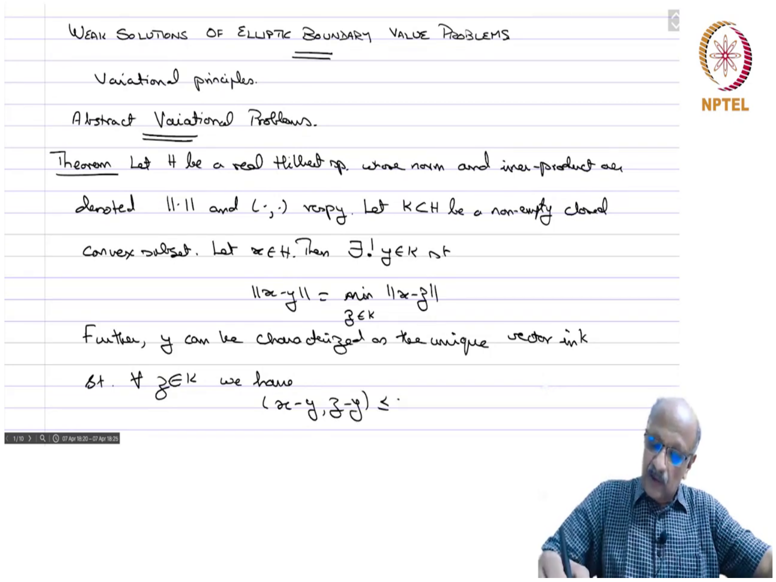
Sobolev Spaces and Partial Differential Equations
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences, Chennai
Lecture 7
Abstract Variational Problems - Part 1

(Refer Slide Time: 00:18)



We will now start a new chapter. This is Weak Solutions of Elliptic Boundary Value Problems. Most partial differential equations which come in engineering or physics arise out of what are called variational principles. So you must have heard of expressions in physics like the principle of least work. So there is an energy associated with any system and in order that you have equilibrium this energy has to be minimized.

So, when we do this minimization of energy, there is a functional which we write down in some Hilbert space, for instance, and then when we try to minimize it. So if for instance in finite dimensions if you are given F, minimize the function F of x then you first do the first derivative equal to 0. Now corresponding to this you have what is called the first variation or the vanishing of the first Frechet derivative of the functional.

That will give you an equation called the Euler-Lagrange equations which will be, which will turn out to be the differential equation we started with. So many differential equations which we come across arise in this way. So they come as the Euler-Lagrange equations of the minimization of some energy functional.

So, we will see several examples of this in the days to come, but first we will see a set of abstract variational problems and then most of these, all these problems which we look at will fit into this abstract framework.

So, so Abstract Variational Problems. So one of the classical variational problems is to given a Hilbert space, and a vector in it, and a closed convex set, find a point which is closest to it in the convex set. For instance, if you have a point and a line in the plane then you know that the closest point is the foot of the perpendicular which you draw.

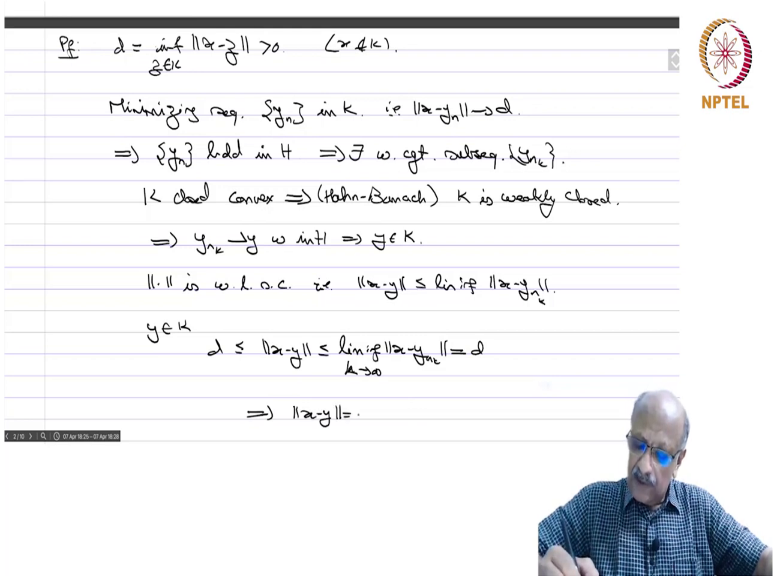
So, in the same way given a closed convex set in the Hilbert space we show that there is a point always and there is only one point which is closest to it in this. So we have the following theorem. So this is one of the first theorems we proved in Hilbert space theory.

Theorem:

Let H be a real Hilbert space whose norm and inner product are denoted respectively. Let $K \subset H$ be a non-empty closed convex subset. Let $x \in H$. Then there exists a unique $y \in K$ such that $||x - y|| = Inf_{z \in K}||x - z||$

So in, actually it is, since it is, you write inf, but then since it is actually realized we write minimum. Further, y can be characterized as the unique vector in K such that $\forall z \in K$ we have, $< x - y, y - z > \leq 0$. So this is the theorem.

(Refer Slide Time: 04:55)



Proof. So let us take $d = Inf_{z \in K} ||x - z||$, which is strictly positive if x is not in K. If x is in K then there is nothing to do because you have x itself is the unique point and the distance is 0, and therefore you have, you do not have anything to worry about. So let us take which is this, so x is not in K.

So, then you take a minimizing sequence, namely

$y_n \in K$ that is $||x - y_n|| \rightarrow d$.

So you can always do that. Because you have an infimum you can always take this out. So this implies, since d is a positive real number, $||x - y_n|| \rightarrow d$, so this implies that $\{y_n\}$ is bounded.

H is a Hilbert space so it is reflexive, so implies there exists a weakly convergent subsequence $\{y_{n_K}\}$. But K is closed convex and this implies by the Hahn-Banach theorem, K is weakly closed. That is, it is closed in the weak topology. So this, since x n, so $\{y_n\}$ in K converges to y weakly in H$\Rightarrow$ $y \in K$.

Also, the norm is weakly lower semi-continuous. So that means, that is,

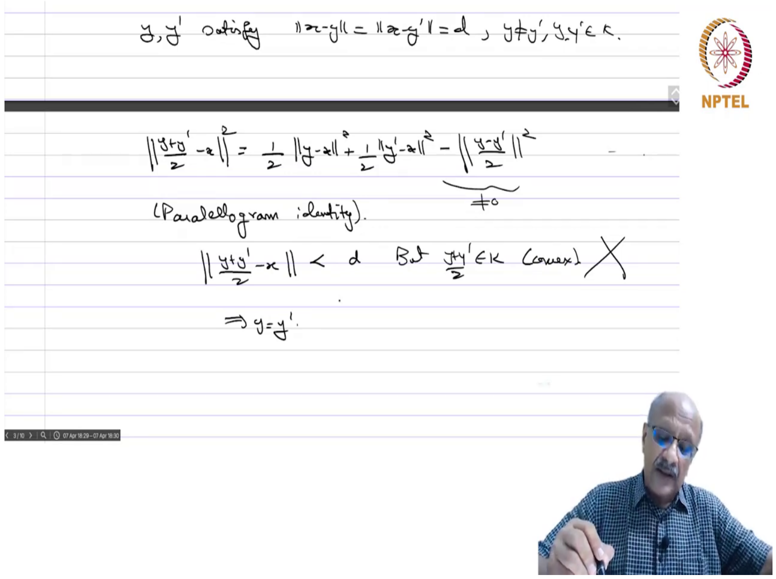
$$||x - y || \leq lim\, inf\, ||x - y_n||$$

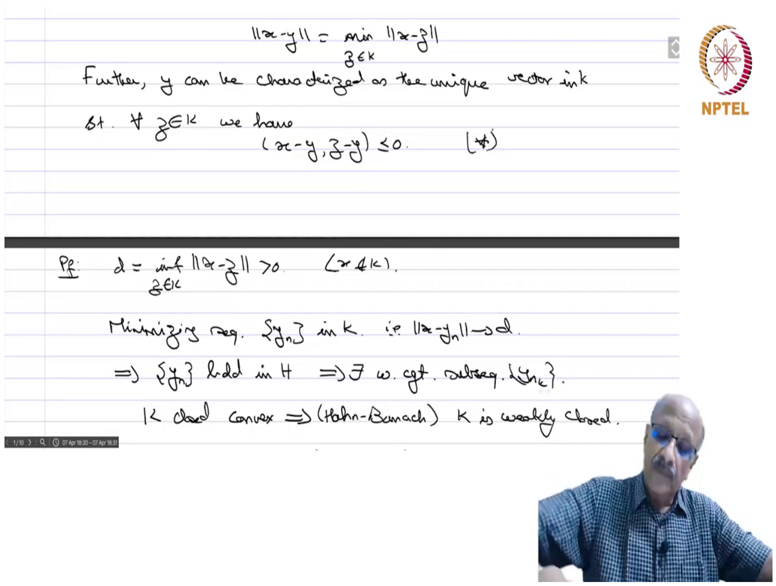
So this is the, one of the definitions of weak lower semi-continuity, namely if $\{y_n\}$ converges to y weakly then the norm of x minus y is less than equal to lim inf of this. But then y belongs to K, therefore

$$d \leq ||x - y|| \leq lim\, inf\, ||x - y_n|| \leq d$$

But that is equal to d because $\{||y_n - x||\} \rightarrow d$, and therefore this equal to d, and therefore this implies that $||x - y|| = d$. Therefore, we have established the existence of a closest point.

(Refer Slide Time: 08:00)



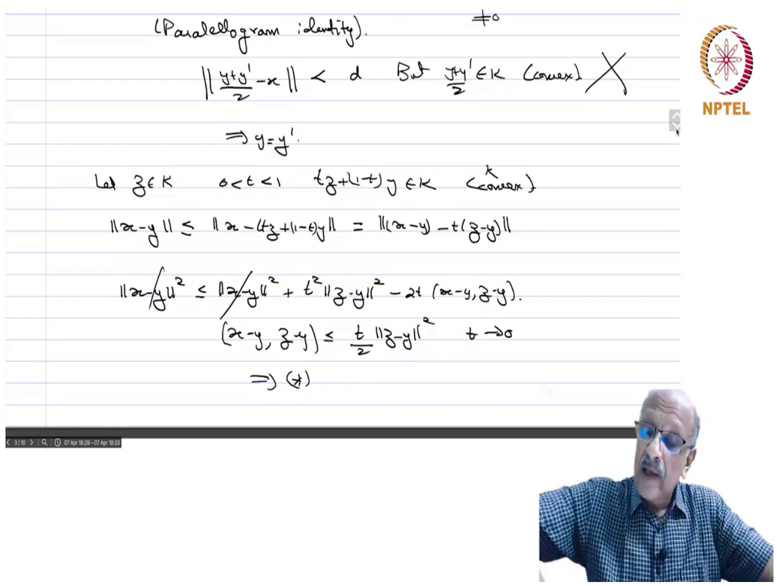


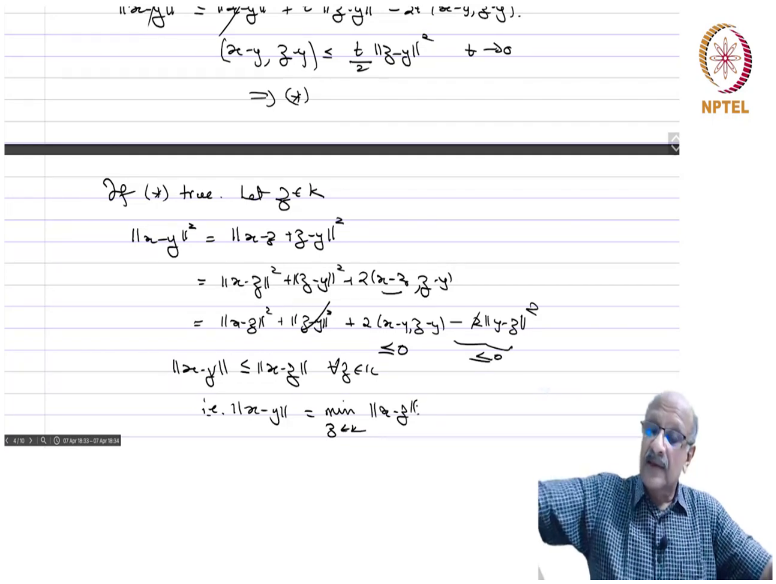
Now you, if you have two such points, so if y and $y^{'}$ satisfy norm of x minus y equals norm of $x - y^{'}$ equal to d, y not equal to $y^{'}$, $y^{'}$ in K, then what happens is you take norm of y plus $y^{'}$ by 2 minus x is less than or equal to, is equal to one half of norm of y minus x square plus one half of norm of $y^{'}$ minus x squared minus norm of y minus $y^{'}$ by 2 whole squares. So this is nothing but the parallelogram law, parallelogram identity.

Now this is d, this is d. So norm of $y + y'$ by 2 minus x is strictly less than whole, sorry, square, square is less than equal to d square since this y minus y' is not 0, and therefore you have that norm of $y + y'$ is strictly less than d, but $y + y'$ by 2 belongs to K since it is convex, and that is a contradiction because you have an element which is less than d from x in K. But that is not possible because d is in fact the infimum.

Therefore, you have that, so this implies that $y + y'$. So we have a unique solution. So now we will show that this can be characterized. Namely, if you have a minimum point, the point which is closest then it will satisfy the inequality which is given here. Let me call this star. So we have to show the star is also equivalent to saying the norm of x minus y is the minimum of norm of x minus z.

(Refer Slide Time: 10:30)





So let $z \in K$. And, so then you have

for $0 < t < 1$, we have by the convexity of K that $tz + (1 - t)y \in K$.

Again, we are using the convexity, K convex. So then

$$||x - y|| \leq ||x - tz + (1 - t)y|| = ||x + y + t(- z - y)||$$

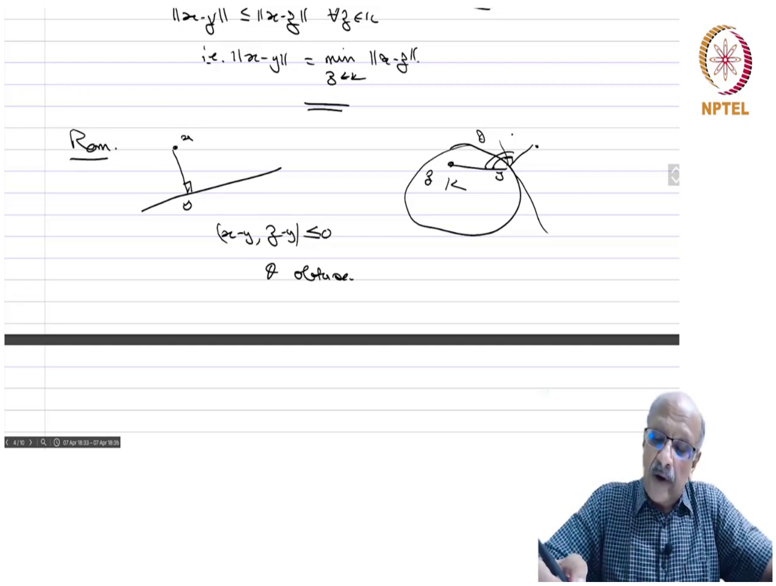
So now you develop this.

So, the norm of x minus y square is less than equal to, if you take the square you have norm of x minus y squared plus t square norm z minus y squared minus 2 t x minus y z minus y. So these two cancel out, you bring the other thing to this side, so you have x minus y, z minus y is less than equal to t by 2 times norm z minus y square. And now you let it go to 0, so you get a star. Namely x minus y minus y less than equal to 0.

Conversely, if star is true let z belong to K. Then you have norm of x minus y square equals norm of x minus z plus z minus y square, which is equal to x minus z squared plus x minus y square plus 2 times x minus z, z minus y, which is equal to norm of x minus z squared plus norm of z minus y square plus 2 times, now I am going to add and subtract a y in this term.

So, you have 2 times x minus y, z minus y. And then y minus z, z minus y is minus 2 times y minus z square. So this will cancel with this. And therefore, you now have, this is less than equal to 0 is given, this is anyway less than 0. And therefore, the norm of x minus y is less than equal to the norm of x minus z for all z in K. That is, x minus y equals minimum norm of x minus z, z in K. So this characterizes it completely. So this proves the theorem.

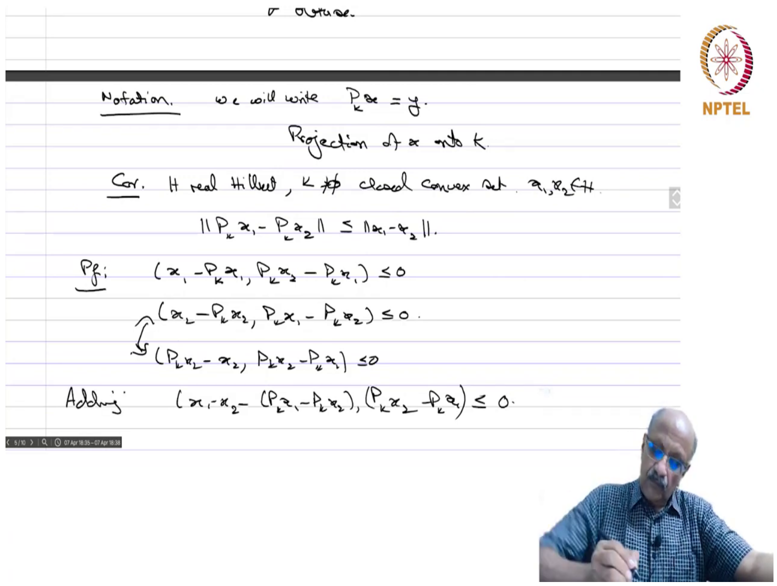
(Refer Slide Time: 13:40)



So, remark. So as I already said if you are in the plane and you have a point and a straight line because it is a closed convex set, then, so if this is x, this will be y where you have the foot of the perpendicular is the thing. More generally if you have a closed convex set K, and you have a point here, so the, so this point y will be the foot of the perpendicular to the tangent at y. And then if you take any z you have this angle.

So, you have x minus y, z minus y less than or equal to 0, that is, theta will be obtuse. So the angle theta is always obtuse. So that is, there is a unique point which for which, it could be a right angle also, but it, and that is the geometric interpretation of this, this particular result.

(Refer Slide Time: 14:50)



So, notation. So we will write P K x, the projection onto K, as, so this is the thing. Now, this is, so projection of x onto K. So now,

Corollary.

You have that x, H real Hilbert K non-empty closed convex set. Then, if $x, y \in H$, remember this is not linear, the projection is non-linear mapping so you have

$$||P_k x_1 - P_K x_2|| \leq ||x_1 - x_2||$$

So this is nice, it is a non-expansive mapping.

Proof. You have that, from the characterization in terms of the inner product, you have$P_k x_1, P_K x_2 \in K$. So x_1 is P_K, $||x_1 - P_K x_1|| \leq 0$. Similarly, x_2 minus $P_K x_2$, $P_K x_1 - P_K x_2$ is less than equal to 0.

So let us add these two. So I want to get $P_K x_2 - P_K x_2$ same, in here also. So I will put a minus sign here and transfer it 1ere. So I will write this. So this, this is same as saying

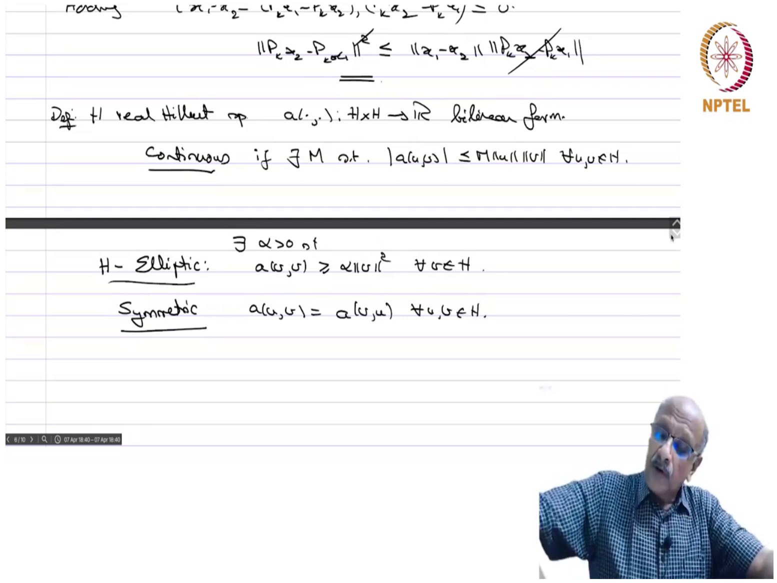
$$< P_K x_2 - x_2, P_K x_2 - P_K x_1 >\leq 0.$$

So this is the same as this. I have just put a minus sign in each of the things. So now let us add these two things. So you get

$$< x_1 - x_2 - P_K x_1 - P_K x_2, P_K x_2 - P_K x_1 > \leq 0.$$

So we are just adding.

(Refer Slide Time: 17:37)

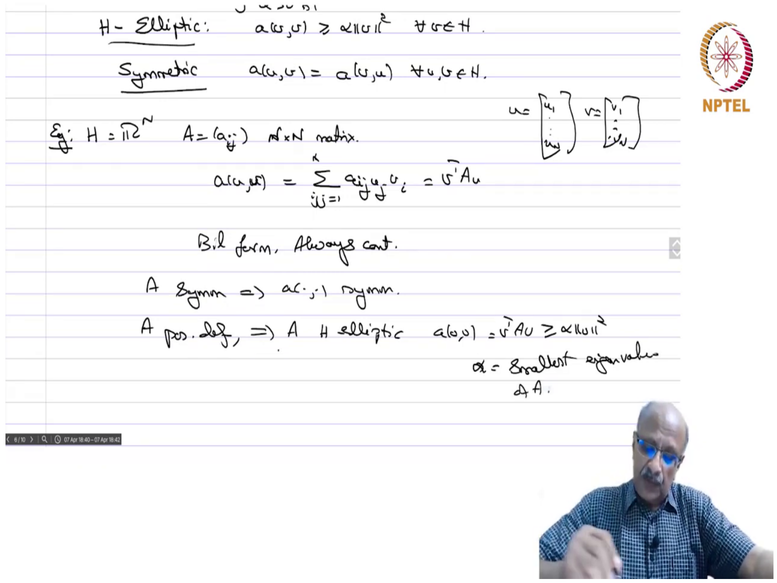


So, from this you get a norm of $P_K x_2 - P_K x_1$ square is less than equal to, by the Cauchy-Schwarz inequality, norm of $< x_1 - x_2 - P_K x_1 - P_K x_2$. Cancel one of them and then you get the result for the thing.

So, so H, so definition. Real Hilbert space and a bilinear form. That means it is linear in each of the variables. So it is said to be continuous if there exists an M such that mod a u v is less than equal to m times norm u norm v for all u, v in H.

It is said to be elliptical. And if you want to say, specify the vector space which you are talking about then you say H is elliptic if a v, v, if there exists an alpha positive such that a v, v is greater than equal to alpha times norm v square for every v in H. So then you say that it is elliptic. It is said to be symmetric, a u v equals a v, u for all u, v in H.

(Refer Slide Time: 19:40)



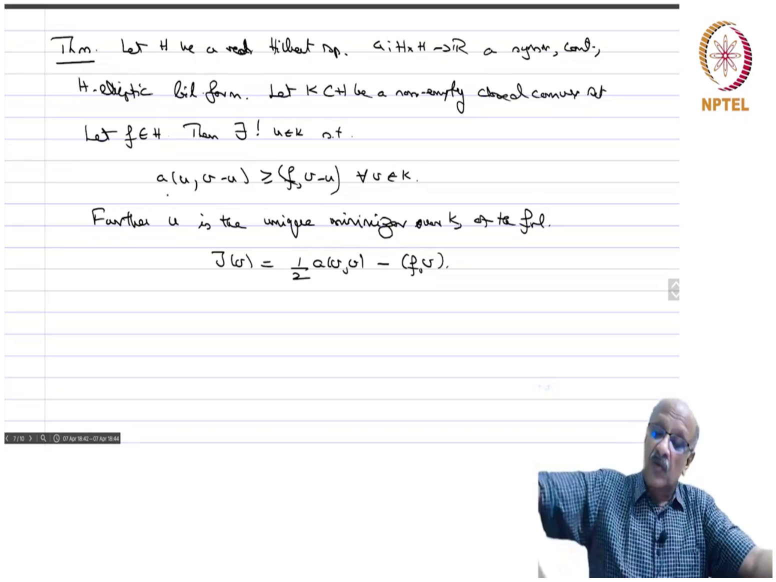
So, if you want, so if you take $H = \mathbb{R}^N$, for example, and you have A to be equal to $A = (a_{ij})_{N\times N}$, then you have a

$$< u, v >= \sum_{i,j=1}^{N} a_{ij} u_i v_j$$

So this is nothing but v transpose a u. So u is the u 1 to u n, and v equals $v_1 \; to \; v_n$, and v transpose is the transpose. So a u, v, sorry. So this is a bilinear form and it is always continuous, because in finite dimensional space everything, anything linear is in fact continuous.

Now, if A is symmetric implies A will be symmetric. If A is positive definite, then you have A will be H elliptic. And we have A v, v equals v transpose A v, which is A v, v, of course, will be in alpha times norm v square where alpha is the smallest eigenvalue of A. So this is just linear algebra. So if it is positive definite then all the eigenvalues will be positive and therefore you have. So this is a typical example of the thing.

(Refer Slide Time: 21:35)



So now we will come across several examples of elliptic bi-linear forms in the future. So now we have a

Theorem.

Let H be a real Hilbert space and, a $H \times H$ to R symmetric continuous H elliptic bi-linear form. Let $K \subset H$ be a non-empty closed convex set. Let $f \in H$. Then there exists a unique $u \in K$ such that

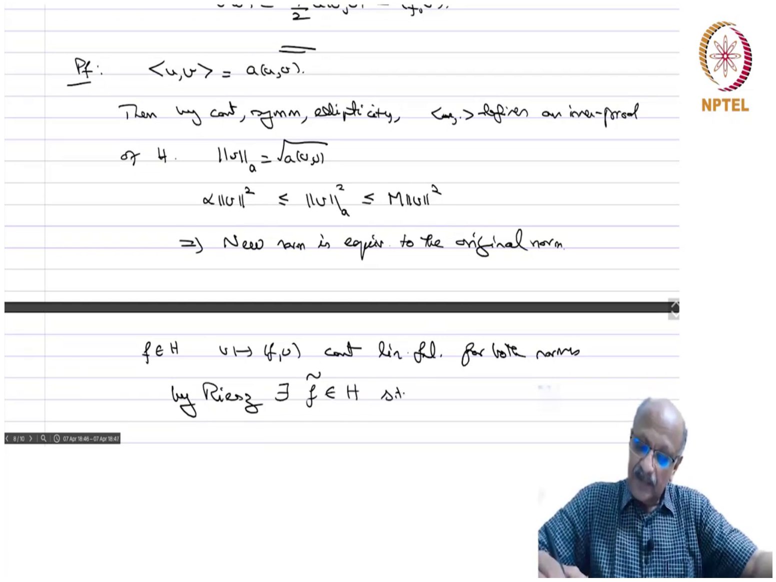
$$a(u, v - u) \geq < f, v - u > \quad \forall v \in K$$

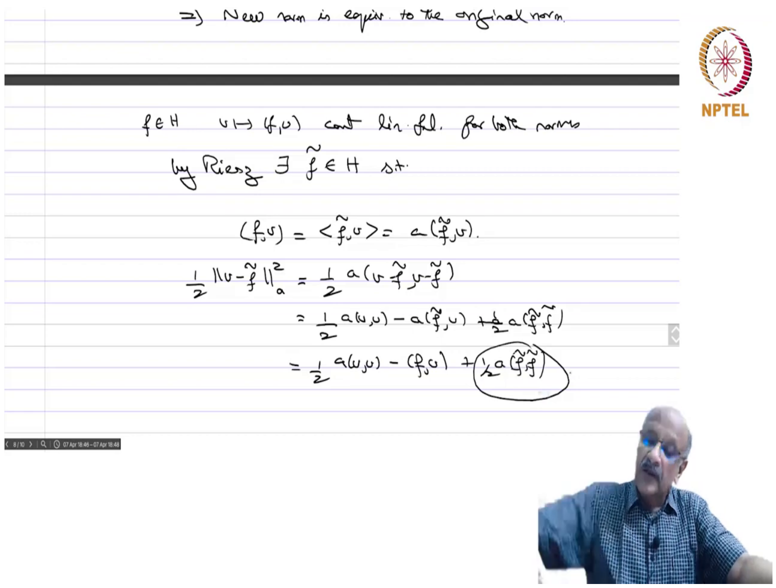
Further, u is the unique minimizer over K of the functional

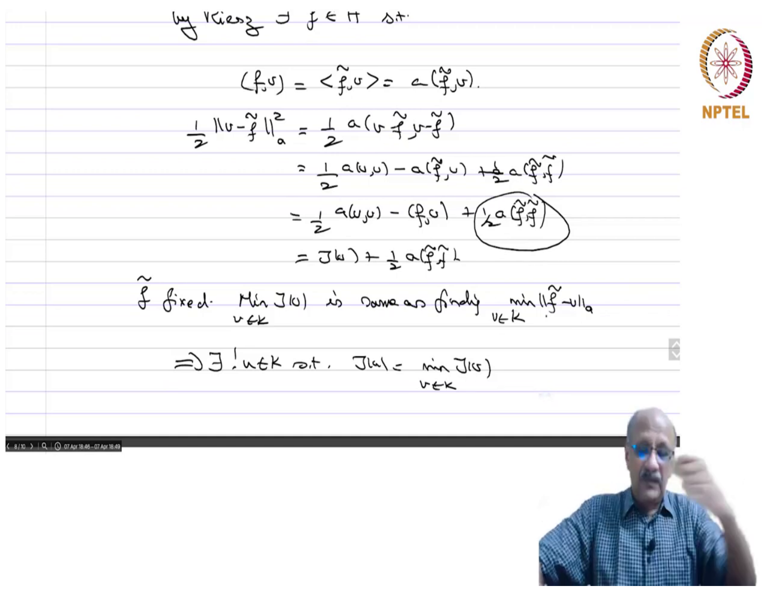
$$J(v) = \frac{1}{2} a(u, v) - < f, v >$$

and then you will get that the minimizer will be characterized by this set of inequalities which is true, which are true for every $\forall v \in K$.

(Refer Slide Time: 23:48)







Proof. So you define a new inner product $< u, v >= a < u, v >$.

Then by continuity, symmetry and ellipticity defines an inner product on H. Also, we have that $||v||^2 = a < v, v >$.. So then you have norm v of a is greater than equal to alpha norm v square, that is, ellipticity. And by continuity this is M times norm v square. So this implies the new norm is equivalent to the original norm. So the space does not change.

So then, if f belongs to H then v going to f v is a continuous linear functional for both norms. So by Riesz representation theorem there exists f tilde in H such that f v equals f tilde v equals a of f tilde v. So we can find such a thing. So now you consider one half of the norm of v minus f tilde in the norm square.

$$\frac{1}{2}||v - f^{\sim}|| = \frac{1}{2}a(v - f^{\sim}, v - f^{\sim}) = \frac{1}{2}a(v, v) - a(f^{\sim}, v) + \frac{1}{2}a(f^{\sim}, f^{\sim})$$

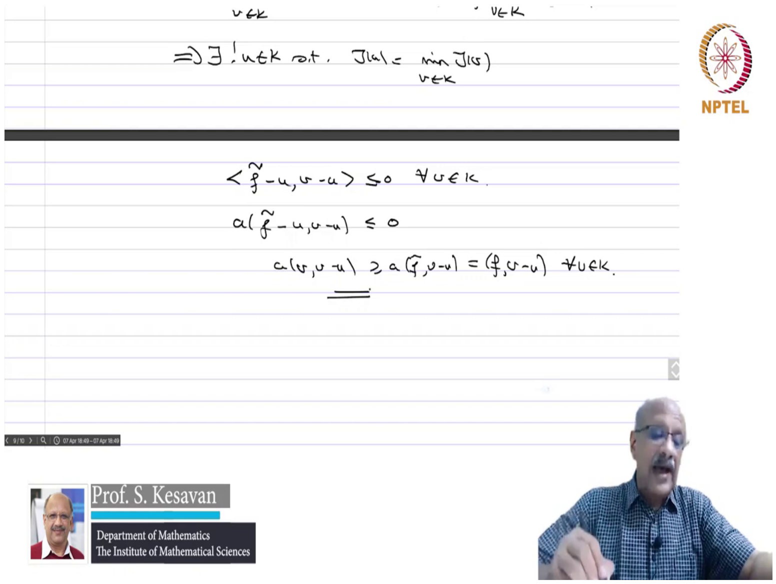
$$\frac{1}{2}a(v, v) - < f\ , v > + \frac{1}{2}a(f^{\sim}, f^{\sim}) = J(v) + \frac{1}{2}a(f^{\sim}, f^{\sim})$$

$f^{\sim}$ is fixed , $Min_{v\in} J(v)$ is same as $Min_{v\in K}||f^{\sim} - v||$

And you know we have already shown that, so implies there exists a unique $u \in K$ such that $J(v) \leq Min_{v \in K} J(v)$

because it is from the minimum distance theorem.

(Refer Slide Time: 28:00)



And then we know that by the characterization, so this is nothing but

$$< f^{\sim} - u, v - u > \leq 0, \;\; v \in K$$

$$\Rightarrow a(f^{\sim} - u, v - u) \leq 0$$

$$\Rightarrow a(v, v - u) \geq a(f^{\sim}, v - u) = (f, v - u), \quad v \in K$$

And therefore, you have that this characterizes the solution of the, the minimizing property of K.

So, we will now see that we can actually relax this condition on the symmetry. The symmetry is not really needed. We needed it if you want to interpret it as the solution of a minimization problem. But still this inequality will always have a solution. So that is the next theorem which we are going to show. We will see that in a moment.