Sobolev Spaces and Partial Differential Equations
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Trace theory - Part 2

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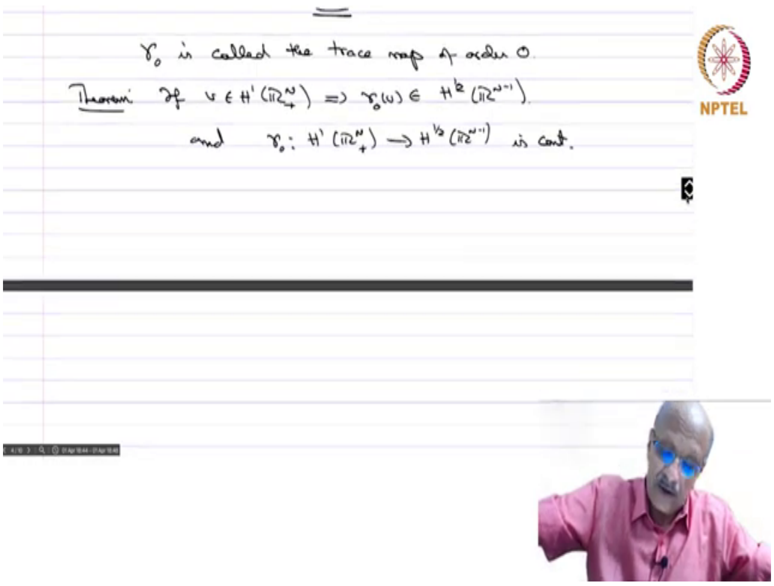

So, now the γ_0 is called the trace map of order 0. So, our next theorem states that the range is not all of $L^2(\mathbb{R}^{N-1})$. So, the

Theorem: If $v \in H^1(\mathbb{R}^N_+)$ this implies that $\gamma_0(v) \in H^{\frac{1}{2}}(\mathbb{R}^{N-1})$ and

$$\gamma_0 : H^1(\mathbb{R}^N_+) \to H^{\frac{1}{2}}(\mathbb{R}^{N-1})$$

is continuous. So, this is what we would like to show. And in fact, later we will show, I will not prove it fully but we it is known that in fact this is a subjective map. That will come to it at the end.

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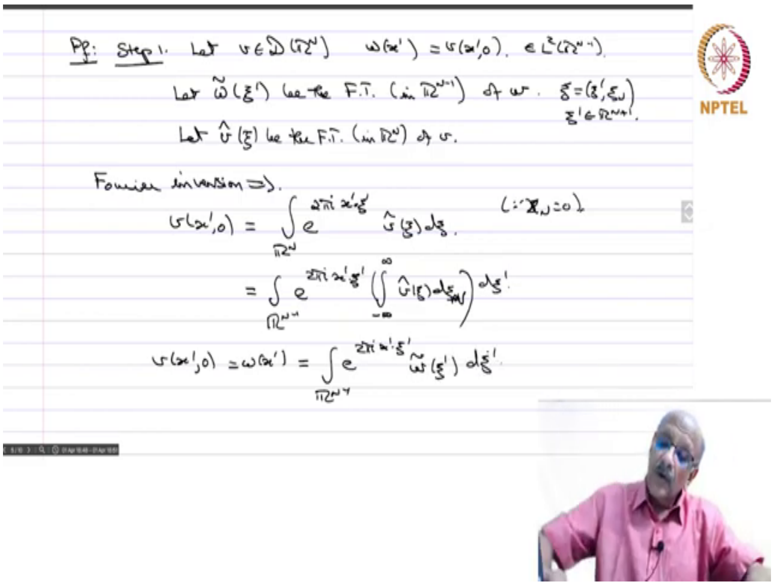

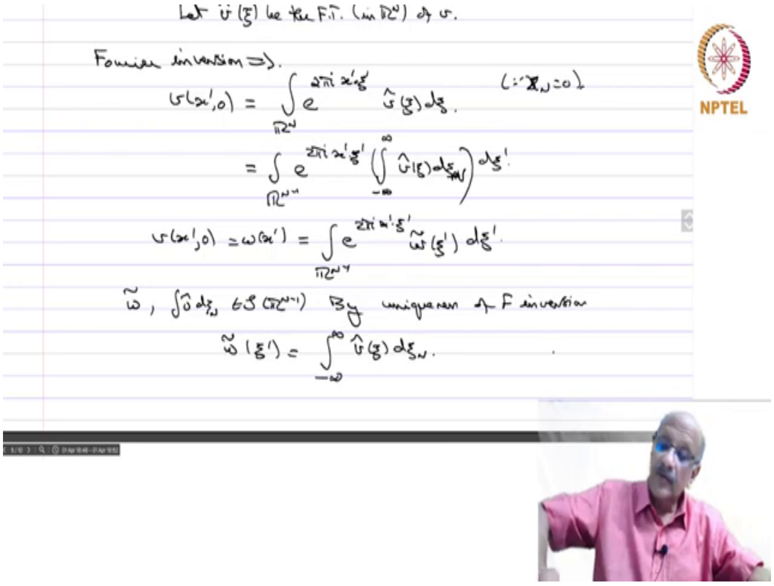

So proof: Step one, so let v belong to $D(\mathbb{R}^N)$ that is always our starting point and you say $w(x^{'}) = v(x^{'}, 0)$. So, now we are going to we can it is in $L^2(\mathbb{R}^{N-1})$, therefore we can apply the Fourier transform to this so let w tilde of psi dash be the Fourier transform in $\mathbb{R}^{N-1}$ of w. So, again x_i, I am going to write as $x_i^{'}$, $x_i^{'}$ in $\mathbb{R}^N_-$.

Let $\hat{v}(x_i)$ be the Fourier transform in $\mathbb{R}^N$ of v. So, tilde usually I have been reserving for an extension by 0, but now there is no confusion because we are working in all of $\mathbb{R}^N$ and therefore extension does not arise, so I am using the symbol tilde for the Fourier transform in

Rn minus 1. So, we want to connect these two what is the connection between these two things?

By the Fourier inversion formula

$$v(x',0) = \int_{\mathbb{R}^N} e^{2\pi i x'.\xi'} \hat{v}(\xi) d\xi = \int_{\mathbb{R}^{N-1}} e^{2\pi i x'.\xi'} \left(\int_{-\infty}^{+\infty} \hat{v}(\xi) d\xi_N \right) d\xi'$$

$$v(x',0) = w(x') = \int_{\mathbb{R}^{N-1}} e^{2\pi i x'.\xi'} \tilde{w}(\xi') d\xi' \quad ; \; \tilde{w}(\xi') = \int_{-\infty}^{+\infty} \hat{v}(\xi) d\xi_N$$

Now, apply the Fourier inversion formula in $\mathbb{R}^{N-1}$. So, this is nothing but w of x dash and that is equal to integral over $\mathbb{R}^{N-1}$ e power 2 pi i x dash ξ dash into W tilde ξ dash d ξ'. So, now W tilde and integral v hat d ξ_N all belong to s of $\mathbb{R}^{N-1}$, this is a very easy thing to check and therefore by uniqueness of Fourier inversion we have the W tilde ξ dash is nothing but integral, minus infinity to infinity we had ξ dξ_N. So, that is the first step where we have related the two.

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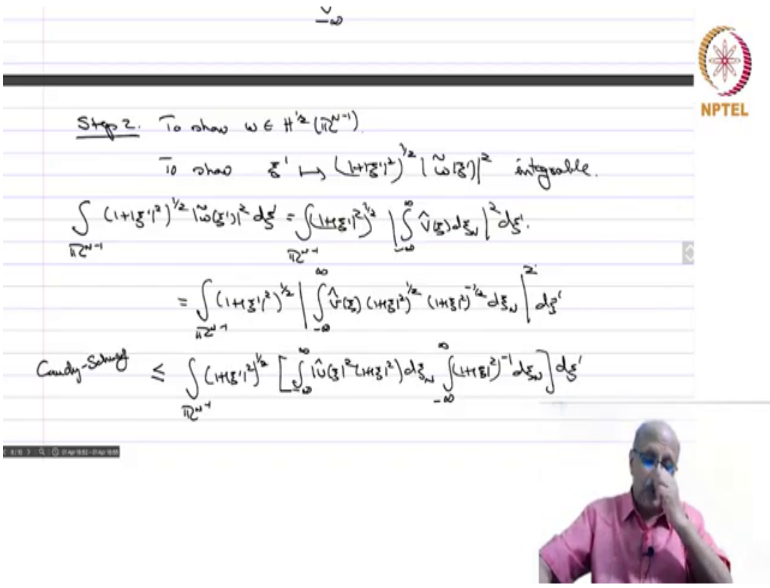

So, now we go to step two. So, what is the claim? We claim that it is in H half of $\mathbb{R}^{N-1}$ one. So, we want to show that to… so to show W belongs to $H^{\frac{1}{2}}(\mathbb{R}^{N-1})$, so in other words we need to show that xi dash going to 1 plus |ξ'| square power half W tilde ξ' square integrable. So, we just go ahead and try to compute that integral. So, integral Rn minus 1 of 1 plus mod ξ dash square power half mod w tilde psi dash square dc dash that is equal to integral over $\mathbb{R}^{N-1}$.

1 plus mod ξ dash square power half into mod integral, from minus infinity to infinity we had ξ dξ n square d ξ dash; that is equal to integral over $\mathbb{R}^{N-1}$, 1 plus mod x_i dash square power half into modulus integral minus infinity to infinity, we had xi into 1 plus mod z square power half into 1 plus mod ξ square or minus half d ξ_N square d ξ'.

Now, for the term inside the modulus I will apply Cauchy-Schwarz inequality, so Cauchy-Schwarz, this is less than equal integral over $\mathbb{R}^{N-1}$, 1 plus mod ξ dash square power half into integral minus infinity to infinity mod v hat xi square into 1 plus mod ξ square Dξ dash, d ξ_N, sorry, into integral 1 plus mod z square power minus 1 d ξ_N minus infinity to infinity into d ξ'. So, the power half which should have come is taken up by the square here and therefore we have this.

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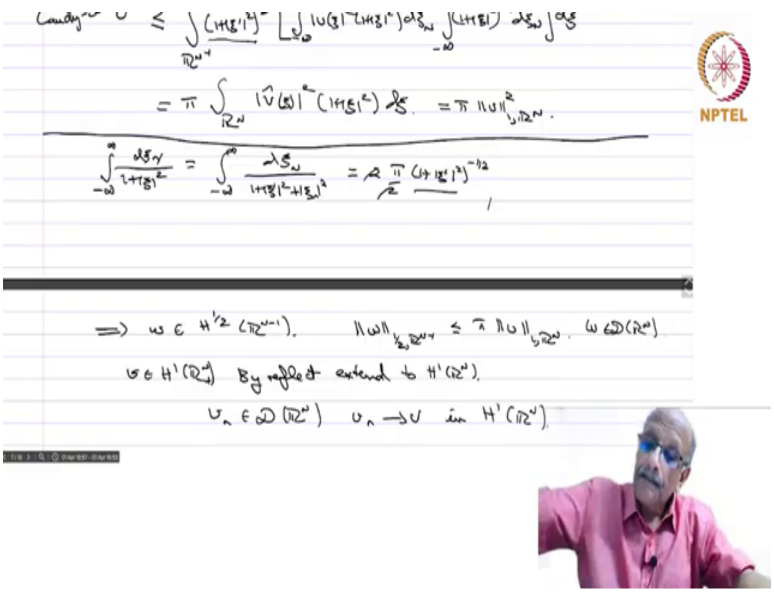

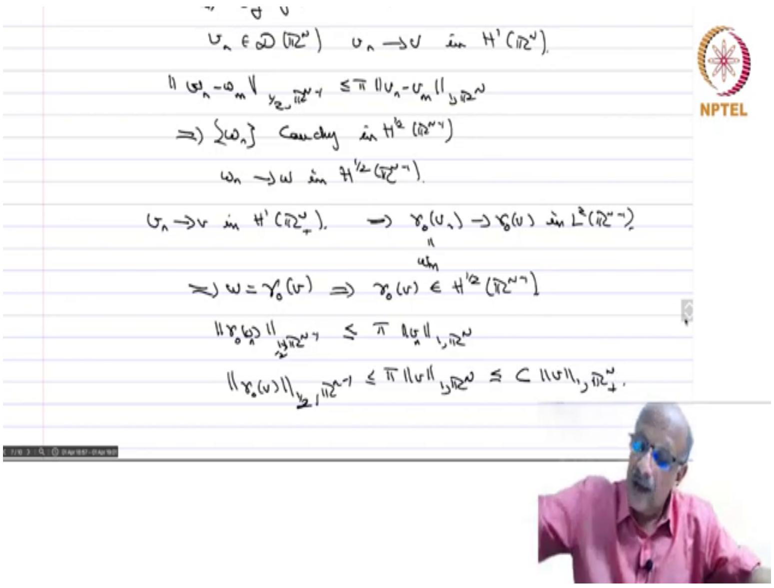

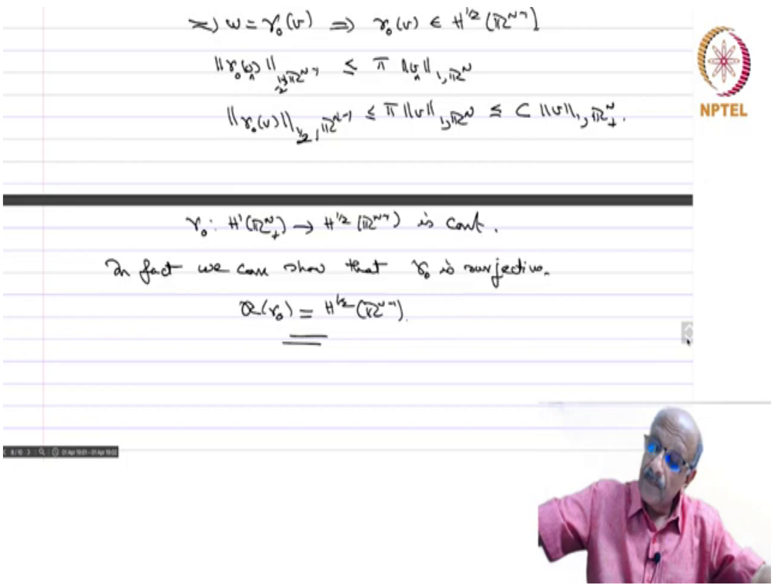

So, now let us lo at this integral minus infinity to infinity d ξ_N by 1 plus mod ξ square, this equal to integral minus infinity to infinity dξ_N by 1 plus mod ξ dash square plus mod ξ_N square. So, this is like the form a square plus x square in the denominator that is 1 by a tan inverse x by a and then you have 2 times 0 to infinity, so that is equal to 2 times π by 2 into 1 plus mod ξ square, ξ dash square power minus half.

So, if you substitute that here that will give you, so 2 into this got canceled, so there will be a pi, 1 plus mod xi square to the minus ξ dash square to the minus half will cancel with 1 plus mod xi square, xi dash square to the 1 half, so that gets canceled and then you have just this integral here $d\xi_N$ and the $d\xi'$ and therefore you get integral of $\mathbb{R}^N_+$, sorry, mod v hat z square into 1 plus mod ξ square dx, dξ

That is not here, pi times integral over $\mathbb{R}^N$ of mod v hat ξ square into 1 plus mod ξ square d xi and what is that that is nothing but pi into norm v square 1 $\mathbb{R}^N$ so this implies that w belongs to H half of $\mathbb{R}^{N-1}$, and you have norm W in half $\mathbb{R}^{N-1}$ is less than equal to pi times norm v in 1 $\mathbb{R}^N$ and this is for v in D of $\mathbb{R}^N$. So now if you take v in $H^1(\mathbb{R}^N_+)$ then by reflection extend to $H^1(\mathbb{R}^N)$.

Now, if you extend it to $H^1(\mathbb{R}^N)$, then you know D of $\mathbb{R}^N$ is dense there, so you take Wn, sorry, vn in D of Rn, vn converging to v in H1 of R1. Then by what we have just see, we saw that if you take Wn minus Wm in half $\mathbb{R}^{N-1}$, this is less than equal to pi times norm v n minus v m in 1 $\mathbb{R}^N$ and therefore this implies that wn is Cauchy in H half $\mathbb{R}^{N-1}$, so Wn converges to some W in H half $\mathbb{R}^{N-1}$.

But v n also converges to v in H1 of $\mathbb{R}^N$ plus and that implies that gamma naught vn converges to gamma naught v in L2 of $\mathbb{R}^N$minus 1, but gamma naught v n is nothing but Wn and that goes to some W and therefore this implies that W equal to gamma naught v and this implies that gamma naught v belongs to H half $\mathbb{R}^{N-1}$; also you have that norm gamma naught v in h half $\mathbb{R}^{N-1}$ is less than or equal to pi times norm v gamma naught v n…

We have this 1, Rn and therefore norm gamma naught v half $\mathbb{R}^{N-1}$ is less than equal to pi times norm v 1 $\mathbb{R}^N$, but v is nothing but the reflection thing so that is a prolongation operator, so that is C times norm v in $\mathbb{R}^N$ and therefore you have that the map gamma naught from H1 of $\mathbb{R}^N_+$ to H half of $\mathbb{R}^{N-1}$ is continuous.

So, in fact, we can show that gamma naught is surjective; that is, the range of gamma naught equals H half of Rn minus 1. I am not going to prove this, you can check the proof in the book Topics in Functional Analysis and Applications which I mentioned to you and you have that in fact you have, you can actually construct a pre image for this one.

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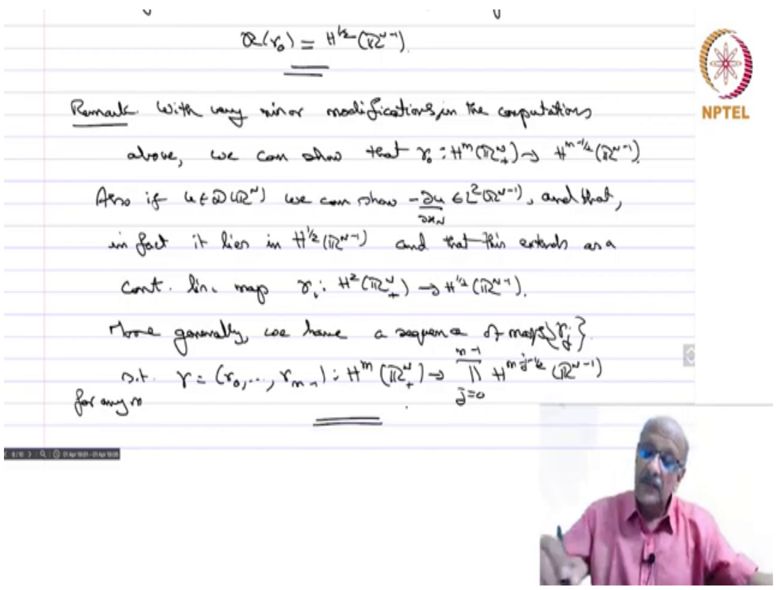

So, now, so remarks, so with very minor modifications in the computations above, we can show that

$$\gamma_0: H^m(\mathbb{R}^N_+) \to H^{m-1}(\mathbb{R}^{N-1}).$$

So, it is very easy to show this. Also if u belongs to $D(\mathbb{R}^N)$, we can show minus $-\frac{du}{dx_n}$ belongs to $L^2(\mathbb{R}^{N-1})$ and in that in fact it lies in H half Rn minus 1 and that this extends as a continuous linear map gamma 1 from $H^2(\mathbb{R}^N_+)$ to H half $\mathbb{R}^{N-1}$

More generally, we have a sequence of maps gamma j such that gamma equals gamma naught gamma m minus 1 for a, such that for any m positive integer maps $H^m(\mathbb{R}^N_+)$ to product j equals 0 to m minus 1, $H^{m-j-1/2}(\mathbb{R}^{N-1})$. So, all this can be done. This just modifications of the calculations which we have been doing. Our next aim is to lo at the kernel of this trace maps gamma and we will see that subsequently.