Sobolev Spaces and Partial Differential Equations
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Lecture 5
Exercises – Part 6

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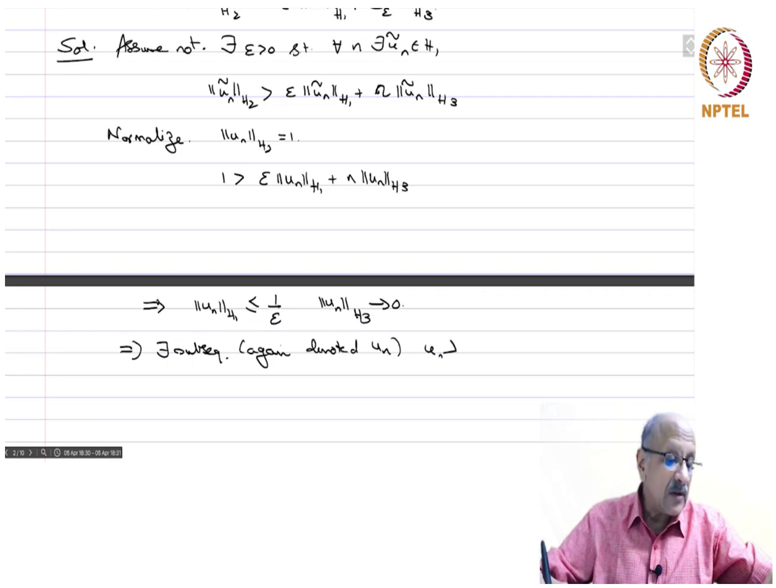

So, now we will do some more exercises. One, so let H_i, i equals 1, 2, 3 be Hilbert spaces such that we have the continuous inclusions H_1, H_2, H_3. Assume that the inclusion H_1 *in* H_2 is compact. a, show that for every epsilon positive, let us, this itself is a. There exists c epsilon greater than 0 such that for every u in H_1, we have norm u in H_2 is less than equal to epsilon times norm u in $H_2$1 plus c epsilon times norm u in H_3.

So, if $u \in H_1$ then it is in on in all the three spaces H_1, H_2, H_3. What we are saying is we can estimate norm of u in the middle space H_2 by epsilon times norm u in H_1. So, we are putting a very small coefficient for H_1 but then we pay a price, c epsilon will be somewhat large for H_3. So, this is the idea.

So, solution. Assume not, then there exists an epsilon positive such that for every n you have there exists a u tilde n in H_1 which violates this with the constant n. So, however bigger the constant n you put for c epsilon it will still violate. So, norm u_n tilde of H_2 is bigger than epsilon times norm u n tilde H_1 plus n times norm u n tilde H_2.

So, we can normalize and so you can assume that norm u_n H_2 equal to 1, and then therefore you have 1 is bigger than, so you divide throughout by norm u_n tilde in H_2. So, that will give you a new vector u n. So, that is u n by norm u n tilde, u n tilde by norm u n tilde H_2, that will have norm 1 in H_2. So, 1 is bigger than epsilon times norm $u_n \in H_1$ plus n times norm $u_n \in H_3$.

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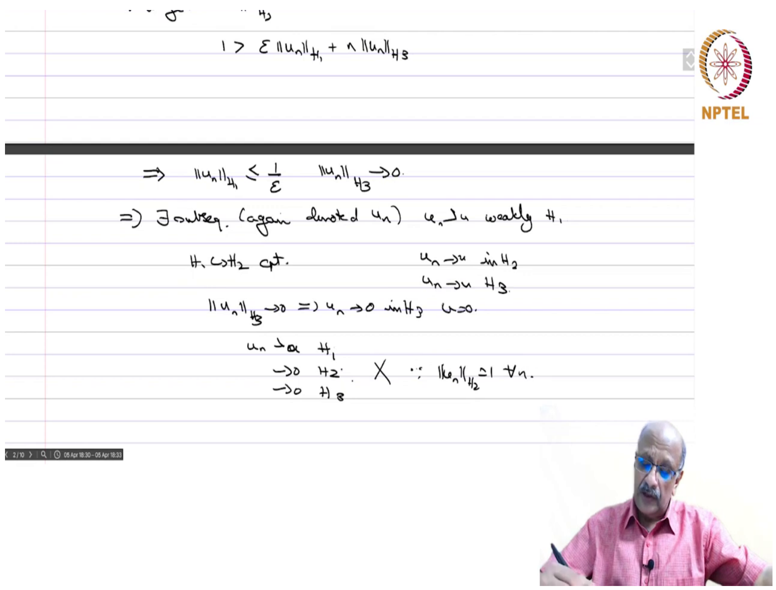

So, this will imply that norm u n in H 1 is less than 1 by epsilon. So, it is bounded, and if you want, and then norm u n in H 3 it goes to 0 because it is less than 1 by n and therefore it goes to 0.

So, you have, we are in the Hilbert space, we are in a bounded sequence in the Hilbert space, so there exists a subsequence again, denoted u n, I have done this often, so I am going to work with that subsequence, so I do not want to put the u and k, and so on because I am not interested in the original sequence anymore. So, again I define it as u n such that u n converges to u weakly in H_1

But then $H_1 to\ H_2$ is compact, so $H_1 to\ H_2$ compact, so weak convergence becomes strong, so u n goes to u in H_2 and therefore obviously u n will go to u in H_3. But norm u n goes to 0 in H_3, and this implies, norm u n H_3 goes to 0, so implies u n goes to 0 in H_3 and therefore u equal to 0. So, u n goes to, weakly to u in H_1, weakly to 0 in H_1, in norm in H_2 and also in norm in H 3. But this is a contradiction since norm u n 2 is 1 for all n. So, it cannot go to 0 in norm. So, this is a contradiction, and therefore we have that the inequality is valid.

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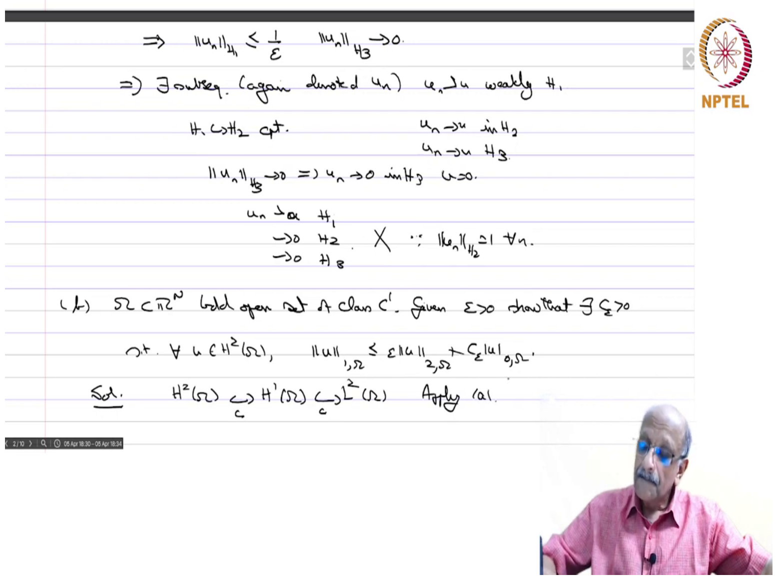

So, **(b),** $\Omega \subset \mathbb{R}^N$ bounded open set of class C^1 given epsilon positive. Show that there exists a c epsilon which is positive such that for every u in $H_2(\Omega)$, we have norm u in 1 omega is less than epsilon times norm u in 2 omega plus c epsilon times mod u in 0 omega.

Solution is one line. So, you have $H_2(\Omega)$ is continuously included in $H_1(\Omega)$ and that is completely in $L^2(\Omega)$. And then we have seen by Rayleigh (())(07:21) in fact both these inclusions are compact. This one is compact by Rayleigh's theorem, and this and this by iteration we said that every W m plus H^m plus 1 will be in H^m compact, so apply a.

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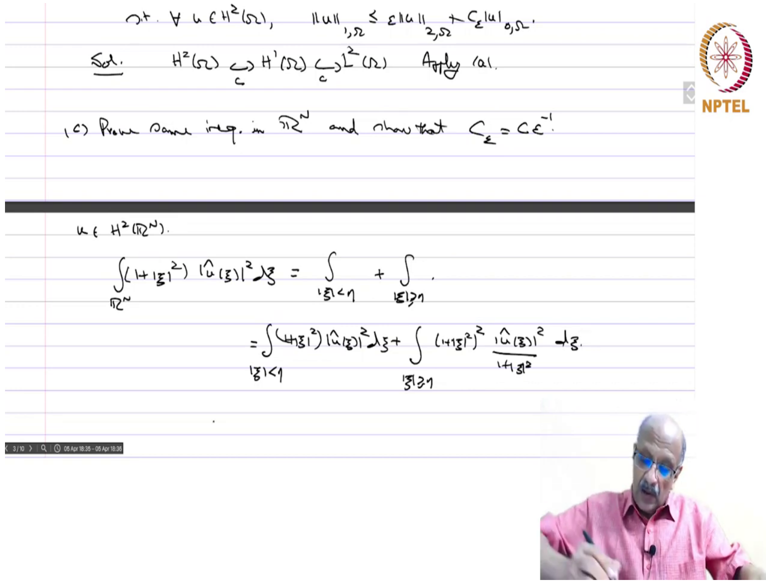

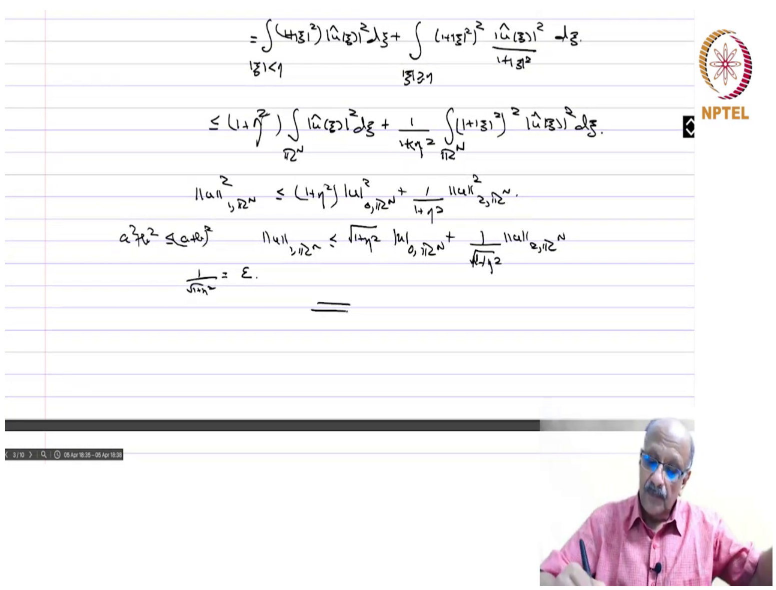

Then c, prove same inequality in $\mathbb{R}^N$, so now no longer you can use the previous theorem because you do not have compact inclusions, and that, and show that c epsilon can be taken as c times epsilon power minus 1.

So, let u belong to $H^2(\mathbb{R}^N)$. And then we want to look at the H^1 norm. So, integral on $\mathbb{R}^N$ 1 plus mod xi square, we are going to use the Fourier interpretation, mod u hat xi square d xi equal to integral some mod xi less than eta plus integral mod xi greater than or equal to eta.

So that is, equal to integral mod xi less than eta of 1 plus mod xi square of mod u cap xi square d xi plus integral mod xi greater than or equal to eta 1 plus mod xi square whole square into mod u cap xi square by 1 plus mod xi square d xi, which is less than or equal to, mod xi is less than eta so 1 plus mod eta square into integral over $\mathbb{R}^N$ mod u cap xi square, so we have got the L^2 norm here.

Plus, again mod xi is bigger than equal to eta and therefore it is in the denominator 1 by 1 plus eta square, mod eta square if you like, but eta is non-negative so the modulus is not necessary. So, here also let me just write eta square. 1 plus eta square integral on, now I am going to write integral on $\mathbb{R}^N$ because I am going to, mod psi greater than equal to eta is what I have to write, I am going to write the bigger integral 1 plus mod xi square, whole square mod u cap xi square d xi.

So, you get that norm u square 1 $\mathbb{R}^N$ is less than or equal to 1 plus eta square mod u square 0 $\mathbb{R}^N$ plus 1 by 1 plus eta square integral, I am sorry, norm u square 2 $\mathbb{R}^N$. So, a square plus b square less than equal to a plus b whole square and therefore you have norm of u 1 $\mathbb{R}^N$ less than equal to root of 1 plus eta square mod u 0 $\mathbb{R}^N$ plus 1 by root of 1 plus eta square norm u 2 $\mathbb{R}^N$. And then you take 1 by root of 1 plus eta square equal to epsilon. And then you are done.

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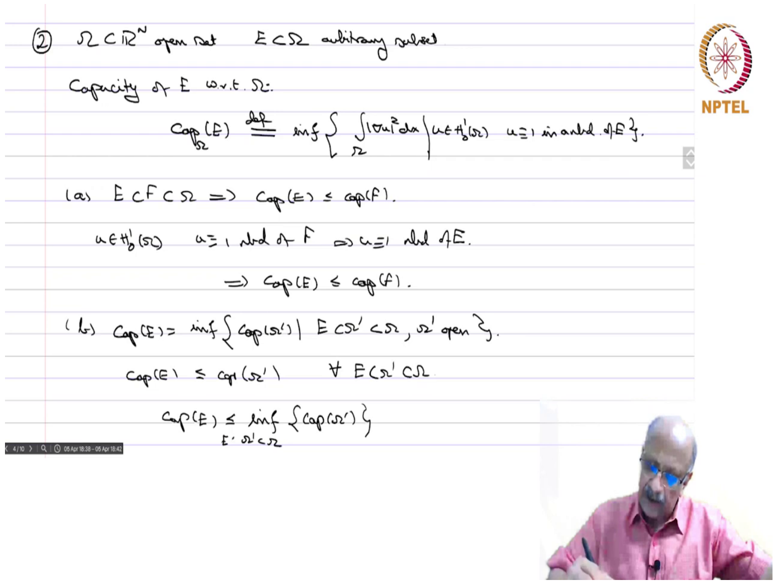

Second one. So, omega contained in $\mathbb{R}^N$ open set and E contained in omega arbitrary subset. So, we define capacity of E with respect to omega. So, we write this as cap of E, so you should put an omega here, cap omega of E but we may, I may drop it when I am talking only of, so this is the definition. inf of integral over omega mod grad u square d x, where u is in H1 0 of omega and u identically 1 in a neighborhood of u.

So, this is called the capacity, sometimes called the electrostatic capacity because this comes from the electrostatic, theory of electrostatics and you in fact these functions will come there so there is a condenser in the center, the E will represent a certain condenser and then around it you have an electrical field, and this is the kind of functions which we will study there.

So, a, if E is contained in F is contained in omega, this implies cap, so I will drop the omega now, so cap E is less than equal to cap F. So, this obvious, if u is in H1 0 of omega, u identically 1 neighborhood of F implies u is identically 1 in a neighborhood of u. And therefore, you are taking the infimum of a bigger set for E, so the infimum will be smaller and therefore this implies that cap E is less than equal to cap F. This from the notion of an infimum.

Now, cap E equals inf of cap omega prime such that E is contained in omega prime contained in omega, omega prime open. So, obviously cap E is, is less than equal to cap omega prime for all E contained in omega prime contained in omega, and therefore cap E is less than or equal to infimum of cap omega prime, omega prime contained in omega, E contained in omega prime So one, one inequality is clear. So, we have to show the reverse inequality.

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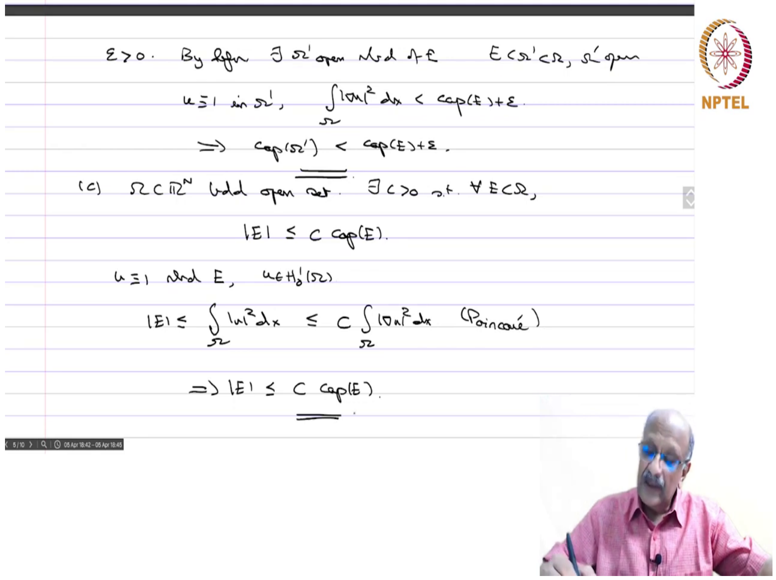

So, let epsilon be greater than 0. So, by definition there exists omega prime open neighborhood of E, so E is contained in omega prime contained in omega, omega prime open, such that integral u is identically 1 in omega prime and integral of omega mod grad u square d x is less than cap E plus epsilon because that is the definition of the infimum. So, the E, cap E is infimum. So, we can find one which you can do like this.

So, now because u is 1 in omega prime, that is the neighborhood of omega prime, its omega prime is an open set and therefore this implies that cap of omega prime which is the infimum of such functions is less than cap E plus epsilon. And that proves the result. So, we have that in fact the infimum is equal to the whole thing.

c, omega in $\mathbb{R}^N$, bounded open set, there exists a c greater than 0, such that for all E contained in omega, you have measure of E this is the Lebesgue measure of E is less than equal to c times capacity of E. So, this capacity is something like the measure. It is comparable to the measure. So, it is another kind of measure of a set. How big a set is can be found in a different way.

So, we have u identically 1 in the neighborhood of E, u in $H^1_0(\Omega)$. Let us take any such function. So, then you have mod E is less than equal to integral on omega mod u square d x because mod u square over omega is greater than or equal to mod u square over E and then it is identical in a neighborhood so it is C1, and so that is equal to mod E. So, this is trivial.

And now by the Poincare inequality this is less than equal c mod grad u square d x and this is nothing but Poincare inequality. So, this is true for any such u. So, you take the infimum so mod u is less than equal to c times capacity of u. So, that these are some elementary facts about the capacity.

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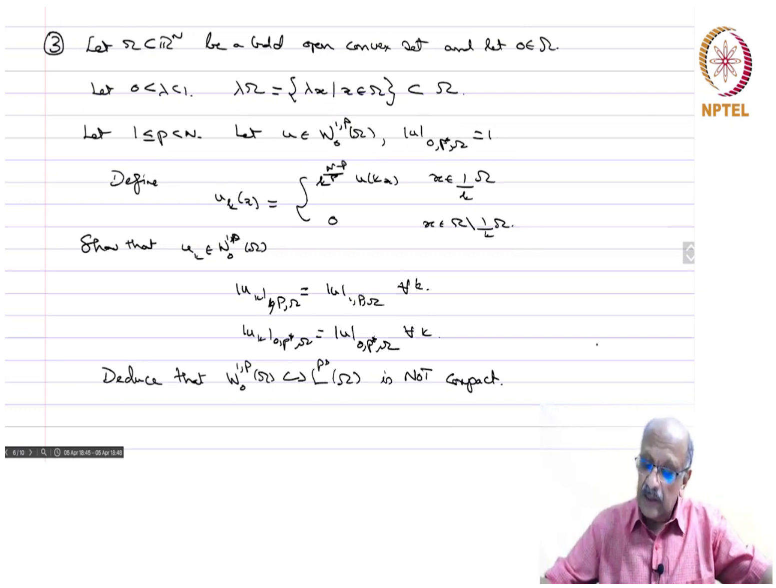

(3), so let $\Omega \subset \mathbb{R}^N$ be a bounded open convex set, and let $0 \in \Omega$. Let $0 \leq \lambda \leq N$, then define lambda omega equals set of all lambda x, x is in omega. So, 0 is in omega and any x is in omega, so lambda x for all 0 to lambda, it is, will be in omega itself, so this is contained in omega. Let 1 less than equal to p less than N. Let u belong to $W^{1,p}(\Omega)$ mod u 0 p star omega equal to 1.

So, we choose such a function. You can always find, you can find any u which is non-zero divided by the L^p star norm because $W^{1,p}$ is contained in L^p star and that will have norm L^p star 1. So, define

$$u_k(x) = k^{\frac{N-p}{p}} u(kx) \quad if\ x \in \frac{1}{k}\Omega$$

$$= 0 \qquad if\ x \in \Omega \backslash \frac{1}{k}\Omega$$

Then show that $u_k \in W^{1,p}{}_0(\Omega)$, $||u_k||_{W^{1,p}{}_0(\Omega)} = ||u||_{W^{1,p}{}_0(\Omega)}$, for all k and mod u_k 0 p star omega equals mod u 0 p star omega for all k. Deduce that $W^{1,p}{}_0(\Omega)$ included in L^p star omega is not compact. So, here we have an example, explicit example of a domain which is a convex open set containing the origin where $W^{1,p}{}_0(\Omega)$ and hence $W^{1,p}$ (Ω) also in L^p star is not compact. We saw in the (())(20:56) theorem the LP star was excluded.

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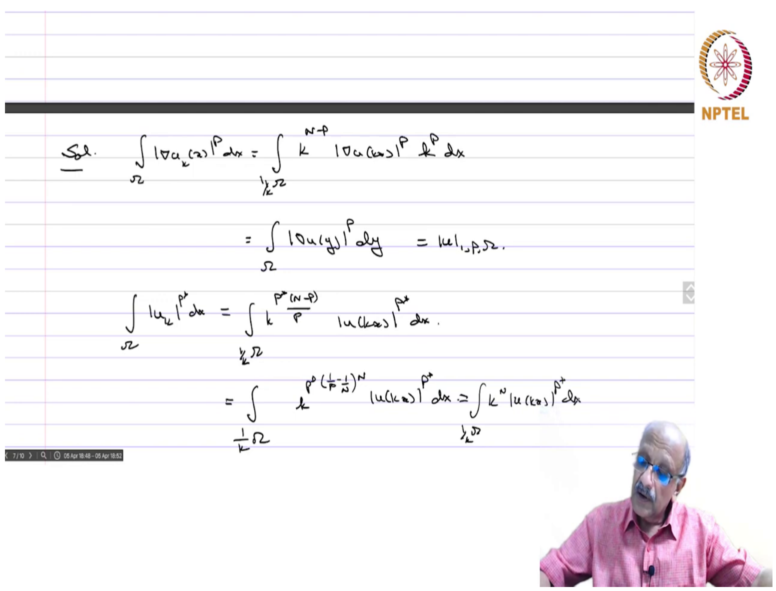

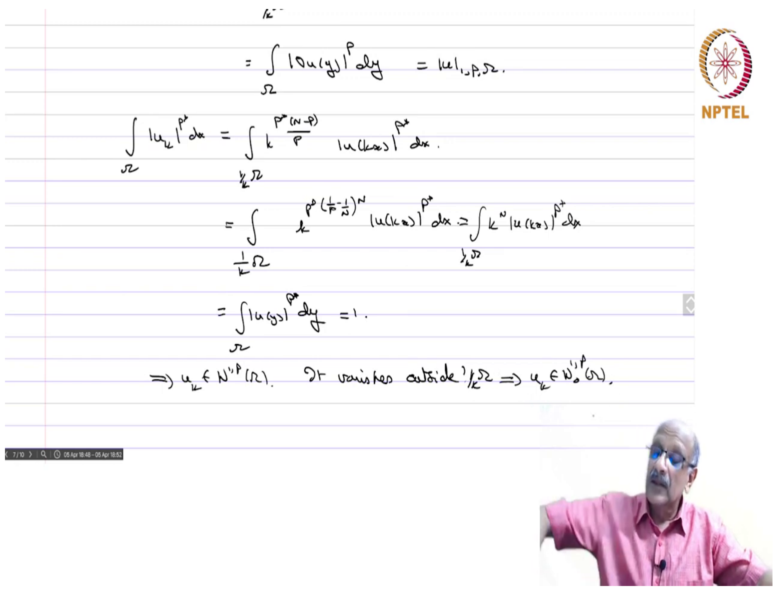

Solution

$$\int_{\Omega} |\nabla u_k(x)|^p dx = \int_{\frac{1}{k}\Omega} k^{N-p} |\nabla u\ (kx)|^p k^p dx$$

$$= \int_{\Omega} |\nabla u\ (x)|^p dx = |u|_{1,p,\Omega}$$

$$\int_{\Omega} |u(x)|^p dx = \int_{\frac{1}{k}\Omega} k^{p^{*}(N-p)/p} |ku\ (x)|^{p^{*}} dx$$

$$= \int_{\frac{1}{k}\Omega} k^{p\ (1/p-1/N)N} |ku\ (x)|^{p^{*}} dx$$

$$= \int_{\frac{1}{k}\Omega} |u\ (kx)|^{p^{*}} dx$$

$$= \int_{\Omega} |u\ (x)|^{p^{*}} dx = |u|_{1,p,\Omega}.$$

So now if you make a change of variable k x equal to y, then k power n d x will be d y, and therefore this will be equal to, and this k power minus P and k power P would have got cancelled so this will become integral on omega mod grad u y power P d y which is equal to mod u 1 P omega.

Integral on omega mod u k P star d x will be equal to integral over 1 by k omega k power P star into N minus P by P mod u of k x power P star d x. So, that is equal to integral 1 over k omega, I will take a k power N outside.

So, if I take k power N outside, I will get k power P star into 1 by P minus 1 by N which is exactly 1 by P star, sorry and what is in the bracket, this 1 by P star will cancel with this P star, so you will just get k, into mod u k x over p star d x. So,rry this is not into k power N, it is just into N. I am just rewriting the expression like this. Now, 1 by P star and 1 by P that is cancelled, so that just becomes integral over 1 by k omega k power N mod u of k x over P star d x.

Now, you again make a change of variable, that is equal to integral omega mod u y P star d y and that is equal to 1. So, this proves, so this implies that u k belongs to $W^{1,p}(\Omega)$ but it vanishes outside 1 by k omega and therefore this implies that u k belongs to $W^{1,p}{}_0(\Omega)$ for all k this, so this 1 by k closure is again, so you can, we have seen this theorem. It vanishes outside a closed set so you can take the closure also, and therefore it is.

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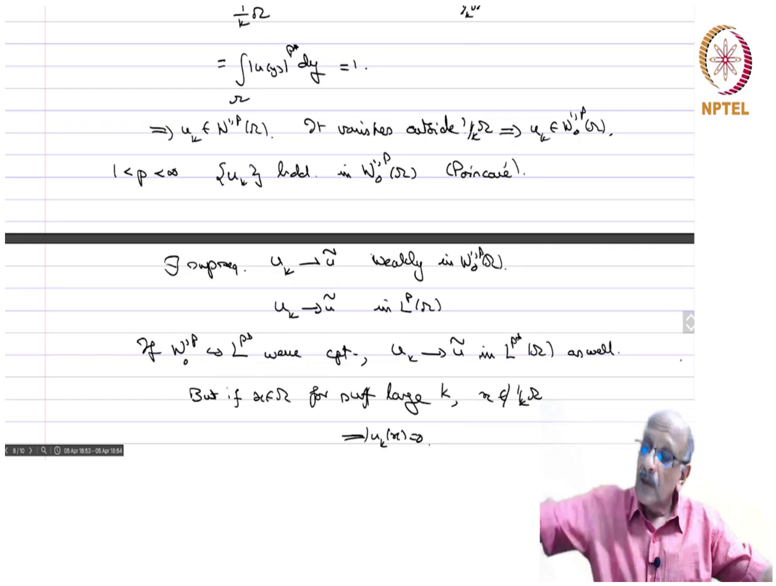

So, let 1 less than p less than infinity, so u k is bounded in $W^{1,p}{}_0(\Omega)$ because its mod 1, is bounded and therefore by Poincare, so this is Poincare, and therefore there exists a subsequence such that, so I will again call it u k, u k converges some u tilde in, weakly, in $W^{1,p}{}_0(\Omega)$. Because it is a reflexive space if you have a bounded sequence in a reflexive space then you have the thing.

So, now if and by (())(26:03) this converges u tilde in LP of omega. So, if $W^{1,p}{}_0(\Omega)$ to L^p star were compact, then u_k will go to u tilde in L^p star omega. But if x belongs to omega for sufficiently large k, x is not in 1 by k omega implies u, u, u k of x equal to 0.

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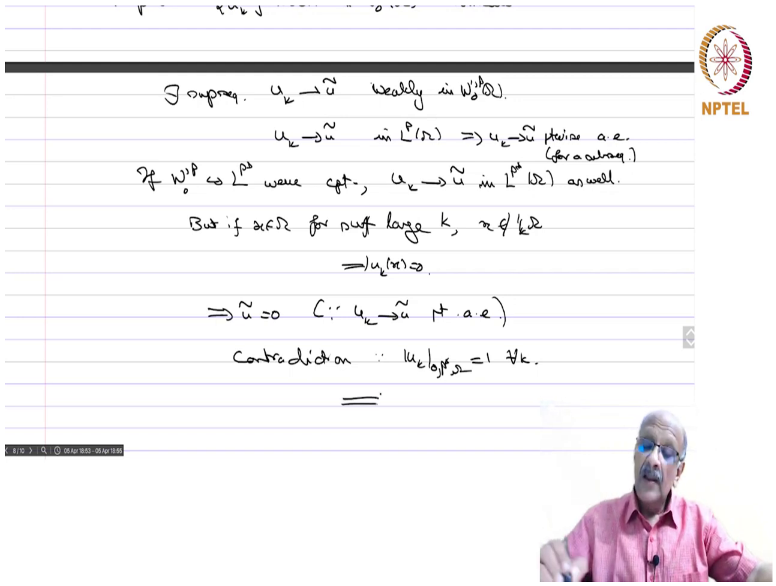

So after, after some level every point $u_k(x)$ becomes 0, so this implies that u tilde equal to 0. Since u_k converges to u tilde, pointwise almost everywhere, almost everywhere, again, for a subsequence. So, I am going, keeping on working with the, smaller and smaller subsequence and therefore you have this. But that is a contradiction since mod u k 0 P star omega equal to 1 for all k. So, it cannot go to 0 in the L^p norm. So, that proves this thing. We will continue with this.