Sobolev Spaces and Partial Differential Equations
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Lecture 3
Trace Theory Part 3

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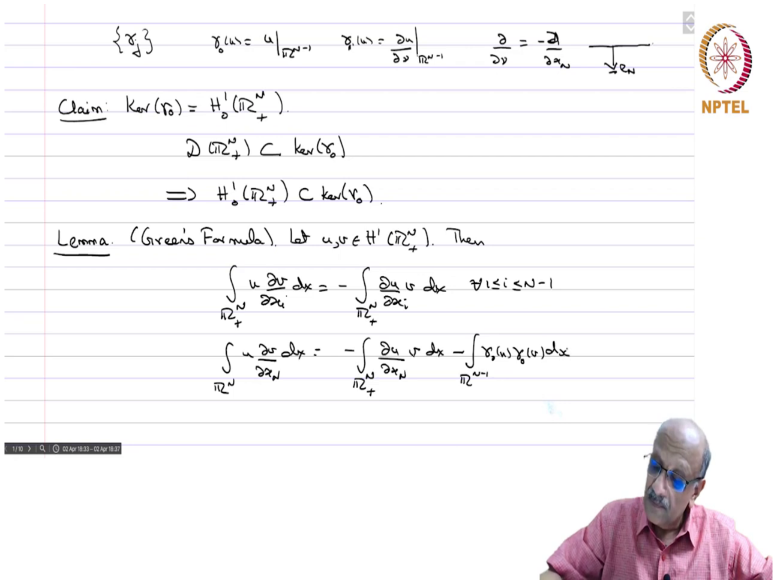

We are looking at trace theory, and we showed that there is a sequence of maps γ_j which you can depend, define on the various Sobolev of spaces $H^m(\mathbb{R}^N)$, and gamma naught of u is nothing but u restricted to $\mathbb{R}^{N-1}$ and gamma 1 of u is nothing but d u by $d\mu$, the exterior normal derivative restricted to $\mathbb{R}^{N-1}$. d by dμ, μ in this case is minus d by d x N because x N is the outer, unit outer normal. So, this is the, minus e N is the unit vector normal to the boundary of $\mathbb{R}^N_+$.

So, this is how, and then gamma 2 would be d 2 u by d n square, d nu square and so on. So, you would get various things. Now we want to show, study the kernel of this map gamma naught. So, now we already have, so the, we will, we will show.

So, the claim kernel of gamma naught is in fact $H^1(\mathbb{R}^N_+)$. I have been talking about this for a long time, various propositions were proved to show that if something vanishes on the boundary or (equivalent), or vanishes outside a compact set then it is in H^1_0 and so on and so forth. So, now we will, finally, this is the last word on it namely, if the boundary value is 0 then it is in fact H^1_0.

So, if you have that, so you know the $D(\mathbb{R}^N_+)$ is clearly in the kernel of gamma naught because $D(\mathbb{R}^N_+)$ has C infinity functions with compact support, so the gamma naught of that is nothing but the value on the boundary and that is of course 0 because it has compact support inside.

And then this is a closed subspace, kernel gamma naught is a closed subspace because it is the kernel of a continuous linear map and therefore this implies that $H^1(\mathbb{R}^N_+)$ is contained in kernel of gamma naught. So, we need to prove the reverse inequality. So, we have to show the reverse inclusion namely kernel of gamma naught is in$H^1_0(\mathbb{R}^N_+)$. That will prove the theorem. For that we need some technical results before.

So, we have lemma, this is Green's formula which we have used many times, integration by parts.

Lemma:(Green's formula) So, let $u, v \in H^1(\mathbb{R}^N_+)$, then integral over

$$\int_{\mathbb{R}^N_+} u\frac{\partial v}{\partial x_i}dx = -\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_i}dx \quad ; 1 \le i \le N-1.$$

And integral over $\mathbb{R}^N_+$ then plus u d by d x n v d x is equal to minus integral over $\mathbb{R}^N_+$ d u by d x n v d x minus integral $\mathbb{R}^{N-1}$ of gamma naught u gamma naught v d x dash. So, this is the Green's formula when you have.

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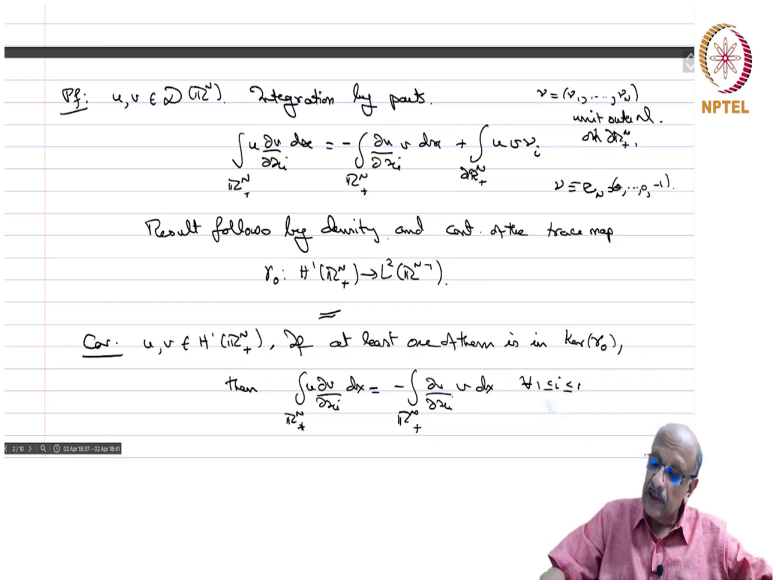

Proof, so let $u, v \in D(\mathbb{R}^N)$. Then both these relationships follow by, this is just integration by parts because you have integral u d v by d x i d x over $\mathbb{R}^N$. So, for this

$$\int_{\mathbb{R}^N_+} u\frac{\partial v}{\partial x_i}dx = -\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_i}dx \quad + \int_{\partial\mathbb{R}^N_+} uv\gamma_i$$

$\mathbb{R}^N_+$ is equal to minus integral d u by d x i v d x $\mathbb{R}^N_+$. Then plus integral on $D(\mathbb{R}^N)$ plus u v into nu i, nui is, so nu equals nu 1, nu n unit outer normal on $D(\mathbb{R}^N_+)$. is nothing but $\mathbb{R}^{N-1}$, and nu is equal to minus e N and therefore is equal to minus, z, so this 0, et cetera 0 minus 1.

So, if you substitute this you will get both these relationships immediately, and u, v on the boundary is nothing but gamma naught, gamma u and now result follows by density and continuity of the trace map gamma naught from $H^1(\mathbb{R}^N_+)$ to$L^2(\mathbb{R}^{N-1})$. So, that proves the first relationship.

Corollary, u, v in $H^1(\mathbb{R}^N_+)$ if one of them at, at least one of them is in kernel gamma naught then integral $\int_{\mathbb{R}^N_+} u\frac{\partial v}{\partial x_i}dx = -\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_i}dx$; $1 \le i \le N - 1$. This is obvious because you have, for 1 to n minus 1 you already have this relationship. And the last one the boundary term vanishes because gamma naught u or gamma naught v is 0. So, therefore, you have this relationship for all the values 1 less than equal to i less than equal to N.

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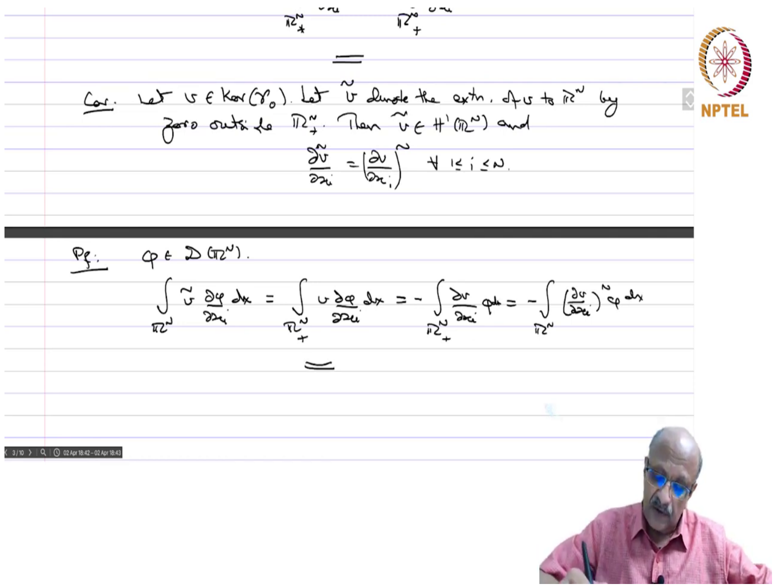

So next

Corollary, let v belong to $Ker\,\gamma_0$. Let $\tilde{v}$ denote the extension of v to $\mathbb{R}^N$ by zero outside $\mathbb{R}^N_+$. Then v tilde belongs to $H^1(\mathbb{R}^N)$ and

$$\frac{\partial \tilde{v}}{\partial x_i} = \widetilde{\left(\frac{\partial v}{\partial x_i}\right)}$$

for all 1 less than equal to i less than equal to N. So, we have already seen this kind of thing. For $H^1_0(\mathbb{R}^N)$ of any omega if you extend by 0 then you know that it is in fact, the extension by 0 is the, is in $H^1(\mathbb{R}^N)$.

Now, we are showing for kernel of gamma naught, so, because after all they are, we, our, our ultimate aim is to show that both these sets are one and the same. So, let phi belong to $D(\mathbb{R}^N)$. Then integral over $\mathbb{R}^N$ v tilde d phi by d x i d x equals integral over $\mathbb{R}^N_+$ of v d phi, sorry, yeah d phi by d x i d x.

And now, because of the previous corollary, we can write this as, because v is in kernel gamma naught so this is integral over $\mathbb{R}^N_+$ d v by d x i times phi which is equal to minus integral over $\mathbb{R}^N$ of d v by d x i tilde phi d x and that proves the result. So, that proves the corollary, and you have immediately the thing.

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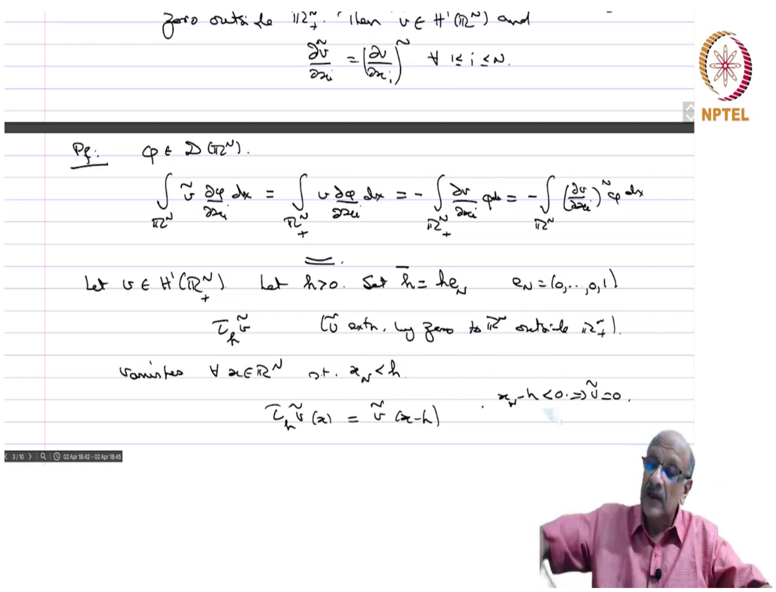

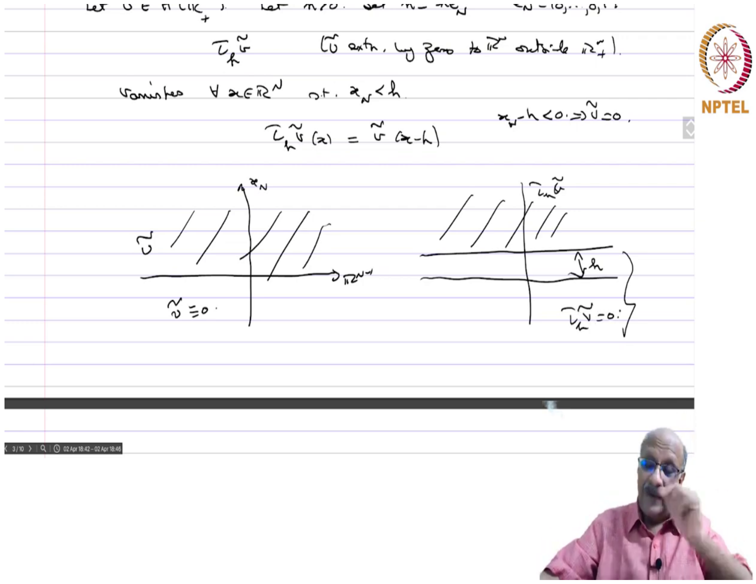

So now, let v belong to $H^1(\mathbb{R}^N_+)$. Let h be greater than 0, and you set h bar, the vector, to be h times e N where e N, of course, equals (0, 0,...,1), the basis vector nth standard basis vector of $\mathbb{R}^N$.

Now, you consider tau h v tilde. v tilde is the extension by 0 outside. So, tau h of v tilde v tilde extension by zero to $\mathbb{R}^N$ outside $\mathbb{R}^N_+$. So, tau h of v tilde vanishes for all x in $\mathbb{R}^N$ such that x_N is less than h because why? Tau h of v tilde at any x is v tilde of x minus h. Now v tilde is 0 if, so x_N minus h is less than 0 implies v tilde is 0. So, v tilde of x minus h and therefore the x_N less than h, it has to vanish.

So, what are we doing? We are just pushing the function. This is $\mathbb{R}^{N-1}$, this is x_N and we have here v tilde, and here v tilde identically 0. So, then when we push, this is, this distance is h, then this will be tau h of v tilde and this is, all this is tau h v tilde equal to 0. So, this is what we, we have here.

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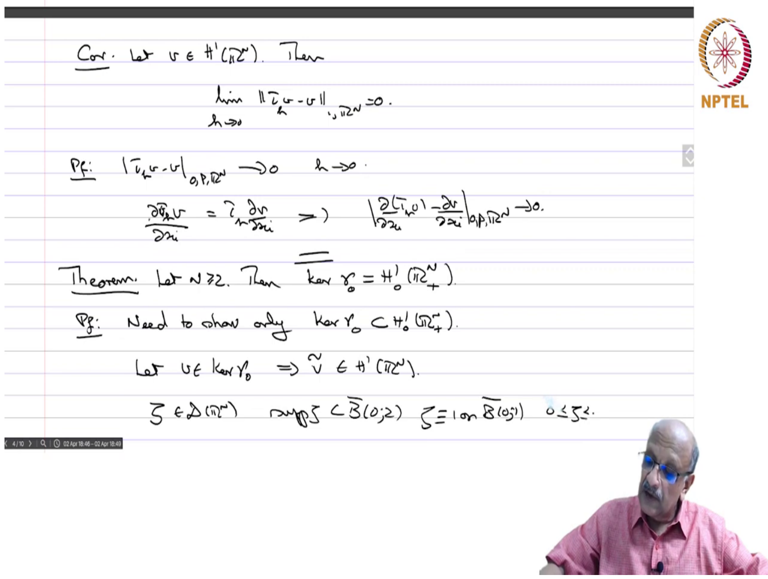

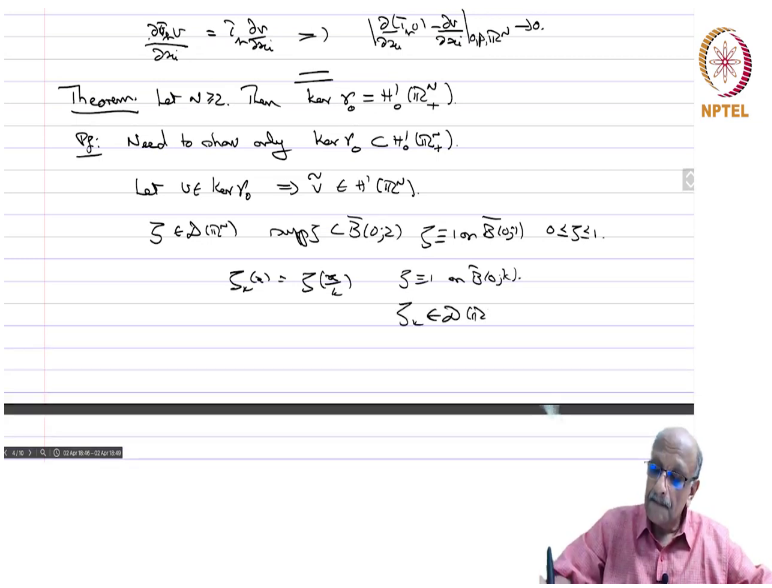

So now, another

Corollary: Let $v \in H^1(\mathbb{R}^N)$. Then limit h tending to 0 norm tau h v minus v 1 $\mathbb{R}^N$ equal to 0. So, proof, we know norm tau h v minus v, sorry, mod 0, P $\mathbb{R}^N$ anyway goes to 0 as h goes to 0. We have seen this long ago. Now, d v, d of tau h v by d x i is nothing but tau h d v by d x i, and therefore this implies that, so the, so now you have the mod d by d x i of tau h v minus d v by d x i 0, p $\mathbb{R}^N$ also goes to 0. So, this is in fact true for all p. I have just stated it for p=2 and therefore this result is immediate.

So, now we have all that we need to, to do this. So,

Theorem. Let $N \geq 2$, then

$$ker\,\gamma_0 = H^1_0(\mathbb{R}^N_+)$$

So,

proof. So, we need to show, so need to show only kernel of gamma naught is contained in $H^1_0(\mathbb{R}^N_+)$ because other inequality, other inclusion is already done.

So, so let v belong to kernel gamma naught. So, then v tilde extension by 0 belongs to $H^1(\mathbb{R}^N)$ as we have already seen. So, now let us go back to our friends, so zeta in D of $\mathbb{R}^N$, support of zeta contained in B(0, 2) zeta identically 1 on B(0, 1), 0 less than equal to zeta, less than equal to 1. And zeta K of x equals zeta of x by k, and therefore zeta is identically 1 on B closure 0 with radius K, and zeta k is in $D(\mathbb{R}^N)$ as well.

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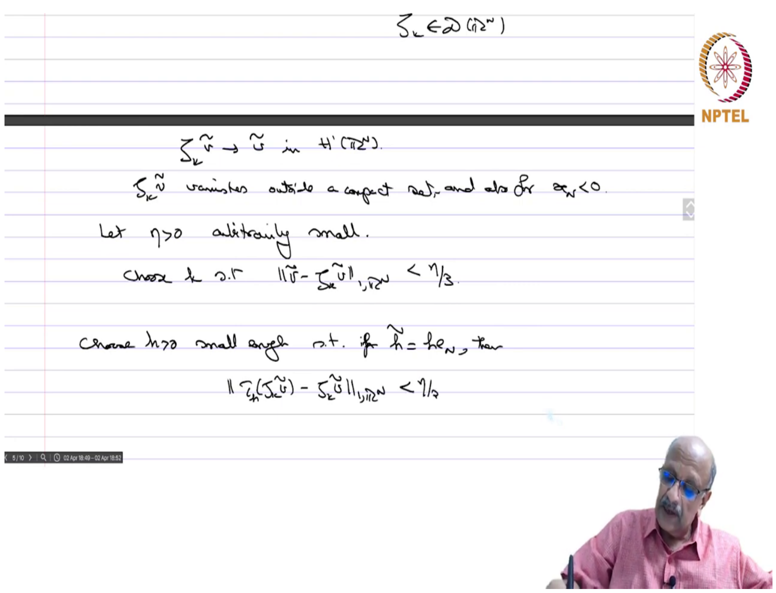

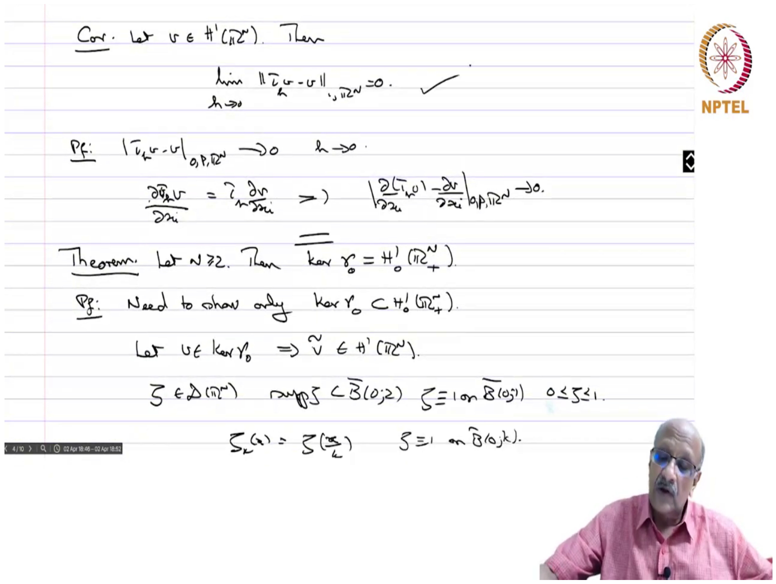

So now we have seen this, again, many times zeta K of v tilde converges to v tilde in $H^1(\mathbb{R}^N)$. Then zeta K v tilde, v tilde is in the kernel of gamma naught, and zeta K has compact support, so vanishes outside a compact set, and also for x_N less than 0. Because v tilde itself vanishes for x N less than 0 so zeta K v tilde will also vanish this.

So let eta be arbitrarily small. Now choose K such that norm v tilde minus zeta K v tilde in 1 $\mathbb{R}^N$ is less than eta over 3. Choose h positive small enough such that if h tilde equals h times h of e N, then norm tau h of zeta K v tilde minus zeta K v tilde 1 $\mathbb{R}^N$ is less than eta by 3. Again, this is possible because K is now fixed and now we are simply applying this corollary here.

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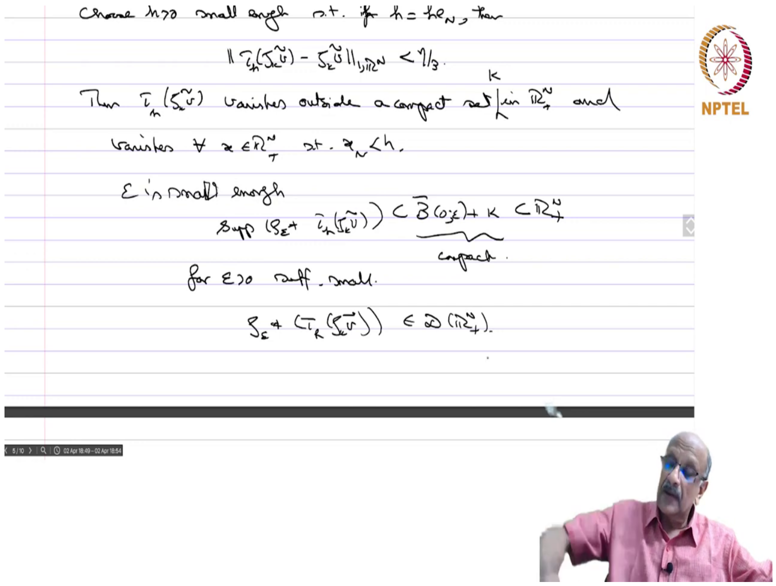

Then tau h of zeta K v tilde has compact support, again, vanishes outside a compact set in $\mathbb{R}^N_+$. And vanishes for all x in $\mathbb{R}^N_+$ such that x_N is less than h. So, let the support of tau h zeta K v tilde, that is not, I will not call it support because say its not a continuous function, outside a compact set K, let us take.

Now, if rho epsilon, if epsilon is small enough, support of rho epsilon star tau h zeta K v tilde is contained in the ball, center origin radius epsilon, plus K and this is still contained in $\mathbb{R}^N_+$ for epsilon greater than 0 sufficiently small. And this is a sum of two compact sets. So, this is compact, and therefore rho epsilon star tau h zeta K v tilde belongs to $D(\mathbb{R}^N_+)$ because it, kind of C infinity function and its support is compact, and therefore this is in $D(\mathbb{R}^N_+)$.

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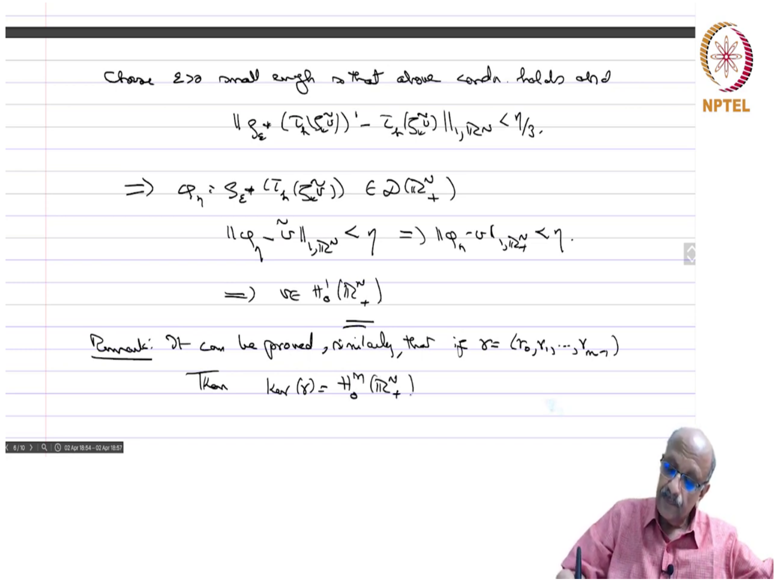

So now you take, choose epsilon small enough so that above condition holds and, that is the support is in d of r, is in $D(\mathbb{R}^N_+)$, and you have that norm of rho epsilon star tau h zeta K v tilde minus tau h zeta K v tilde in 1 $\mathbb{R}^N$ is less than eta by 3. Again, you can do this because rho epsilon star H^1 function goes to this, that function as epsilon goes to 0.

So, if you take so then this implies phi eta equals rho epsilon star tau h zeta K v tilde belongs to $D(\mathbb{R}^N_+)$ and you have norm phi eta minus v in 1 $\mathbb{R}^N$ v tilde 1 $\mathbb{R}^N$ is less than eta because by triangle inequality you have 3 times eta by 3 and this implies that norm phi eta minus v in 1, $\mathbb{R}^N_+$ is also less than eta.

And this implies that v belongs to $H^1_0(\mathbb{R}^N)$. And that completes the proof. So, this proves the trace completely. So,

Remark: It can be proved similarly that if

$\gamma = (\gamma_1, \gamma_{2,\dots} \gamma_{m-1})$

then kernel gamma is in fact $Ker\,\gamma = H^m_0(\mathbb{R}^N_+)$.