Sobolev Spaces and Partial Differential Equations
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Lecture 05
Distribution Derivatives

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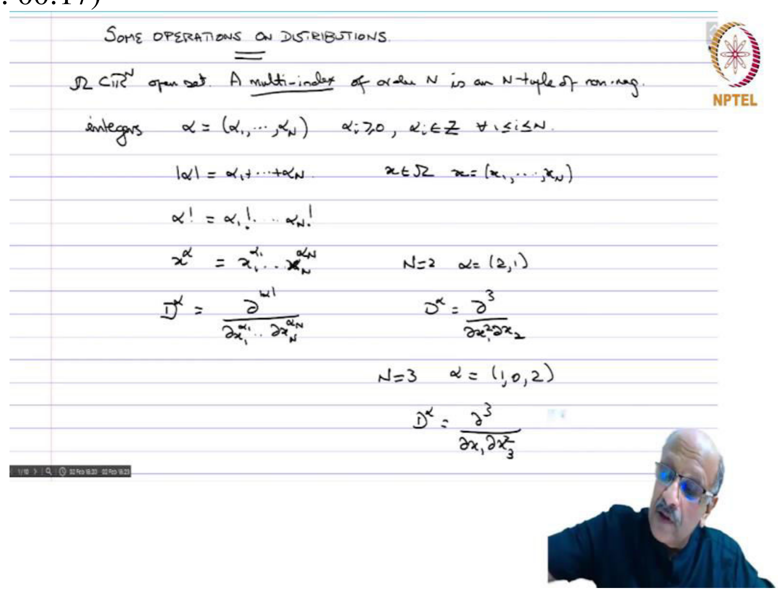

We will now look at some operations on distributions. So, before we get started, we would like to introduce some very useful notation.

$\Omega \subset \mathbb{R}^N$ -an open set. A multi-index of order N is an N-tuple of non-negative integers.

$\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_N)$, $\alpha_i \geq 0$, $\alpha_i \in \mathbb{Z}, \forall 1 \leq i \leq N$.

$|\alpha| = |\alpha_1| + |\alpha_2| + \ldots\ldots + |\alpha_N|$

$\alpha! = \alpha_1!\,\alpha_2! \ldots \alpha_N!$, $\qquad x \in \Omega$; $x = (x_1, x_2, \ldots, x_N)$.

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots\ldots x_N^{\alpha_N}$.

$D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots\ldots \partial x_N^{\alpha_N}}$.

For example: If $N = 2, \alpha = (2,1)$, then

$D^\alpha = \dfrac{\partial^3}{\partial x_1^2 \partial x_2}$.

If □ = 3, □ = (1,0,2), then

$$□^{□} = \frac{□^{3}}{□□_{1}\ □□_{3}^{2}}$$

So, this is how we have very compact notation for introducing derivatives.

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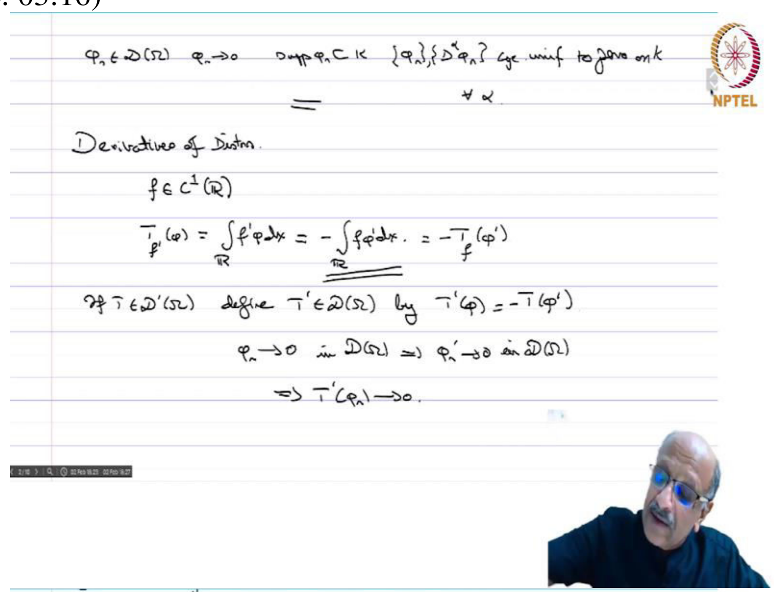

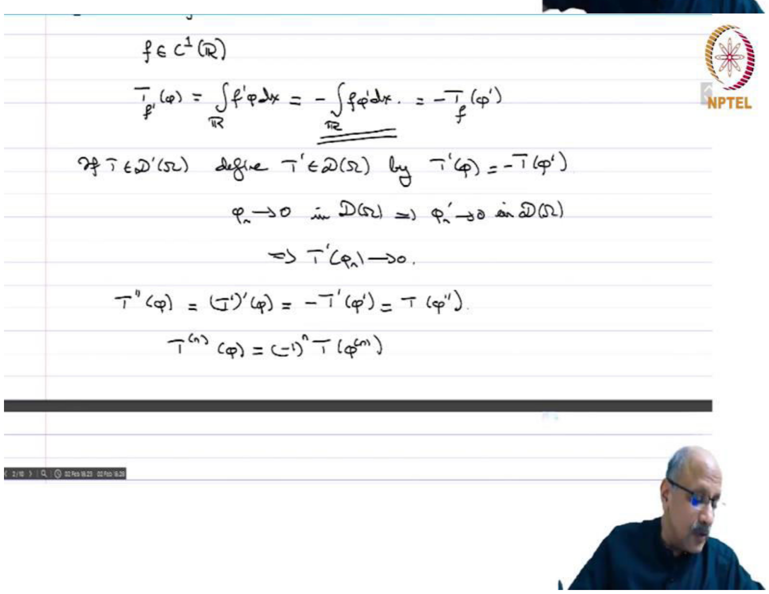

So, for instance in this notation:

$\phi_{□} \in □(□)$, $□_{□} \to 0$ means supp($\phi_{□}$)$\subset$ □ and $\{\phi_{□}\}$, $\{□^{□}□_{□}\}$ converge uniformly to 0 on □ for every multi-index α of order □.

So, this is the way we use this notation here. So, now we want to introduce derivatives of distributions.

Derivatives of distributions: So, let us start with the case $n=1$ and let us take $f \in C^1(\mathbb{R})$.

So, it is a continuously differentiable function. Therefore, f and f' are both continuous and therefore, they are automatically locally integrable and hence define distributions. So, now let us take

$$T_{f'}(\phi) = \int_{\mathbb{R}} f'\phi\, dx = -\int_{\mathbb{R}} f\phi'\, dx = -T_f(\phi').$$

So, we copy this in general. So, if $T \in D'(\Omega)$ is any distribution, we define

$$T'(\phi) = -T(\phi').$$

So, why is this a distribution? Because, it is linear and it is well defined because ϕ' is also a C^∞-function with compact support.

So,

if $\phi_m \to 0$ in $D(\Omega)$ $\Rightarrow$ $\phi_m' \to 0$ in $D(\Omega)$ $\Rightarrow$ $T'(\phi_m) \to 0$.

So, the T' is indeed a distribution. Now, we like to iterate this. So, let us try to define

$$T''(\phi) = (T')'(\phi) = -T'(\phi') = T(\phi'').$$

So, in general, if you want to define the k-th derivative of T :

$$T^{(k)}(\phi) = (-1)^k T(\phi^{(k)}).$$

So, this is why you see having C^∞- functions with compact support is very convenient, namely: we can define all derivatives of T by simply throwing the responsibility of differentiation on to the functions ϕ which can take it because they are all C^∞- functions with compact support. So, this is how we will define derivatives of a distribution.

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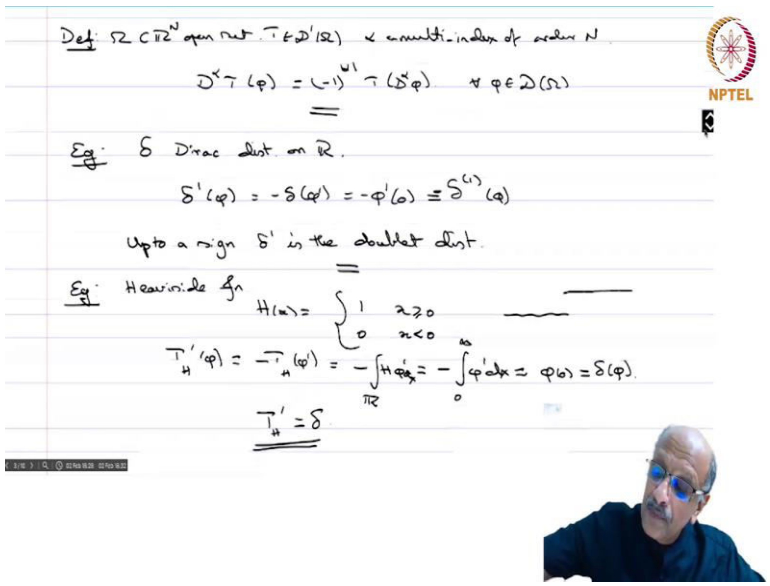

Let us give a formal definition.

Definition: Let $\Omega \subset \mathbb{R}^N$ -an open set and , α multi-index of order N.

$$D^{\alpha}T(\varphi) = (-1)^{|\alpha|}T(D^{\alpha}\varphi) \quad \forall \ \varphi \in D(\Omega).$$

So, this is how we define the differentiation of all these distributions and therefore, we have that every distribution is infinitely differentiable for any multi-index alpha all the derivatives exist.

So, now, let us look at some examples.

Example: *δ Dirac distribution on R*.

$$\delta'(\varphi) = -\delta(\varphi') = -\varphi'(0) = \delta^{(1)}(\varphi).$$

So, the notation was not actually accidentally, it was deliberate, namely up to a sign. So, up to a sign δ' is the doublet distribution.

So, now let us look at another example.

Example: On $\mathbb{R}$, *Heaviside function*:

$$H(x) = 1, \quad x \geq 0,$$
$$= 0, \quad x < 0.$$

So, now let us compute T_H'.

$$T_H'(\phi) = -T_H(\phi') = -\int_{\mathbb{R}} H\phi' dx = -\int_0^\infty \phi' dx = \phi(0) = \delta(\phi).$$

So, $T_H'(\phi) = \delta$.

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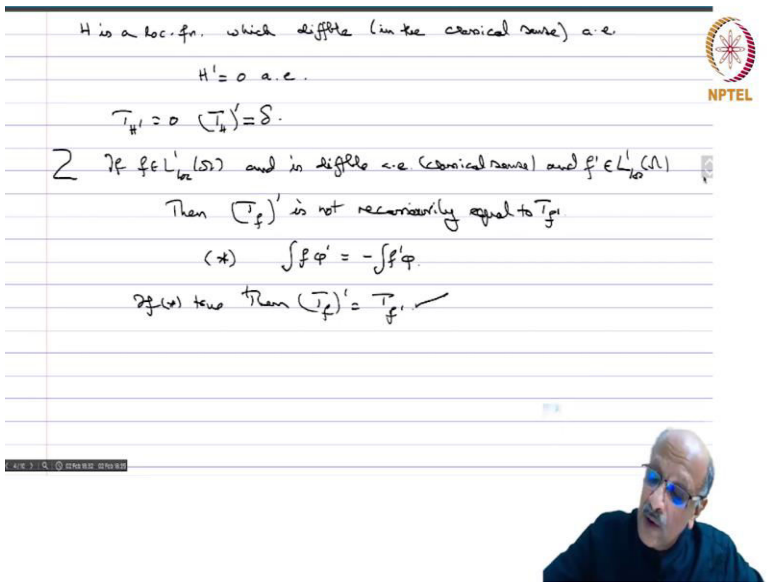

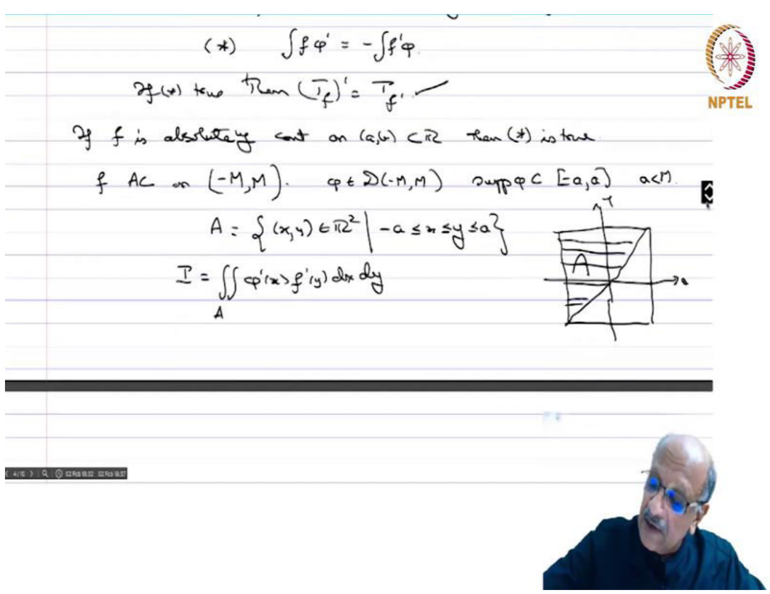

So, now this raises a certain interesting question. So, you remember that H is locally integrable function which is differentiable (in the classical sense) a.e. in fact, you have

$$H' = 0 \text{ a.e.}$$

So, $□_{□'} = 0$, but $(□_{□})' = □$.

So, we have this warning namely: if $□ \in □^{1}_{□□□}(□)$ and is differentiable a.e. (classical sense) and $□' \in □^{1}_{□□□}(□)$, then $(□_{□})'$ is not necessarily equal to $□_{□'}$.

So, earlier we said that we will not say $□_{□}$, $□_{□'}$ etc. to say f is a distribution. Then, when I say $□'$ when I am then I have to be careful with I am talking about the classical derivative or the distribution derivative.

Now, this is a bit disturbing initially because these two objects are not the same. So, we are wondering what we are getting into. However, for all smooth functions there is no problem namely because you have the integration by parts formula:

$$(\ast) \qquad \int □□' = -\int □'□$$

If $(\ast)$ is true, then of course, $(□_{□})' = □_{□'}$. So, the question is when is integration by parts formula? So, if integration by parts formula cannot be taken for granted then of course, you have to check it may still happen that this equation is valid, but you have to be careful you have to check clearly the distribution derivative and see if it comes.

So, there are many examples where that may happen, but still we have to be we have to not take it for granted. So, example where you do have $(\ast)$: if $□$ is absolutely continuous on say an interval $(□, □) \subset \mathbb{R}$, then $(\ast)$ is true. So, let us try to prove that this is true.

proof: So, let us take f absolutely continuous on $(-□, □)$, $□$ is very really large.

$$\phi \in □(-□, □), \text{supp}(\phi) \subset [-□, □], □ < □.$$

So, now, let us look at the set

$$□ = \{(□, □) \in \mathbb{R}^2, -□ \le □ \le □ \le □\}.$$

So, on this we are going to evaluate the following integral:

$$□ = \int \int_{□} □'(□)□'(□)□□□□.$$

So, I am going to evaluate this integral in two different ways.

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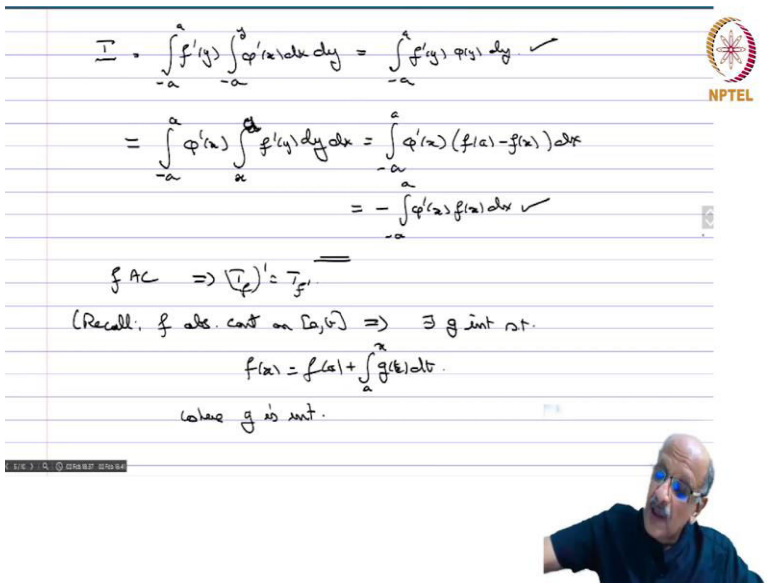

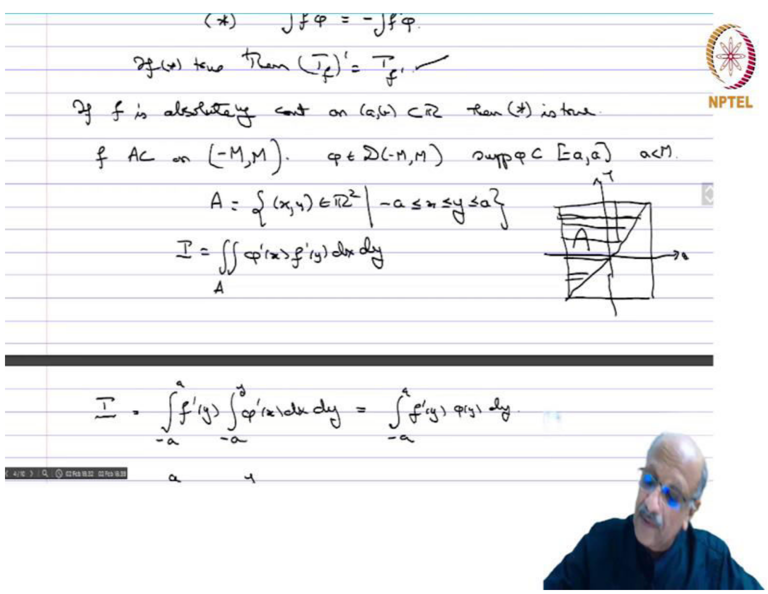

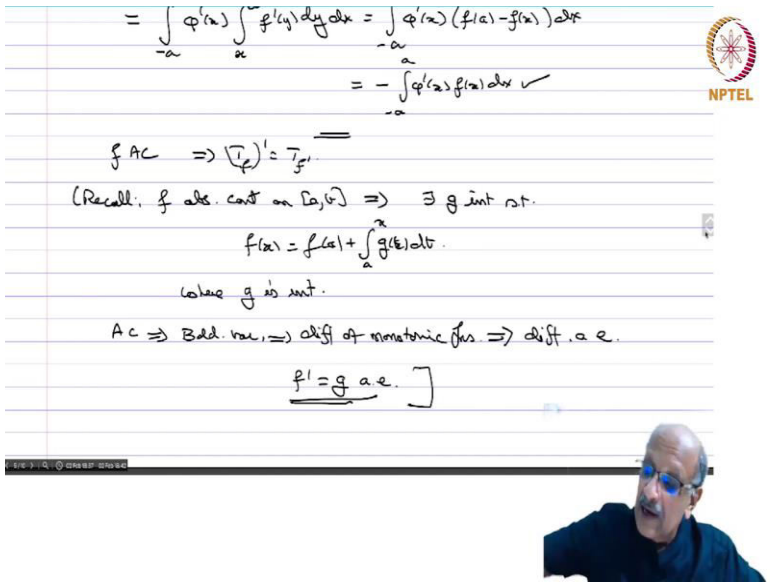

So, on one hand you have

$$0 = \int_{-a}^{a} \varphi'(x) \int_{-a}^{x} f'(y)\,dy\,dx = \int_{-a}^{a} \varphi'(x) f(x)\,dx = \int_{-a}^{a} \varphi'(x) \int_{x}^{a} f'(y)\,dy\,dx.$$

$$= \int_{-a}^{a} \varphi'(x)\{f(a) - f(x)\}dx$$

$$= -\int_{-a}^{a} \varphi'(x) f(x)\,dx.$$

Therefore, f absolutely continuous implies $(T_f)' = T_{f'}$. So, let me recall. So, recall so f absolutely continuous on [a,b] implies , there exist g integrable such that

$$f(x) = f(a) + \int_{a}^{x} g(t)\,dt.$$

Also, absolute continuity ⇒ bounded variation ⇒ difference of monotonic functions ⇒ differentiable almost everywhere in the classical sense and you have in fact that $f' = g$ a.e. So, So, this is the revision of a crash course on absolutely continuous functions.

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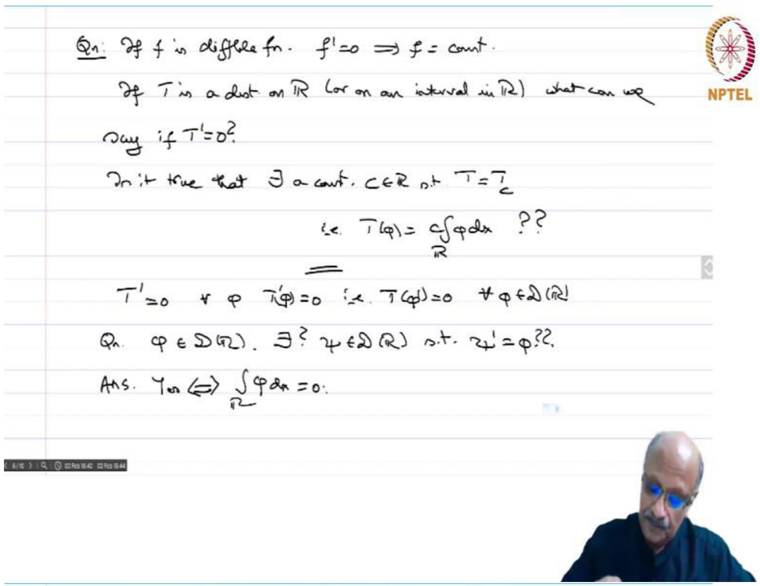

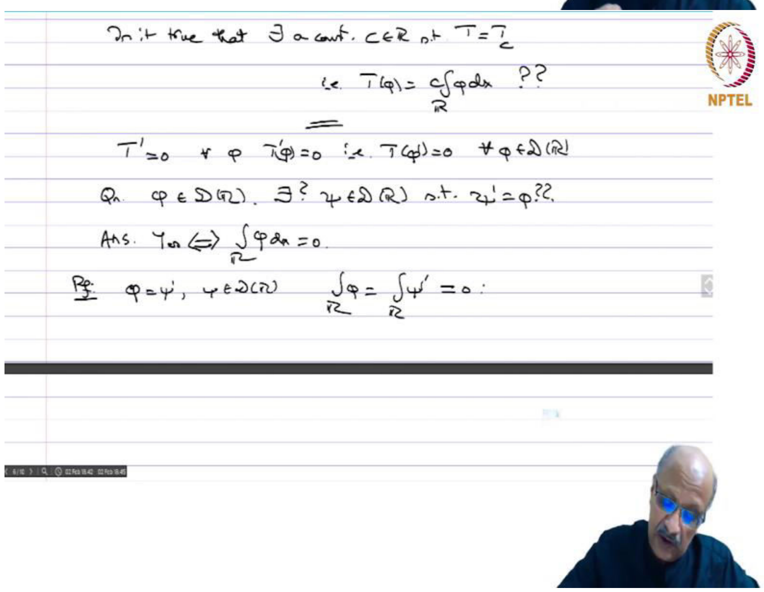

So, we have so before we proceed to other operations here is one interesting question. So, question.

Question: If f is a differentiable function and if $f' = 0$ then f = constant this much, we know. Now, if T is a distribution on $\mathbb{R}$ (or on an interval in $\mathbb{R}$) what can we say if $T' = 0$?

So, what do you mean by saying so, T dash equal to 0 does it mean that T is a constant, but what do you mean by T is a constant T is a distribution generated by a constant. So,

Is it true that there exists a constant c in $\mathbb{R}$ such that $T = T_c$ that is,

$$□(□) = □ \int_{\mathbb{R}} □□□ \, ?$$

This is what we mean by the distribution is a constant. So, if the derivative is 0 does this happen? Answer is Yes.

So, what do you mean by saying $□' = 0$? this means for every ϕ we have $□'(□) = 0$. that is $□(□') = 0, \forall □ \in □(\mathbb{R})$.

Question: $\phi \in □(\mathbb{R})$. Does there exist $\psi \in □(\mathbb{R})$ such that $\psi' = □$?.

Answer is yes if and only if $\int_{\mathbb{R}} □□□ = 0$.

proof: So, if $\phi = □', □ \in □(\mathbb{R})$, then $\int_{\mathbb{R}} \phi = \int_{\mathbb{R}} \psi' = 0$.

So, that is one way we have proved.

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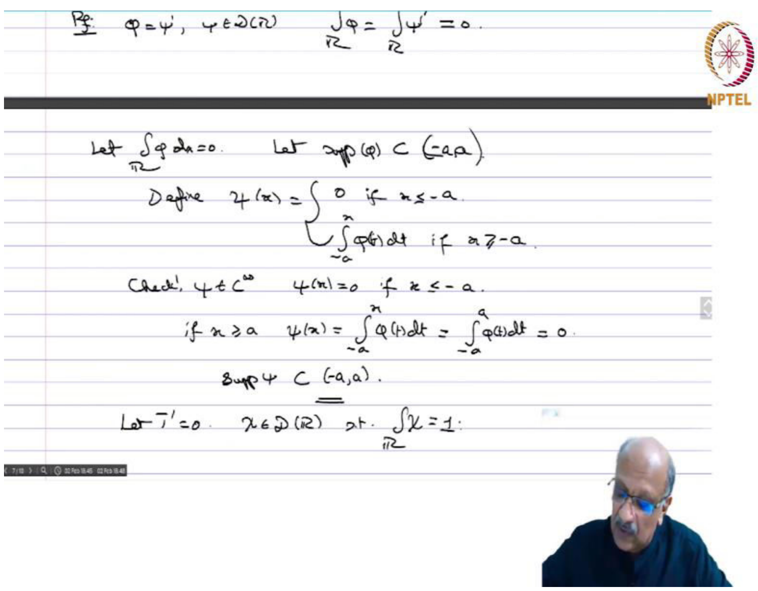

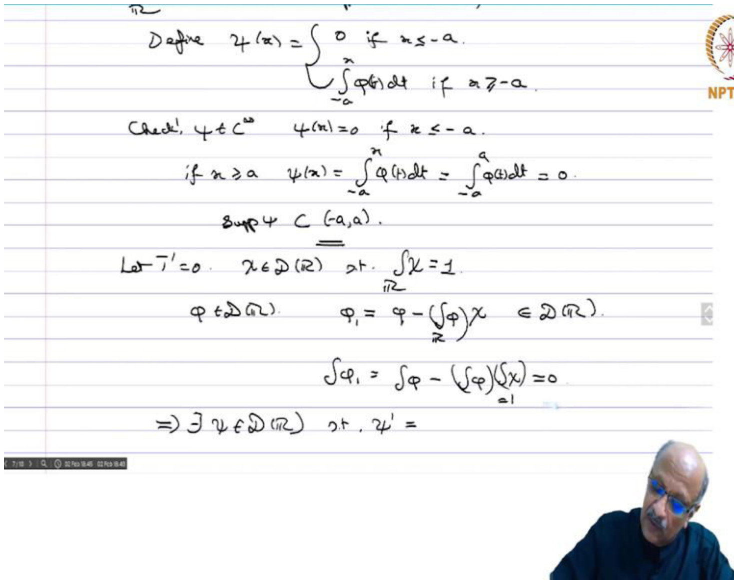

So, now, let us assume let $\int_{\mathbb{R}} \phi = 0$. So, let supp(□) ⊂ (−□, □). Now, define

$$\psi(\square) = 0, \square\square\ \square \leq -\square,$$

$$= \int_{-\square}^{\square} \square(\square)\square\square, \square\square\ \square \geq -\square.$$

So, this is continuous, now we can check that ψ is a C^∞ function , because we have taken the support inside well inside let me say put it as an open interval here.

So, that the support well inside it and therefore this will be 0 even after you cross minus A for some time and therefore, this will be patching up nicely. And afterwards it is derivative by the fundamental theorem of calculus is nothing but phi which is C infinity function and therefore, this function is in fact a C infinity function.

Now, $\psi(\square) = 0, \square\square\ \square \leq -\square$, that we know.

Now, if $\square \geq -\square$, then $\psi(\square) = \int_{-\square}^{\square} \square(\square)\square\square = \int_{-\square}^{\square} \square(\square)\square\square = 0$. And consequently, therefore you have supp(□) ⊂ (−□, □) and therefore, you have this is true therefore, you have this.

So, now let $\square' = 0$ and let us take $\chi \in \square(\mathbb{R})$ such that $\int_{\mathbb{R}} \square = 1$. So, this you can always find any function whose integral is not 0 and divided by the integral then you will have the integral will be equal to 1. So, divided by a constant, you will get the integral equal to 1. So, we can in fact have any number of such functions, which have this, now we would consider.

So, given $\phi \in \mathcal{D}(\mathbb{R})$, then you look at $\phi_1 = \phi - (\int_{\mathbb{R}} \phi)\chi$. So, $\phi_1 \in \mathcal{D}(\mathbb{R})$ and what is the integral?

$$\int_{\mathbb{R}} \phi_1 = \int_{\mathbb{R}} \phi - (\int_{\mathbb{R}} \phi) \int_{\mathbb{R}} \chi = 0$$

$$\Rightarrow \exists \psi \text{ such that } \psi' = \phi_1.$$

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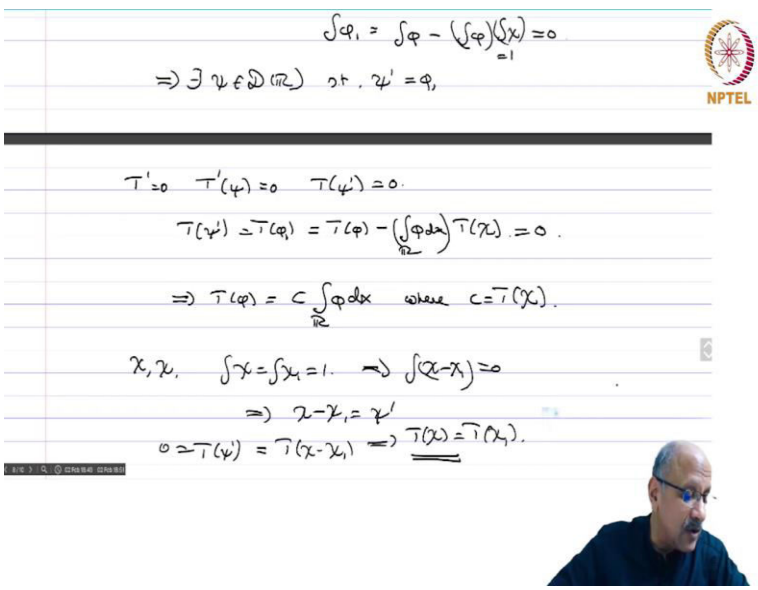

Now $T' = 0$. So, $T'(\psi) = 0$. that is $T(\psi') = 0$ and what is $T(\psi')$?

$$T(\psi') = T(\phi_1) = T(\phi) - (\int_{\mathbb{R}} \phi) T(\chi) = 0$$

$$\Rightarrow T(\phi) = c \int_{\mathbb{R}} \phi \, dx, \text{ where } c = T(\chi).$$

So, we have shown that in fact if the distribution directive is 0, then T has to be generated by a constant function.

There is one point which should however worry you, namely, we have chosen χ to be any function in $\mathcal{D}(\mathbb{R})$ such that it is integral is 1. And so, if I choose some other function will I get a different constant then there is some trouble in the problem. But you would not, because if

$\chi, \square_1$ are such that $\int_{\mathbb{R}} \square = \int_{\mathbb{R}} \square_1 = 1$, then this implies that $\int_{\mathbb{R}} (\square - \square_1) = 0$ and this implies that

$\chi - \square_1 = \square'$, for some ψ'. And therefore,

$0 = \square(\square') = \square(\square - \square_1) \Rightarrow \square(\square) = \square(\square_1)$. So, there is no ambiguity by it, because of the choice of the χ which you chose. So, this is so much about distribution derivatives for the moment. So, our next aim would be to define another operation on distributions there which we will see next.