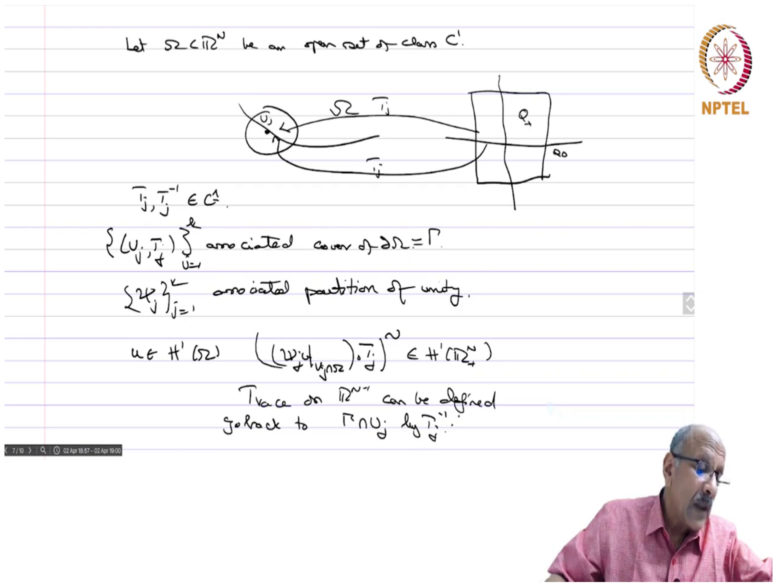
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
The Institute of Mathematical Sciences, Chennai
Lecture 4
Trace Theory Part 4

(Refer Slide Time: 00:18)



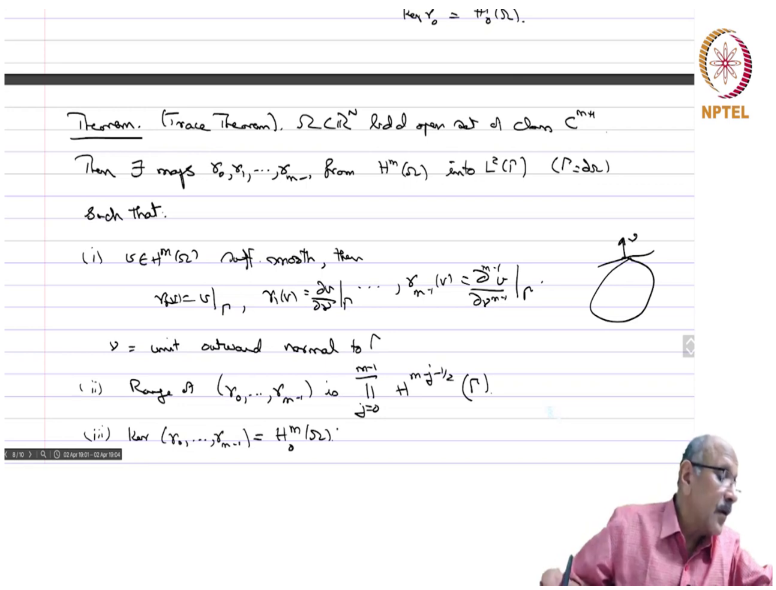
So, now let $\Omega \subset \mathbb{R}^N$ be an open set of class C^1. So, then what do you have? This is Ω, it has bounded boundary, and for any point on the boundary you have this is Q plus, this is Q 0 and this whole cube is called Q and you have a map which, which takes you T_j which goes to this, the boundary goes to this T_j, and T_j and T_j inverse are all C^1 maps.

And these are, so you can cover so U_j, T_j associated cover of $\partial\Omega$. Let us call that as gamma. And then psi j, so j equals 1 to k, associated partition of unity. So, if u belongs to H1 of omega then psi j u, psi j of u restricted to U_j intersection omega composed with T_j and then extended by 0 outside the cube will belong to $H^{1\text{`}}(\mathbb{R}^N_+)$.

And so, we can define its trace, so gamma naught of, can be defined, so is, so trace on $\mathbb{R}^{N-1}$ can be defined, and go back to gamma intersection U_j by T_j inverse. So, this defines the trace on U_j intersection gamma and again use the partition of unity and piece together gamma trace on T_j, sorry, gamma intersection U_j, $j \leq 1 \leq k$, to get trace on gamma.

And we can have range of gamma naught, so we will get in fact gamma naught from H^1 of omega to L^2 of gamma. And then range of gamma naught will be H half of gamma and kernel of gamma naught will be $H^1{}_0(\Omega)$. So, more generally we have the following theorem.

(Refer Slide Time: 04:22)



So,

Theorem(Trace theorem). $\Omega \subset \mathbb{R}^N$, bounded open set of class C^{m+1}, then there exists maps $\gamma_0, \gamma_1, ..., \gamma_{m-1}$ from $H^m(\Omega) \to L^2(\Gamma)$, $\Gamma = \partial\Omega$, such that,

(1) if $v \in H^m(\Omega)$ sufficiently smooth, then

$\gamma_0(v) = v|_\Gamma, \gamma_1(v) = \frac{\partial v}{\partial \mu}|_\Gamma, ... \gamma_{m-1}(v) = \frac{\partial^{m-1} v}{\partial \mu^{m-1}}|_\Gamma,$ $\mu =$ *unit outwrad normal to* Γ

gamm minus 1 v restricted to gamma.

So, these are all the various higher order normal derivatives, where nu is the unit outward normal to gamma. So, you have the boundary here and then at each point you have a tangent and then you have nu, which is unit normal vector, which is there.

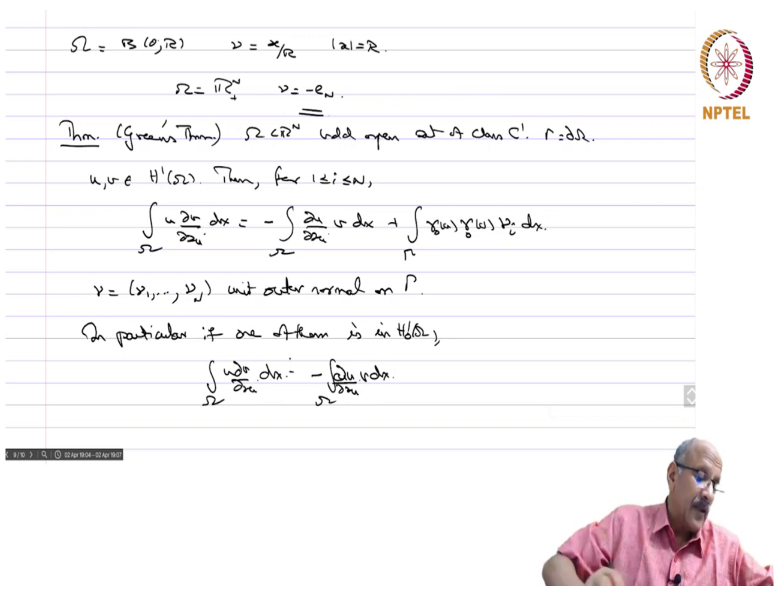
$$\textbf{(2)},\ Range(\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) = \prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\Gamma)$$

And then

$$\textbf{(3)},\quad Ker(\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) = H^m{}_0(\Omega).$$

So, these are, this is the Trace theorem.

(Refer Slide Time: 07:25)



So, if $\Omega = \mathbb{R}^N{}_+$, then nu is nothing but x over R, that is the direction, radius vector itself is the normal for mod x equal to R. This is the unit normal. And if you have $\mathbb{R}^N{}_+$, as I said, omega equals $\mathbb{R}^N{}_+$, then nu equals minus e N. So, the, the, we have, these are the examples of this.

So now we have, theorem, this is

Theorem:(Green's theorem). Omega in $\Omega \subset \mathbb{R}^N$ bounded open set of class C1. $\Gamma = \partial\Omega$, $v \in H^1(\Omega)$, then for $1 \le i \le N$, we have

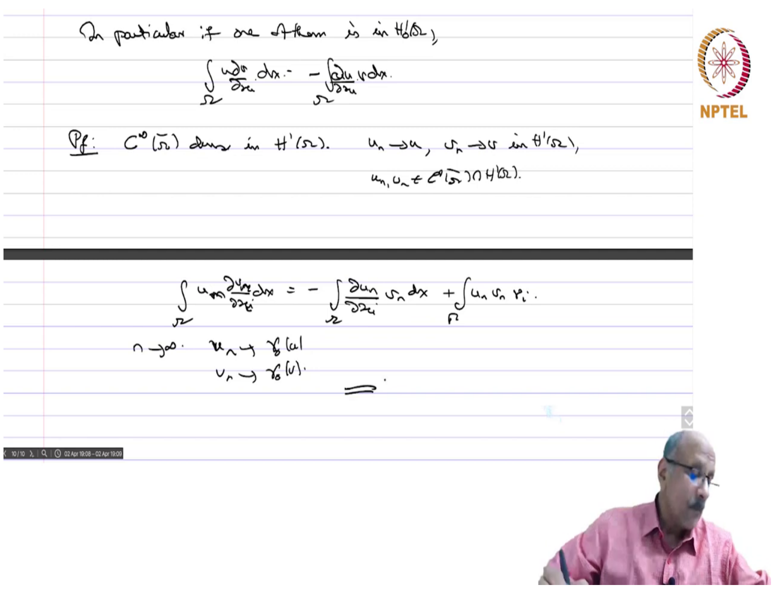
$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = - \int_\Omega v \frac{\partial u}{\partial x_i} dx + \int_\Gamma \gamma_0(u)\gamma_0(v)\gamma_i \, dx \ , \quad \gamma = (\gamma_0, \gamma_1, ..., \gamma_N)$$

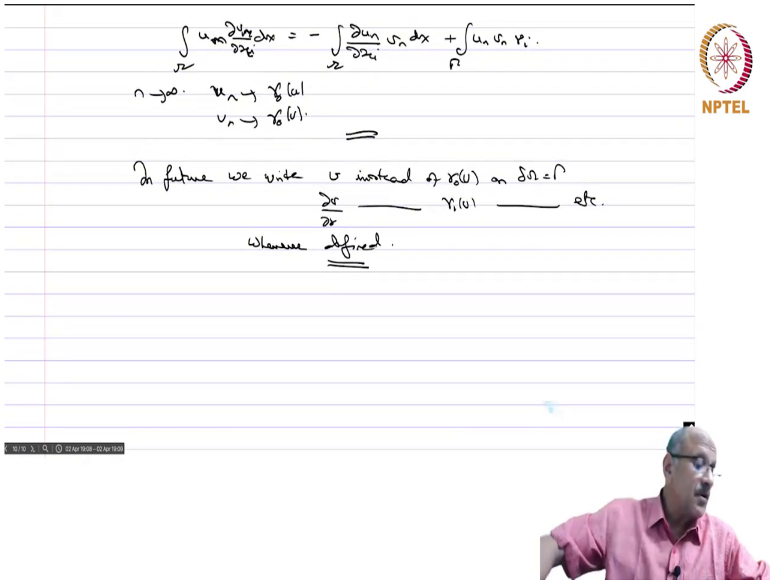
unit outer normal on Ω.

In particular if one of them is in $H^1_0(\Omega)$ then you have integral

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = - \int_\Omega v \frac{\partial u}{\partial x_i} dx.$$ And this is this.

(Refer Slide Time: 10:05)



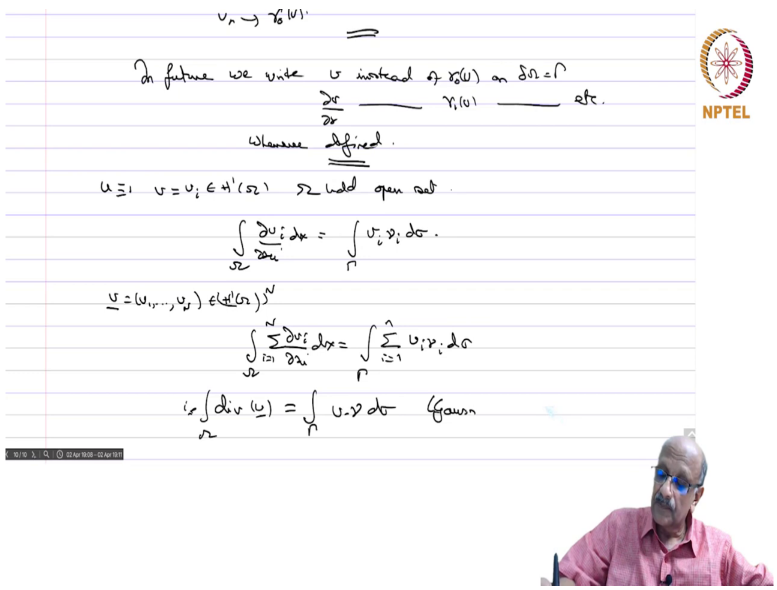


proof. So, we know because of Friedrich's theorem and so on, you have that C infinity of Ω bar is dense in . In fact, $D(\mathbb{R}^N)$ itself will be the dense in $H^1(\Omega)$, and so C infinity Ω bar is dense in this.

So, if u_n converges to u, v_n converges to v in $H^1(\Omega)$, u_n, v_n in C infinity omega bar intersection $H^1(\Omega)$ of omega, then you have, by the classical Green's theorem integral omega $u_n dv_n/dx_i dx$ equals minus integral omega d u n by d x i v n d x plus integral on the boundary of $u_n v_n \gamma_n$. This is the classical Green's theorem.

Now, you let n tend to infinity and you have, this is u n converges to gamma naught u and v_n converges to gamma naught v. So, and you have everything else. So, in future we will write, in future we write v instead of gamma naught v on d omega equals gamma, d v by d nu instead of gamma 1 v on dΩ, et cetera whenever defined. So, we, because we know what it is and, okay.

(Refer Slide Time: 12:25)



So now let us take some simple consequences of this Green's theorem which will be useful in the next chapter. So, we set u identically equal to 1 and $v = v_i \; in \; H^1(\Omega)$ omega bounded open set. So, you get

$$\int_{\Omega} \frac{\partial v_i}{\partial x_i} dx = \int_{\Gamma} v_i \gamma_i d\sigma$$

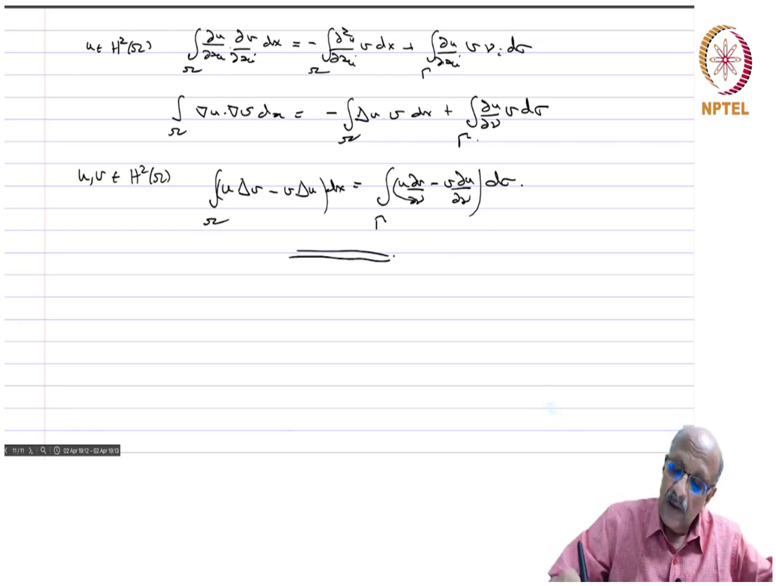
and therefore that will not be there. So, and then integral on gamma u that is this thing and v_i nu i d sigma. So, should write d sigma which is the surface element for the integration on the surface.

Now $v = (v_{1,}..., v_N)$ in $H^1(\Omega)^N$. And then you write this for each i and sum over i you

$$\int_{\Omega} \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i} dx = \int_{\Gamma} \sum_{i=1}^{N} v_i \gamma_i d\sigma$$

And that is exactly the Gauss, that is divergence of v integral on omega equals integral on gamma v dot nu. And that is d sigma, and this is the Gauss Divergence theorem.

(Refer Slide Time: 14:15)



So now let , $v \in H^2(\Omega)$ and so you write

$$\int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = - \int_\Omega \Delta u \, v \, dx + \int_\Gamma \frac{\partial v}{\partial x_i} v \gamma_i d\sigma$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = - \int_\Omega \frac{\partial^2 u}{\partial x_i} v \, dx + \int_\Gamma \frac{\partial v}{\partial \gamma} v d\sigma$$

Now if, if $u, v \in H^2(\Omega)$ then we can write the same thing with v instead of u. Then if you subtract then you will get

$$\int_\Omega (u \Delta v - v \Delta u) \, dx = \int_\Gamma (u \frac{\partial v}{\partial \gamma} - v \frac{\partial u}{\partial \gamma}) d\sigma$$

So, all these are various applications. We will use them. So, we come to an end of this chapter but before winding up we will do some exercises in the next session.