Sobolev Spaces and Partial Differential Equations
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Lecture 6
Exercises – Part 7

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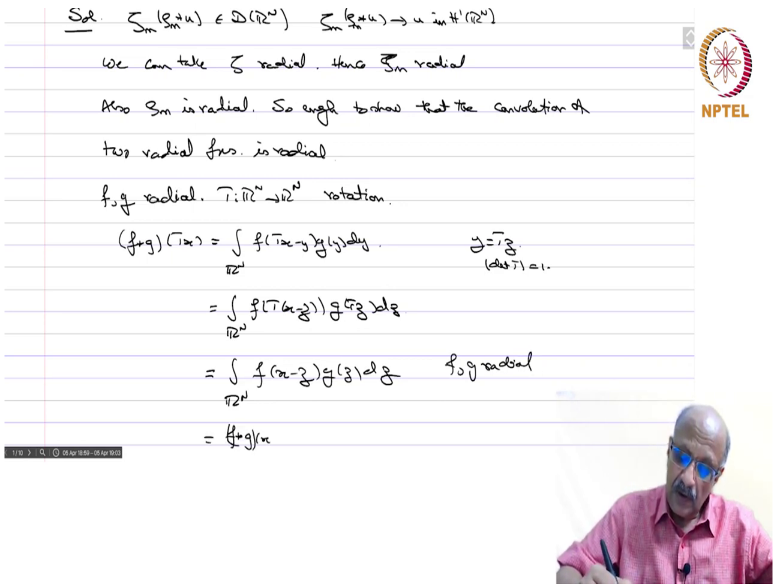

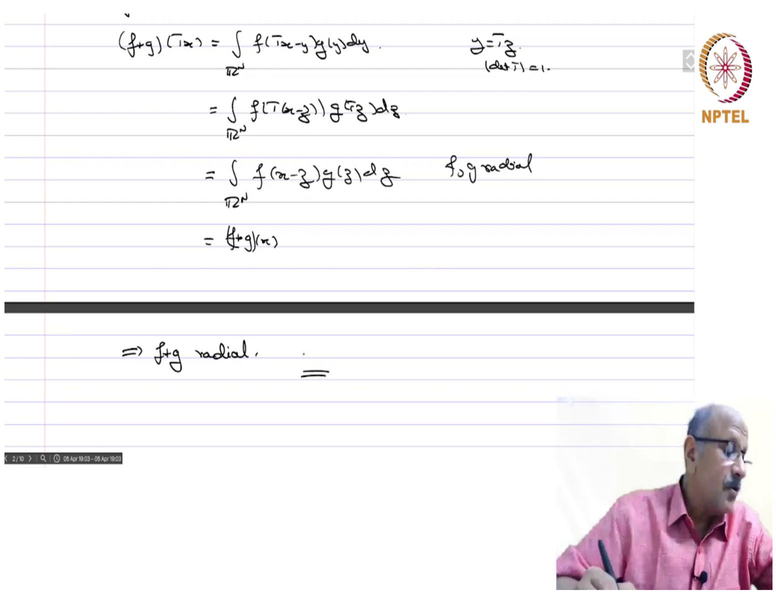

So, we will continue with the exercises. We are now in problem 4, a. So, let

$H^1_{rad}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N)$: u radial $\}$. That means u of x is just a function of the, of mod x it depends only on mod x. So, if u belongs to $H^1_{rad}(\mathbb{R}^N)$ show that there exists u_n in $D_{rad}(\mathbb{R}^N)$ such that u n goes to u in $H^1(\mathbb{R}^N)$. The norm is still the same, this is only a subspace, and therefore you have this.

So, solution. So, we know that zeta m rho m star u belongs to $D(\mathbb{R}^N)$, zeta is the usual function, support of zeta in ball of radius 2, zeta is identically 1 in the radius of 1, 0 less than equal to zeta less than equal to 1, zeta m is zeta of x by m. So, this is the usual standard thing, and zeta m rho m star u converges to u in $H^1(\mathbb{R}^N)$. So, we can take zeta radial, hence zeta m is radial. Also, rho m is radial.

So enough to show that the convolution of two radial functions is radial. So, because, rho m star u will be radial, you are multiplying by another radial function so everything will be radial, and that converges, and that is in $D(\mathbb{R}^N)$ and therefore the whole thing will be true.

So, we, so f ,g radial and T from $D(\mathbb{R}^N)$ rotation. What is radial? Radial means it should be the same when you do any rotation. So,

$$f * g(Tx) = \int_{\mathbb{R}^N} f(T(x - y))g(y)\, dy$$

$$= \int_{\mathbb{R}^N} f(T(x - y))g(T\xi)\, d\xi\ , \quad y = T\xi$$

$$= \int_{\mathbb{R}^N} f(x - \xi)g(\xi)\, d\xi, \ \textit{since } f, g \textit{ are radial.}$$

$$= (f * g)(x)$$

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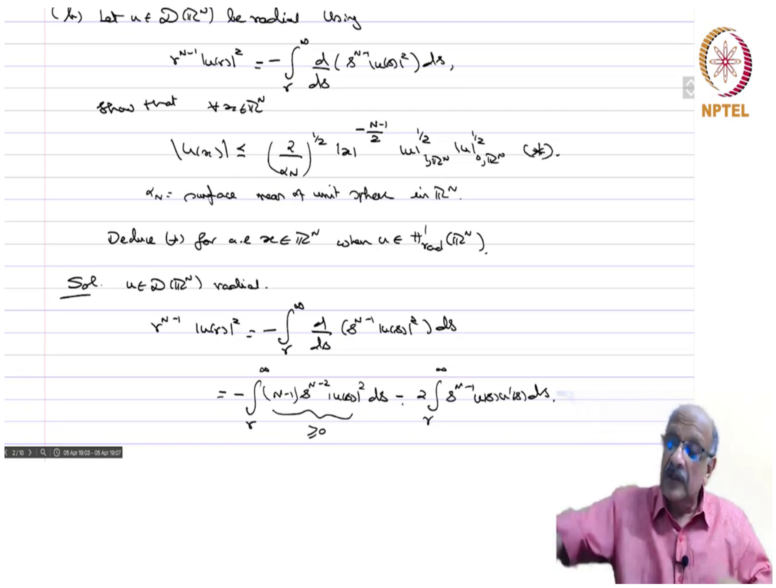

So, **(b)**, let $u \in D(\mathbb{R}^N)$ be radial. Using r power N minus 1 mod u r square equals minus integral r to infinity d by d s, s power N minus 1 u s square d s, show that for all x in $\mathbb{R}^N$ you have u of x in absolute value is less than equal to 2 by alpha N power half mod x power minus of n minus 1 by 2 mod u power half 1 $\mathbb{R}^N$ mod u 0 of $\mathbb{R}^N$. Alpha n equal surface measure of unit sphere in $\mathbb{R}^N$. Deduce star for almost every x in $\mathbb{R}^N$ when u belongs to $H^1_{rad}(\mathbb{R}^N)$. So, this is star.

So, $u \in H^1_{rad}(\mathbb{R}^N)$., and you have r power N minus 1 mod u r square equals minus integral r to infinity d by d s, s power N minus 1 mod u s square d s. Because this is just a integral of a derivative and therefore its minus N values, u is in $D(\mathbb{R}^N)$ so at infinity will give you nothing. With a minus sign, you get exactly the lower limit and that is equal to that.

So that is equal to minus integral over r, so I am going to just differentiate that. So, N minus 1 s to the N minus 2 mod u s square d s minus, so integral r to infinity minus 2 times integral r to infinity s to the n minus 1 u of s u dash of s d s.

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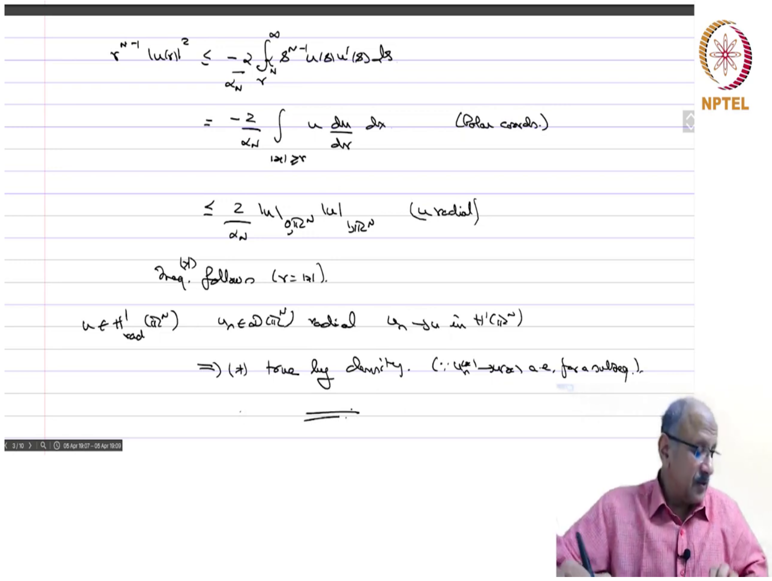

Now this integrand is non-negative, everything is between r to infinity, everything is non-negative and therefore the integral will be non-negative and consequently you have r power N minus 1 mod u r square is less than or equal to minus 2 integral r to infinity s to the N minus 1 u s u dash s d s.

Now if i had an alpha N here, so let me put an alpha N, alpha N. Now if I put an alpha N s power N minus 1 into a function like this, this is nothing but the integral of the radial function. So, this equal to minus 2 by alpha N integral mod x greater than equal to r u d u by d r d x. This is just polar coordinates, the integration of radial function.

And therefore 2 a b, so that is by the Cauchy-Schwartz inequality, less than or equal to 2 by alpha N mod u 0, in this domain I can say less than equal to 0 $\mathbb{R}^N$ mod u_1 $\mathbb{R}^N$ because again u radial, d u by d r, modulus of that will give you modulus of the gradient also. So, and therefore the inequality now follows because you just take, so, so inequality follows. r equals mod x. So, you should just put that then inequality star follows.

So, if u is in H^1 rad $\mathbb{R}^N$ then you have $u_n \in D(\mathbb{R}^N)$ radial and u n converging to $u \in H^1(\mathbb{R}^N)$ implies star true by density. And also, since

$u_n \to u \Rightarrow u_n(x) - u(x)$ almost everywhere for, for a sub sequence, so hence this result follows.

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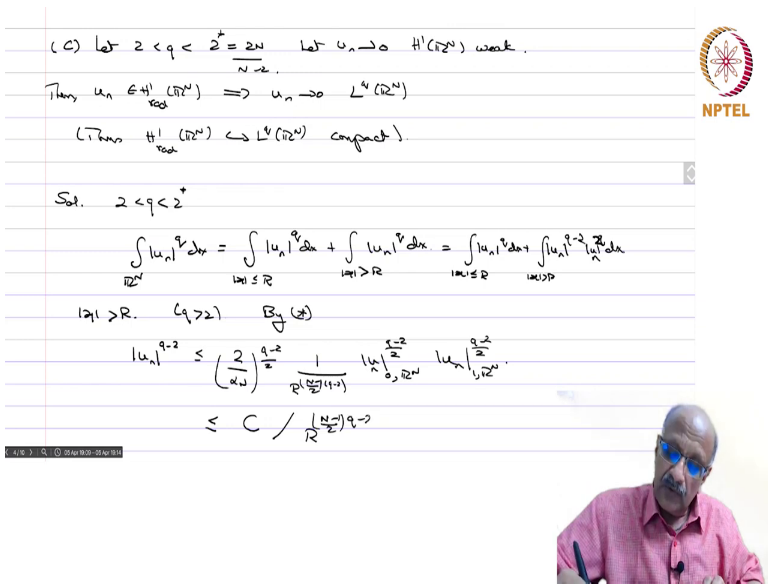

c, let 2 less than q less than 2 star which is 2N by N minus 2. Let $u_n \to 0$ in $H^1(\mathbb{R}^N)$ weak. $u_n \in H^1_{rad}(\mathbb{R}^N)$ will then imply that $u_n \to 0$ in $L^q(\mathbb{R}^N)$. So, this is, then, so this is what, so thus you have $H^1_{rad}(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$ is compact. So, you see this case, even though $\mathbb{R}^N$ is an unbounded domain and in general you do not have Rayleigh theorem, if you reduce it to radial functions then it becomes a compact thing because radial function means it essentially depends only in one dimension.

And therefore, so if you have radial functions then if you restrict your attention then $H^1_{rad}(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$ becomes compact. So, this is a nice observation. So, let us take, so, solution. So, 2 less than q less than 2 star. So, you take integral over $\mathbb{R}^N$ mod u_n

q d x. That is equal to integral mod x less than equal to some r, which we will choose. Mod u n q d x plus integral mod x greater than r mod u n q d x.

So let us take mod x greater than R. And you have q is bigger than 2, therefore mod u n to the q minus 2, so this I will write as integral mod x less than equal to r mod u n q d x plus integral mod x greater than r mod u n q minus 2 mod u power q, u n power q d x. 2, sorry, not q, 2.

So mod u n 2 minus q is less than equal to, by star, so we are using the inequality star, this is almost every x, so this you have less or equal to 2 by alpha N to the power of q minus 2 by 2 into 1 over capital R, mod x is bigger than R so, here in the denominator, so this is less than R power N minus 1 by 2 into q minus 2.

And then you have mod u q minus 2 by 2 0 $\mathbb{R}^N$ mod u n q minus 2 by 2 1 $\mathbb{R}^N$, which is less than equal to, so mod u N, it converges weekly so it is bounded in $H^1(\mathbb{R}^N)$ and therefore, this is, all this will be less than C, it is in, L2 norm will also be less than equal to C. This two, all this stuff I am going to put it in one thing, so this is less than C divided by R to the power of N minus 1 by 2 into q minus 2.

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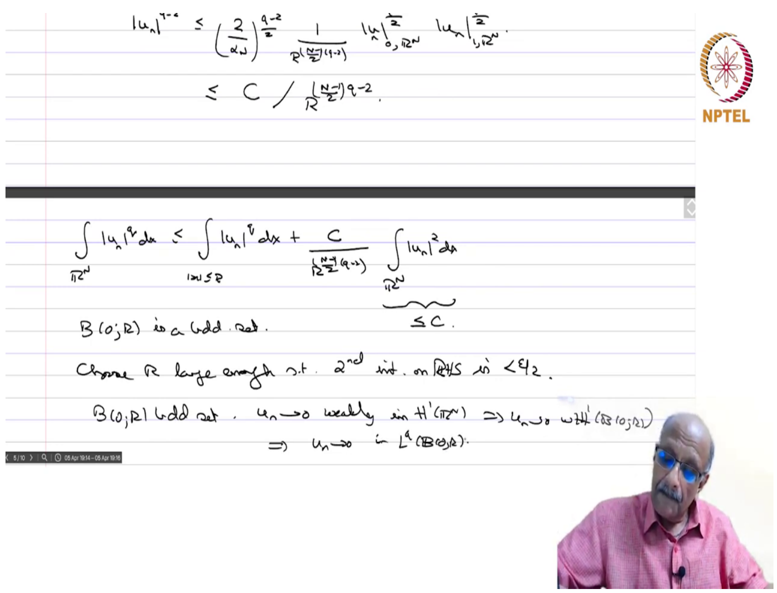

So integral on $\mathbb{R}^N$ mod u_n to the q d x is less than or equal to integral mod x less than or equal to R mod u_n q d x plus C by R to the N minus 1 by 2 into q minus 2. So, q is, so we are having all that. And then integral over $\mathbb{R}^N$, I can say mod u_n square d x, and again this is bounded by some constant because u_n is bounded in H1 and therefore in L2 also.

Now B 0, R is a bounded set. So, choose R large enough such that second integral on RHS is less than epsilon by 2. Now $B(0, r)$ is a bounded set, so u_n goes to 0 weekly in $H^1(\mathbb{R}^N)$, that is what is given, right, in $H^1(\mathbb{R}^N)$. And therefore, it will go, implies u_n goes to 0 weekly in $L^q(B(0, r))$ also. It is true for anything because that is contained in the other one. And, and therefore you have that u_n goes to 0 in, in $L^q(B(0, r))$.

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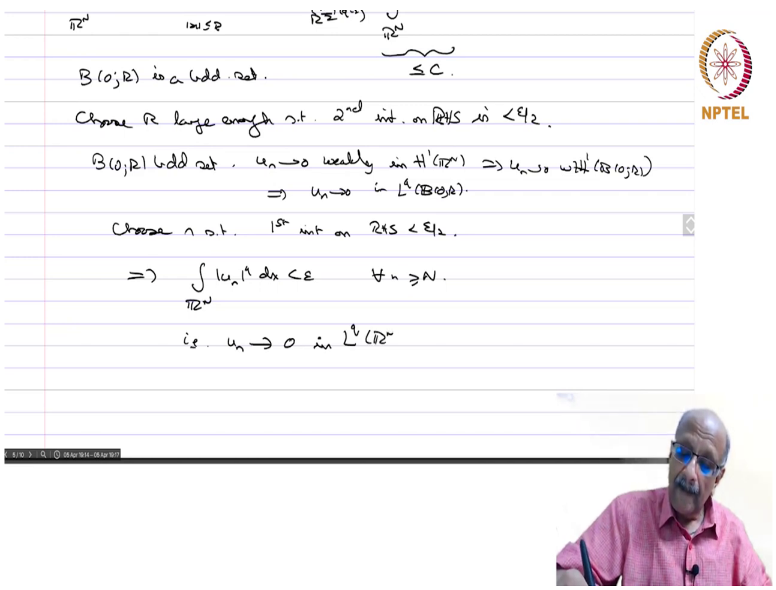

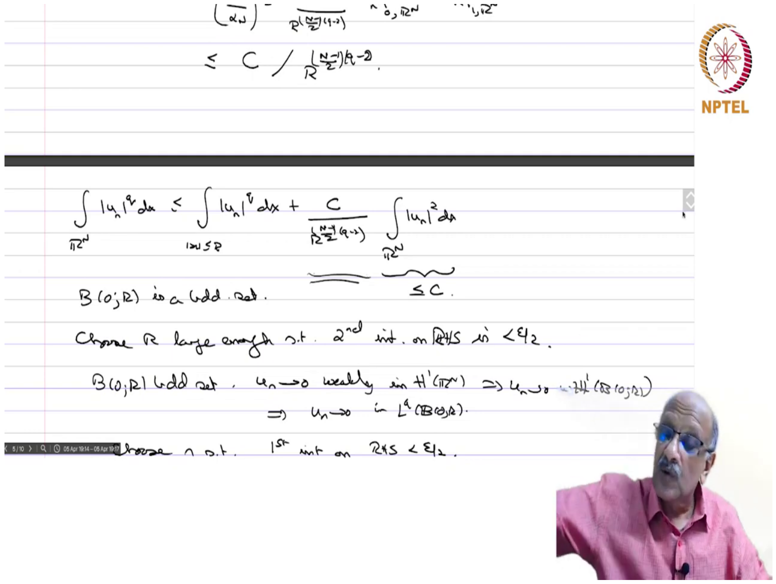

So, choose n such that first integral on RHS is less than epsilon by 2. So, this implies integral over $\mathbb{R}^N$ mod u_n q the x is less than epsilon for all n greater equal to capital N, and that is u_n goes to 0 in L^q.

So, this will not work for q equal to 2 because you have a q minus 2 factors here, and if you want R, this integral, this to be made small this term, and therefore you will need that q is bigger than 2. So, it will not work for q equal to 2.

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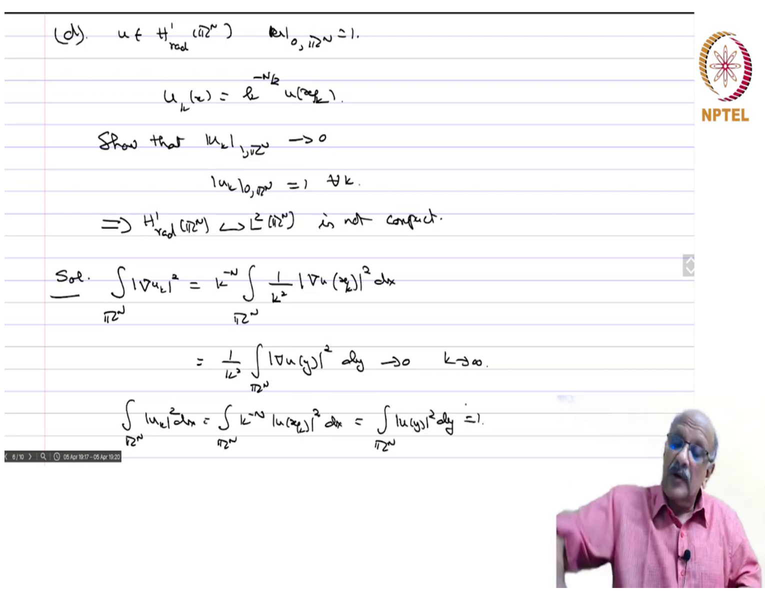

So that is the part d of this exercise. So, u in $H^1_{rad}(\mathbb{R}^N)$ 0, such that mod ||u|| $\mathbb{R}^N$ equal to 1. And you define $u_k(x)$ equals k power minus N by 2 u of x over k. Then show that mod u_k 1 $\mathbb{R}^N$ goes to 0 and mod u_k 0 $\mathbb{R}^N$ equal to 1 for all k. And that will imply that rad $H^1_{rad}(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ is not compact because this u_k will go weakly to 0 in H^1_{rad} but then it will not go in L^2_{rad} because its norm is completely 1.

So, solution. So, integral on $\mathbb{R}^N$ mod grad u_k square equals k power minus n of integral $\mathbb{R}^N$. So, I am going to define, differentiate so I will get 1 by k square mod grad u_k by k square d x. So, d x, k power minus N, if you put x by k equal to y, this will give you 1 by k square integral $\mathbb{R}^N$ mod grad u y square d y. And that goes to 0 as k tends to infinity.

And integral over $\mathbb{R}^N$ mod u_k square d x equals integral over $\mathbb{R}^N$ k power minus N mod $u(x/k)$ square d y d x, and that is equal to integral over $\mathbb{R}^N$ mod u y square d y, and that is equal to 1.

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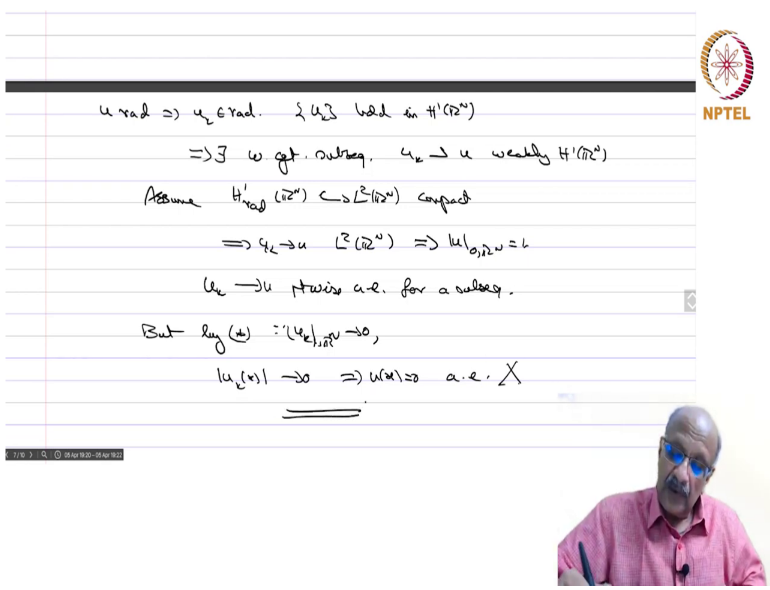

So now if u is radial, this implies u_k is also radial. And u k is bounded in $H^1(\mathbb{R}^N)$ because its 1 norm goes to 0 and the L^2 norm is constant. So, this is bounded. So, implies there exists weakly convergent subsequence. So, converges to u weakly in $H^1(\mathbb{R}^N)$.

Assume u_k converges to, assume $H^1_{rad}(\mathbb{R}^N)$ into $L^2_{rad}(\mathbb{R}^N)$ is compact. So, this will imply that u_k converges to $u \in L^2(\mathbb{R}^N)$, implies mod u 0 $\mathbb{R}^N$ equal to 1. And u_k converges to u point wise almost everywhere for the sub sequence.

But by star, since u_k goes to 0, we have mod u_k of x goes to 0. And this implies that u of x equal to 0 almost everywhere, and that is a contradiction because u is 0 almost everywhere, mod u 0 $\mathbb{R}^N$ is equal to 1, and that is not possible. So, you have that in even the L^2 thing, and L^2 star both of them are not compact.

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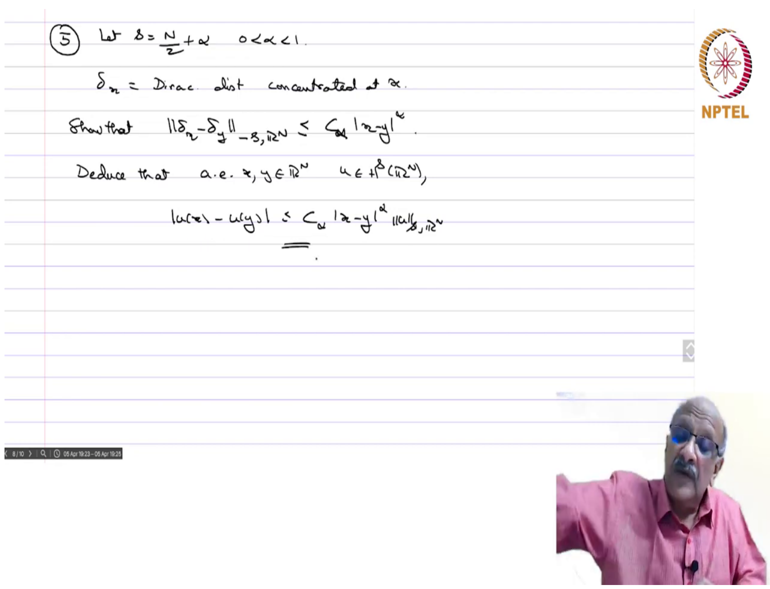

5. Let s equal to N by 2 plus alpha, 0 less than alpha less than 1. So, delta x equals Dirac distribution concentrated at x. Show that norm delta x minus delta y minus s $\mathbb{R}^N$ is less than equal to C N times mod x minus y to the alpha. Deduce that almost every $x, y \in \mathbb{R}^N$ you have, and u is in Hs, Hs of $\mathbb{R}^N$, we have mod u x minus u y is less than equal to C N,

C alpha, so this is, sorry, C alpha, C alpha mod x minus y to the alpha norm u 1 $\mathbb{R}^N$, s $\mathbb{R}^N$, sorry.

So, this tells you that, norm u s $\mathbb{R}^N$, so this tells you that it is Holder, so this is a different proof of the Holder continuity of Hs we know that if s is bigger than N by 2 delta x, del, Dirac distribution belongs to H of minus s.

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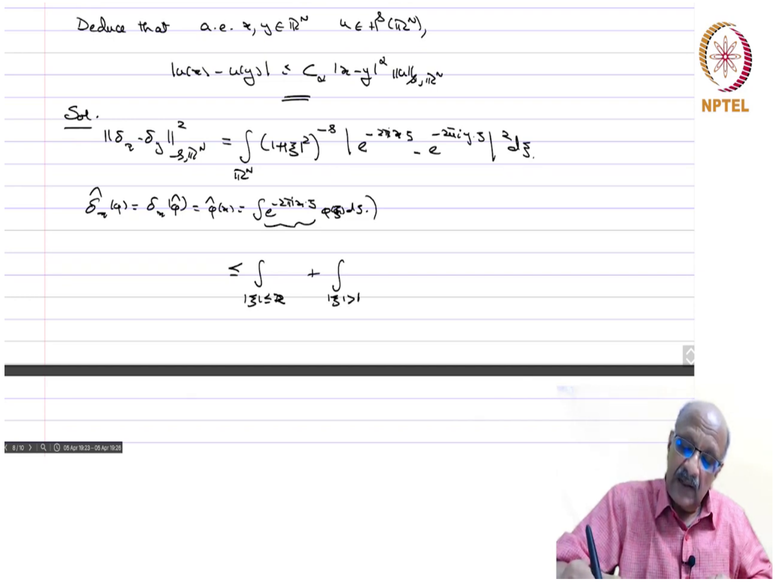

So, the solution. You have norm delta x minus delta y minus s $\mathbb{R}^N$, whole square is equal to integral on $\mathbb{R}^N$ of 1 plus mod xi square to the minus s into mod, so we have e power minus 2 pi i x i minus e power minus 2 pi i y dot psi square d xi.

Why is this? Because delta x hat phi equals delta x of phi hat, and that is equal to phi hat of x, and that is equal to e equal minus 2 pi i x dot psi phi x, phi psi d psi, and therefore you have the delta x hat is nothing but this expression here. So, this is how we have already seen this before. Fine so now we will say this is less than equal to integral over mod xi less than equal to R, and plus integral mod xi greater than R. So, we will split it into two, two parts.

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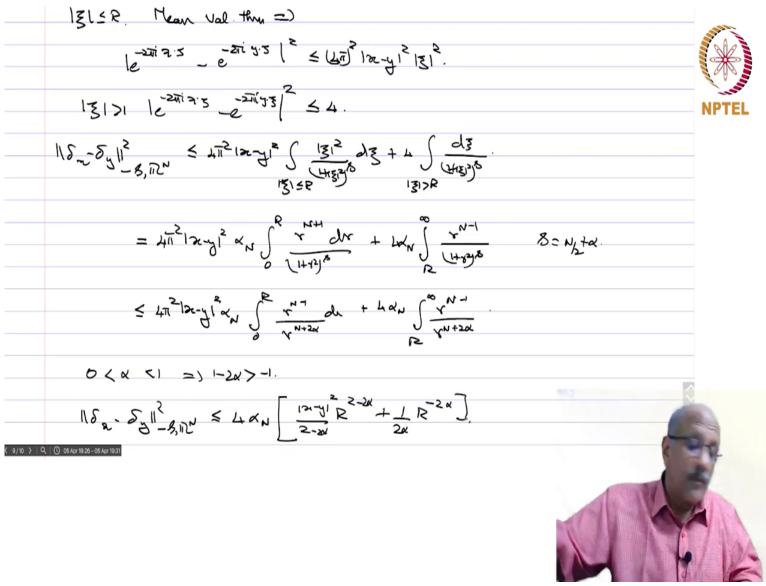

So, $|\xi| \leq R$, we will assume, we will prove mean value theorem implies

$$|e^{-2\pi i x\cdot\xi} - e^{-2\pi i y\cdot\xi}| \leq (2\pi)^2|x - y|^2|\xi|^2$$

It is just the mean value theorem.

$|\xi| > 1,\ |e^{-2\pi i x\cdot\xi} - e^{-2\pi i y\cdot\xi}| \leq 4$

So, we have

$$||\delta_x - \delta_y||^2_{-s,\,\mathbb{R}^N} \leq (2\pi)^2|x - y|^2\alpha_N \int_{|\xi|\leq R} \frac{|\xi|^2}{(1+|\xi|^2)^s} d\xi + 4 \int_{|\xi|>R} \frac{|\xi|^2}{(1+|\xi|^2)^s} d\xi$$

$=$

$$4\pi^2|x - y|^2\alpha_N \int_0^R \frac{r^{N+1}}{(1+r^2)^s} dr + 4\alpha_N \int_{|\xi|>R} \frac{r^{N-1}}{(1+r^2)^s} d\xi,\ s = N/2 + \alpha$$

$0 < \alpha < 1 \Rightarrow 1 - 2\alpha >- 1,$

$$||\delta_x - \delta_y||^2_{-s,\,\mathbb{R}^N} \leq 4\alpha_N\left[\frac{|x-y|}{2-N} R^{2-2\alpha} + \frac{1}{2\alpha} R^{-2N}\right]$$

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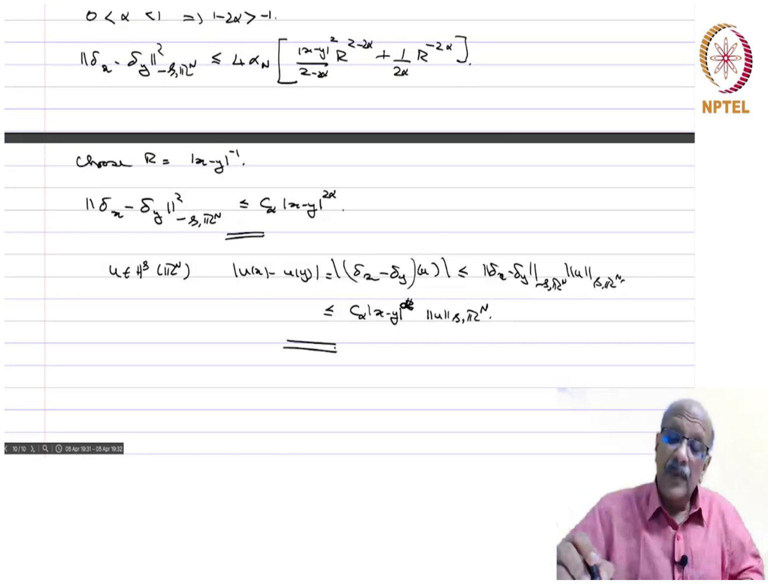

Now choose

$R = |x - y|^{-1}$,

$$||\delta_x - \delta_y||^2_{-s,\,\mathbb{R}^N} \leq C_N |x - y|^{2\alpha}$$

So, now if

$u \in H^s(\mathbb{R}^N)$,

$$|u(x) - u(y)| = |(\delta_x - \delta_y)(u)| \leq ||\delta_x - \delta_y||_{s,\,\mathbb{R}^N} ||u||_{s,\,\mathbb{R}^N} \leq C_\alpha |x - y|^\alpha ||u||_{s,\,\mathbb{R}^N}.$$

So, with this we complete the exercises for this chapter, and this chapter also. So, next time we will take up weak solutions of elliptic boundary value problems.