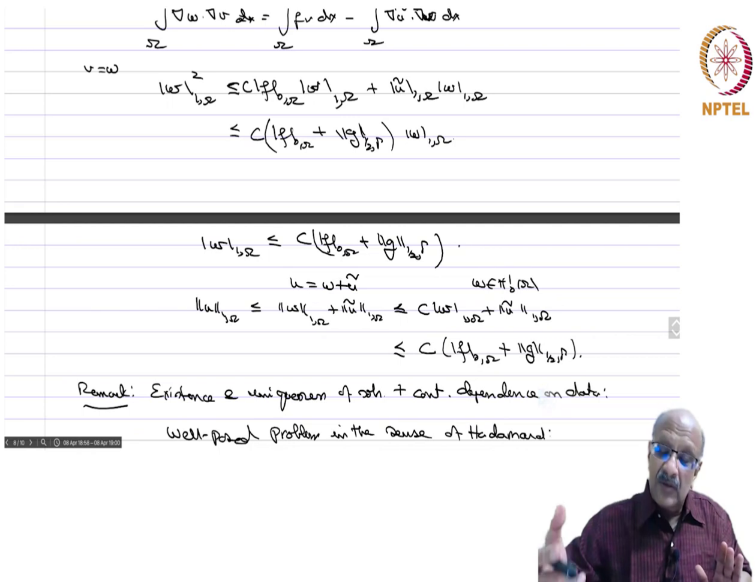


So,

**Remark:**

Existence and uniqueness of solution plus continuous dependence on data, so this is called Well-Posed problem in the sense of Hadamard. So, Well-Posed problem means it will have a solution, solution will be unique and the solution will depend continuously on the data. These are the three characteristics of a Well-Posed problem.

So, the Dirichlet problem, homogeneous, there is no, that straightforward we have already seen in the abstract setting, that the mapping g which maps the solution to the, f to the solution is the continuous operator. And therefore the, in the, the homogeneous case we had no difficulty and in the inhomogeneous case we have shown that there exists a unique solution which depends continuously on the data. So, our next thing is to look at other second order elliptic operators which are not necessarily the Laplacian. So, we want to know what is the situation in those cases.