Sobolev Spaces and Partial Differential Equations
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Lecture 8
Abstract Variational Problems - Part 2

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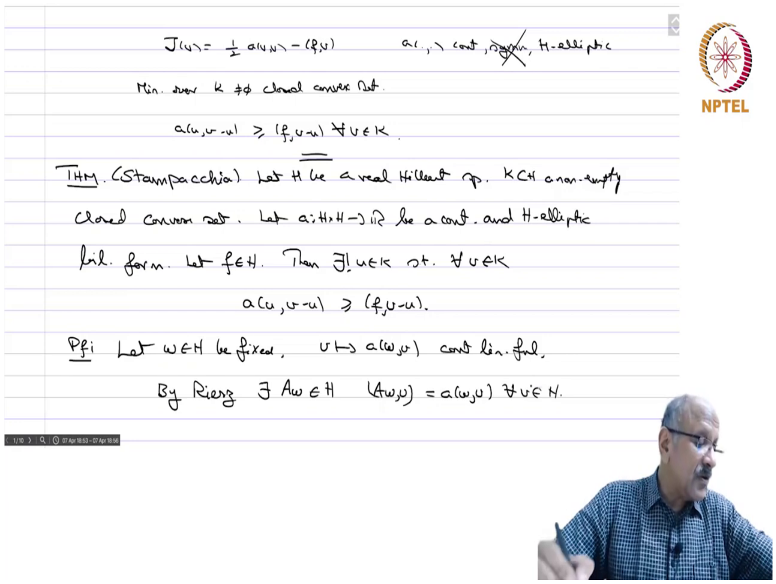

So, we just saw a theorem where if you had a symmetric continuous elliptic bi-linear form then the energy functional $J(v) = \frac{1}{2}a(v,v) - < f, v >$ could be minimized over K non-empty closed convex set, and the minimizer satisfies

$$a(u, v - u) \geq \; < f, v - u >, \quad v \in K.$$

So, a is continuous symmetric H elliptic.

So, we will now try to get rid of the symmetry hypothesis. So, that is not needed. But then of course, the price we pay is we will not get this variational characterization, as we call it. We will only get a solution to these inequalities, which is itself a useful thing to have.

So, so this is the theorem of Stampacchia. So,

Theorem(Stampacchia):

let H be a real Hilbert space and let $K \subset H$ a non-empty closed convex set. Let $a: H \times H \rightarrow \mathbb{R}$ be a continuous and H elliptic bi-linear form. Let f belong to H. Then there exists a unique $u \in K$ such that for every $u \in K$ you have

$a(u, v - u) \geq < f, v - u >$.

Proof:

So, this is the theorem. So, we do not need the symmetry but the price we pay is that this is not the solution to some minimization of energy.

So let $w \in H$ be fixed. And then $v: \rightarrow \ a(w, v)$ is a continuous linear functional. So, by Riesz, there exists $Aw \in H$ such that $a(Aw, v) = a(w, v)$ for every v in H. So, this is the thing.

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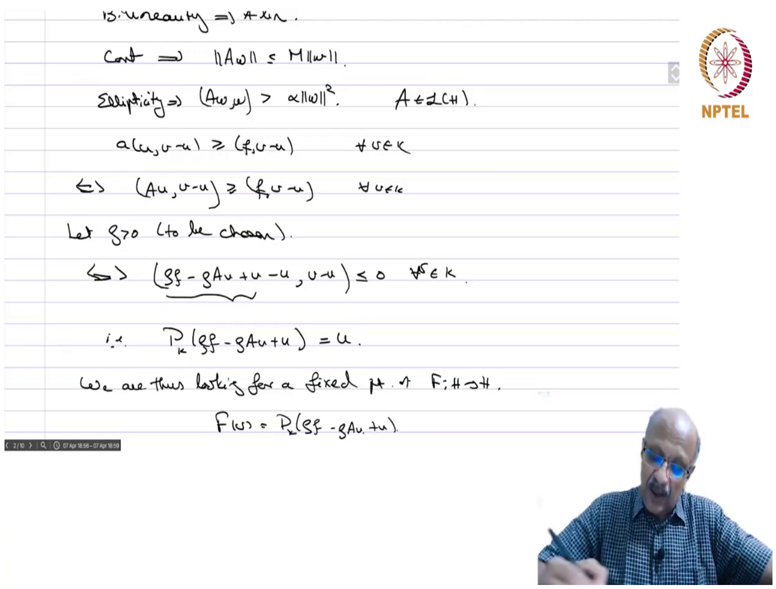

So, by the bi-linearity, so by, so bilinear, bi-linearity implies A is linear. Continuity implies norm of A w is less than equal to M times norm w. And also, you have

$$< Aw, w > \quad > \quad \alpha ||w||^2$$

So, we get these properties immediately from the continuity and by, so ellipticity.

So, A is a bounded linear operator, so $A \in L(H)$, that is, it is a bounded linear operator on H. So, now we can write

$$a(u, v - u) \geq < f, v - u > \quad \Leftrightarrow \quad < Au, v - u > \quad \geq \quad < f, v - u >, \ \forall v \in K$$

So, we can write this equivalently as saying, so

let $\rho > 0$ to be chosen. So, this is same as saying that

$$< \rho f - \rho A v + v - u, \ v - u > \quad \geq 0 \ , \ \forall v \in K.$$

But this resembles something which we know. So, you have something here minus u v minus u less than equal to for every, less than equal to 0 for every v in K. That is, we have that the

$$P_K(\rho f - \rho A v + u) = u$$

So, we are thus looking for a fixed point of f from H to H where f of v is given by projection to the K of rho f minus rho A u plus u. So, and obviously the projection, because it is a projection on K, it means the range is in K so any fixed point is going to be in K itself.

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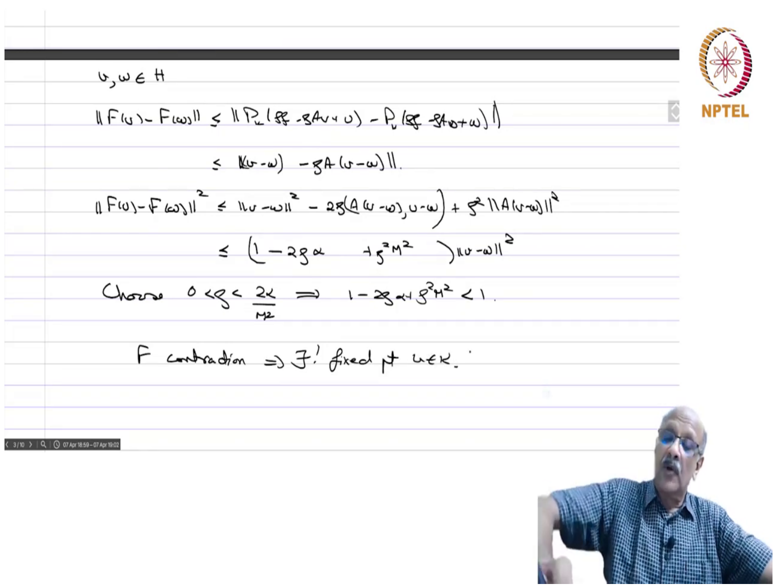

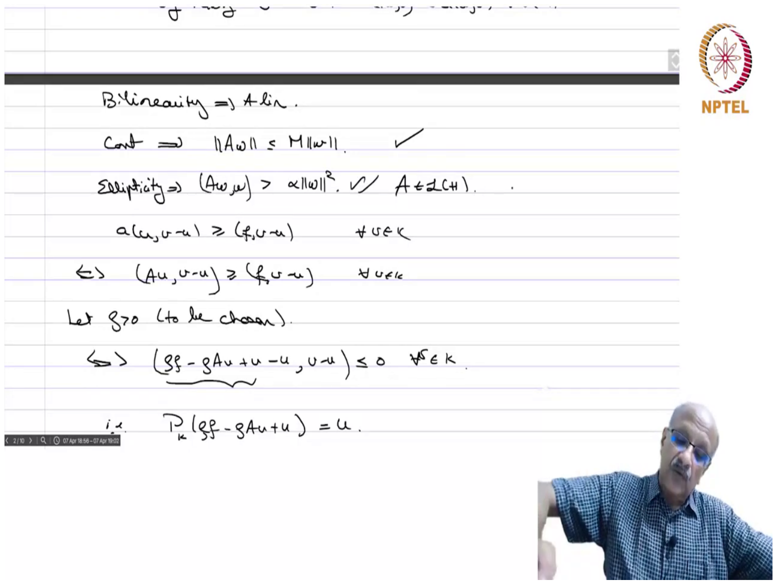

So now let v and w, so when you, the first fixed point theorem which we like to see whether we can apply is a contraction mapping theorem. So, we are trying to see if we can apply like that. So, norm of F v minus F w, so is P_K of rho F minus rho A u plus v, rho A v plus v minus P_K of, so norm of P_K of rho F minus rho A v plus v minus P_K of rho F minus rho A v plus w plus w.

But the norm of P_K of u 1 minus u 2 is less than equal to norm u 1 minus u 2. So, when we take the difference you get, these get subtracted out and therefore you will get that norm of v minus w minus rho Av minus w. So, the norm of F v minus F w square is less than the equal norm of v minus w square minus 2 rho Av minus w v minus w plus rho square norm of Av minus w square.

So now that is less than or equal to, I am going to write everything in terms of norm v minus w square. So, this will give me 1. The last one, norm of A of anything is less than M times that vector and therefore you have that plus rho square M square into norm v minus w square.

Now rho of Av w v minus w is greater than equal to alpha times v minus w square. And therefore, from that, since we are having a minus sign here, minus 2 rho alpha times norm v. So, you can now choose 0 less than rho less than 2 alpha by M square.

And so this part will become negative and therefore you have 1 minus 2 rho alpha plus rho square M square will be strictly less than 1. And therefore, F is a contraction, implies there exists a unique fixed point u in K. And that is a point which we are looking for. And that completes the proof of the theorem.

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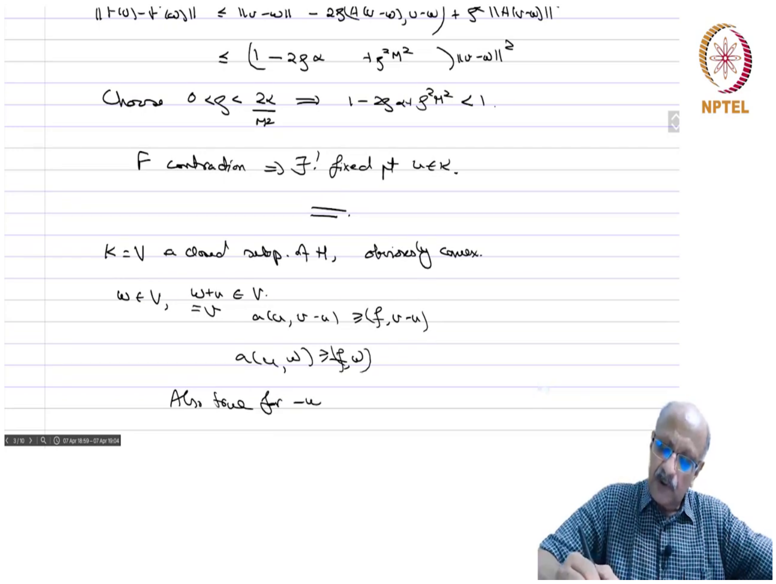

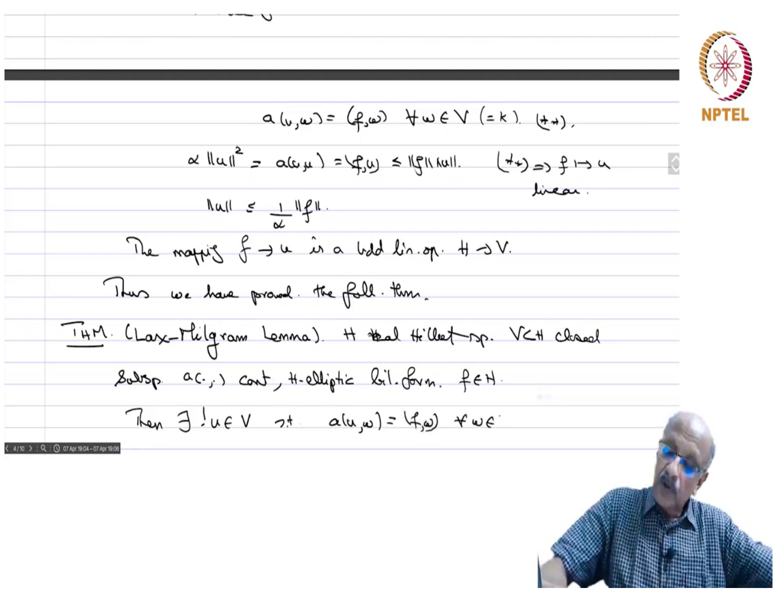

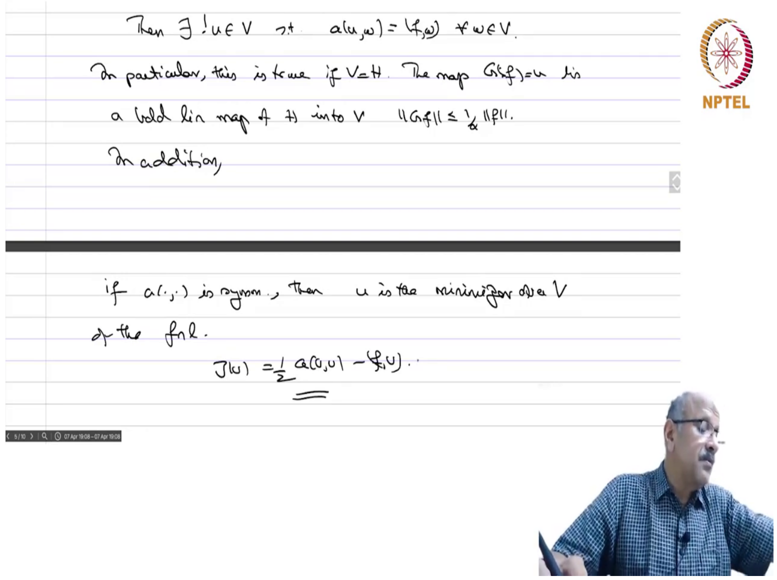

So, if K equals v, a closed subspace of H, then obviously convex non-empty, anyway it is not empty because 0 is always there. So, it is closed, so I do not have to say that, yeah, closed subset, so obviously convex, and non-empty. So, if $w \in V$ then w plus u also belongs to V. So, you can, you can substitute that.

So a u, v minus u is greater than equal to f of V minus u and that will give you a u w, so call this as V, and therefore a u w greatly equal than f of w. Also true for minus w. Therefore, from this you get a of u w equals f of w for every w in V.

Also, alpha times norm u square equals a of u u equals f u less than equal to norm f times norm u, and therefore norm u is less than equal to 1 over alpha times norm f. And therefore, the mapping, u going to, sorry, f going to u is a bounded linear operator H to V. So, we have actually proved, so thus we have proved.

So, this, this implies, so let, call me, let us call this double star, double star implies that f going to u is linear because of the bi-linearity of A and the inner product. So, this implies that this is linear and therefore you have this bounded linear operator. Thus, we have proved the following theorem. So, this is a very important theorem which we will use again and again.

So, this is called the

Lax-Milgram Lemma.

H Hilbert, real Hilbert space, $V \subset H$ closed subspace, a continuous and H elliptic bi-linear form, $f \in H$, then there exists a unique $u \in V$ such that

$$a(u,\ w) = \ < f,\ w >,\ \ \forall w \in V.$$

In particular this is true if $V = H$ itself. And the map G which takes $Gf = u$ is a bounded linear map of H into V and norm $||Gf|| = \frac{1}{\alpha}||f||$. In addition, if a is symmetric then u is the minimizer over V of the functional

$$Jv = \frac{1}{2}a(v,\ v) - < f,\ v >.$$

So, this is the Lax-Milgram Theorem.

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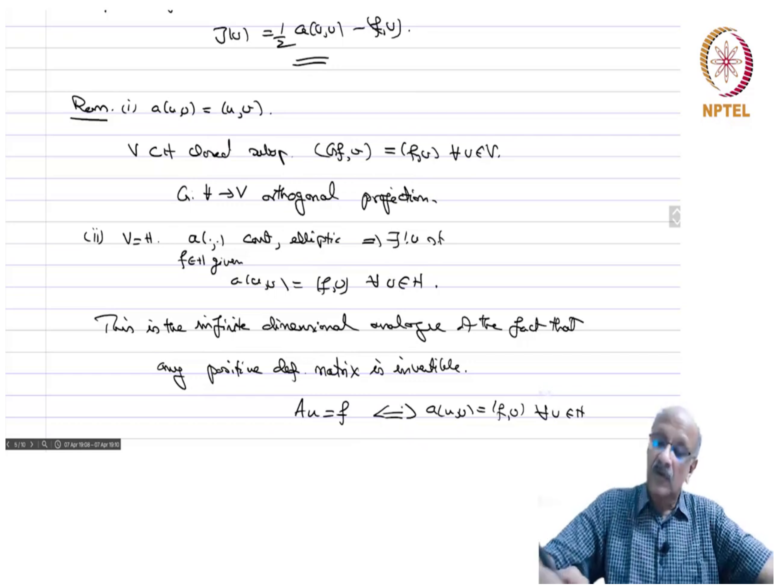

Remarks. 1,

You take $a(u, v) = < u, v >$.

So, so if you have V contained in H close subspace, so

$$< Gf,\ v > = < f,\ v >,\ \ \forall v \in V$$

And this is called, $G: H \to V$ is the orthogonal projection.

Remark. 2,

so let us take $V = H$ and then you are saying that a continuous elliptic implies there exists a unique $u \in H$ such that

$a(u, v) = < f, v >, \ \forall v \in H$

in any case, and f, f in H given. So, this is the infinite dimension, dimensional analog of the fact that any positive definite matrix is invertible. Because in finite dimensions onto equals 1 1 and so you have a unique solution. So, this is, it tells you that the matrix is invertible. Here you are saying that, so the A u equal to f is the linear system. So, that is the same as saying $a(u, v) = < f, v >, \ \forall v \in H$. So, the this is how you relate the two.

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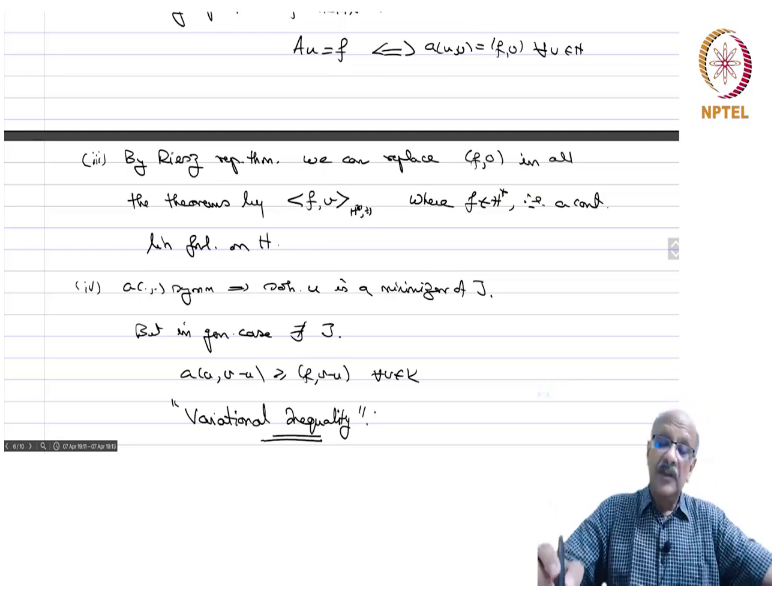

Remark. 3,

By Riesz representation theorem we can replace $< f, v >$, in all the theorems by $< f, v>_{H^*,H}$, the duality product H^* $and\ H,\ where\ f \in H^*$. That is, a continuous linear functional on H.

Why I am saying this is that we may not, as I, we studied in the Sobolev spaces, we have various Sobolev of spaces which we are dealing with, and we always said we will only identify L^2 with its dual but will keep the duals of H^1_0 as H^{-1} and so on. So, we will not, we will not always have that f v, but we will have in fact the duality bracket. Namely, f is in the, if f is a continuous linear function in fact.

Remark. 4,

By the Riesz representation theorem, this can be written as $< f^{\sim}, v >$ inner product for some $f^{\sim}$. So, we does not matter what we are doing this. For a symmetric implies, so solution u is a minimizer of J. But in the general case you do not have the, that does not exist energy that, does not exist J, which is minimized.

But $a(u, v - u) \geq < f, v - u > \ , v \in K$

this is called variational inequality. So, this is an example of what we call a variational inequality. So, our next aim is to see several examples of this situation, and then connect it with various boundary value problems.