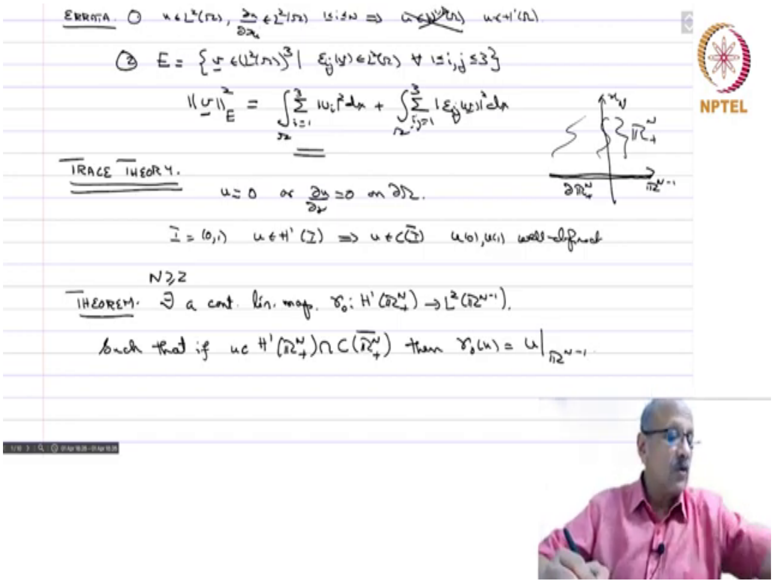
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Institute of Management Science
Trace theory - Part 1

(Refer Slide Time: 0:21)



Before starting I just want to make a few corrections in the previous video. So, first one is when motivating Lion's lemma I wrote the following $u \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ 1 \leq i \leq N \Rightarrow u \in W^{1,p}\ (\Omega)$. Now, you would have figured out that this is not what I meant. I meant $u \in H^1(\Omega)$. So, that is the first correction.

Second correction is when we define this in proving Korn's inequality we define the space E which is set of all $E = \{v \in (L^2(\Omega))^3 : \epsilon_{i,j}(v) \in L^2(\Omega),\ 1 \leq i, j \leq 3\}$. And here I define norm of

$$||v||_E^2 = \int_\Omega \sum_1^3 |v_j|^2 dx + \int_\Omega \sum_1^3 |\epsilon_{i,j}(v)\ |^2 dx.$$

So, what I wrote last time, the sigma ij equals 1 to 3 was missing, so that has to be added, so these are the minor corrections which I made, wish to make. Now, we are going to start an important topic this is called trace theory. So, I have been always saying that the Sobolev spaces form the natural functional analytic framework in which we look for solutions of partial differential equations.

So, when solving partial differential equations we are often encountering boundary value problems, so omega will be a bounded domain and in the domain the solution u will satisfy some differential equation and on the boundary it will satisfy some conditions like $u = 0,\ or \frac{\partial u}{\partial \gamma} = 0,\ on\ \partial\Omega.$ this is the outer normal derivative on the boundary. So, we wish to make give a meaning to these expressions.

This need not be 0, it could be other functions also. So, when if you for instance if you are if $I = (0, 1)$ and if $u \in H^1(I)$ of i for instance or $W^{1,p}(I)$, then this implies that $u \in C(\bar{I})$. Therefore, u(0), u(1) well defined and if you are in higher Sobolev order, Sobolev spaces like h m then you will be in space of continuously differentiable or more functions, so successive derivatives of u at the boundary points are also well defined.

Now, if you are in higher dimensions say $N \geq 2$, then if you, you do not always have this, in order to have this continuity inclusion, inclusion space of continuous or differentiable functions you need to go to very high order Sobolev spaces $m > \frac{N}{p}$ as you know and if you are in H^1 etcetera it may not always be true.

So, what do we mean by u on the boundary? Now, if you take a function $u \in L^p(\Omega)$ and H^1 functions or $W^{1,p}$ functions are all in L^p and therefore these are only defined almost everywhere and the measure of the boundary is 0 and therefore it does not make sense to talk of the value of u or the value of derivatives of u on the boundary.

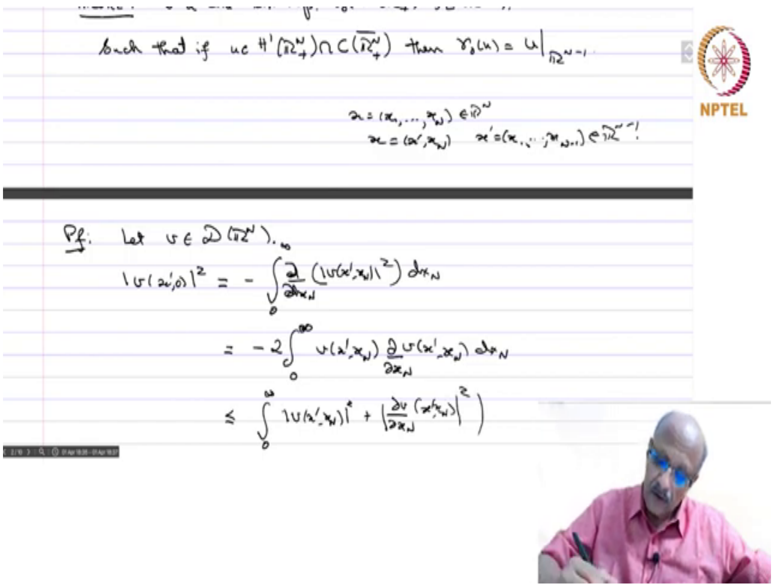
But we want to make use of the fact that we know something more about the derivatives of the function, namely they are also in L^p spaces and using that we wish to make, give a meaning to what is meant by u restricted to the boundary or du by d nu restricted to the boundary and so on and these are called traces of the function on the boundary and that is why we call this trace theory.

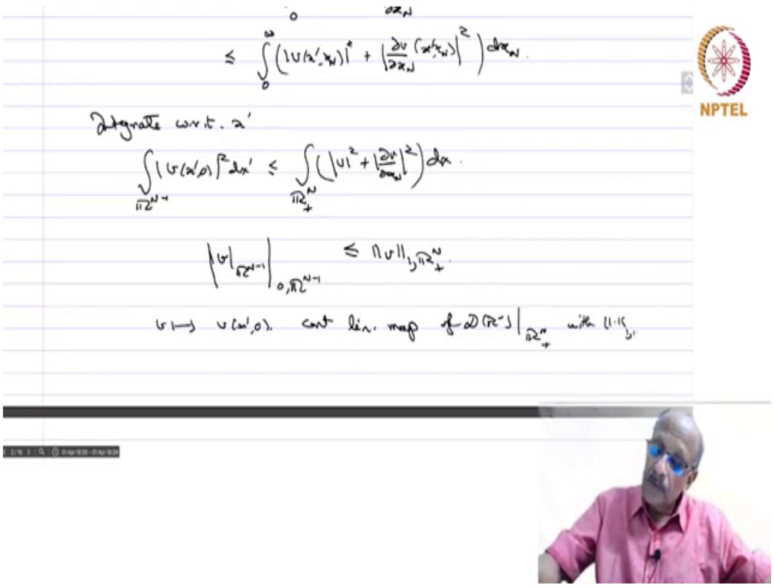
In what follows I will do everything for $p = 2$ but one can easily change it to any other p if you like but it is more convenient for me therefore I will, it is enough to do this. So, theorem and I will do it for the domain Rn plus and then the transition from that to any other omega of class C^k, C^1 and so on is using the coordinate charts, the mappings from q to the neighborhoods on the boundary which we have used many times.

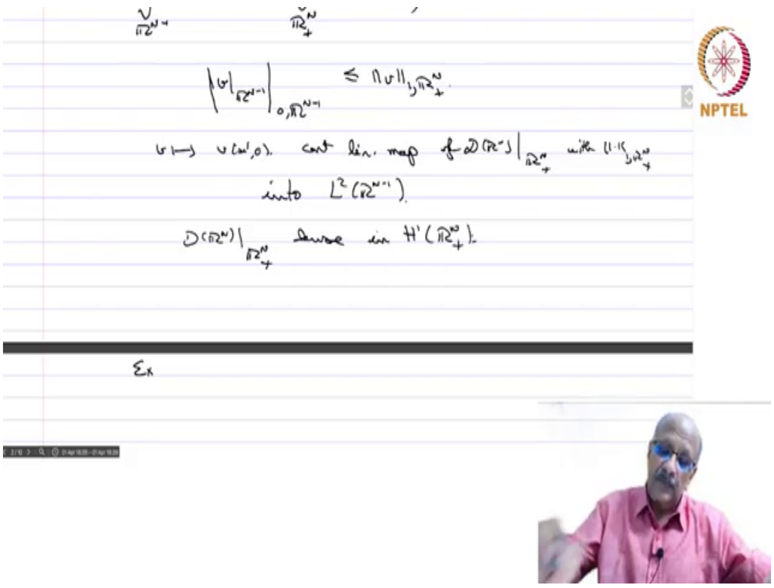
And therefore, we will wave our hands about that, but we will try to do in as much detail as possible the theory for Rn plus because that is where the key thing lies and everything else can thereafter be easily done from that.

Theorem: So, there exists a continuous linear map $\gamma_0 : H^1(\mathbb{R}^N_+) \to L^2(\mathbb{R}^{N-1})$ So, recall you have that $\mathbb{R}^N$, so this is $\mathbb{R}^{N-1}$ and this is x_n and this is $\mathbb{R}^N_+$ and then and this is the boundary of Rn plus, so d of $\mathbb{R}^N_+$, $\mathbb{R}^{N-1}$ So, we have a function mapping from $H^1(\mathbb{R}^N_+)$ to the boundary such that if $u \in H^1(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$, so it is a continuous function on the closure of $\mathbb{R}^N_+$, then $\gamma_0(u) = u|_{\mathbb{R}^{N-1}}$. So, this is the first trace theorem which we want to see.

(Refer Slide Time: 7:04)







So, now let us try to give a

Proof: So, as usual we start with let $v \in D(\mathbb{R}^N)$ and we are going to use the fundamental theorem of Calculus every time. So

$$|v(x^{'}, 0)|^2 = -\int_0^{\infty} \frac{\partial}{\partial x_N} (|v(x^{'}, x_N)|^2) dx_N$$

differentiating the x_N and then integrating with respect N which means you are just going to get the difference of the end values at infinity because we are then $D(\mathbb{R}^N)$, it will be 0 and at the lower end point it is be x_N=0.

So, we get v of x dash 0 and that is why we have picked up a minus sign in the process. So, this is equal to,

$$=- 2 - \int_0^\infty v(x', x_N) \frac{\partial}{\partial x_N} v(x', x_N) dx_N$$

$$\leq \int_0^\infty (|v(x', x_N)|^2 + |\frac{\partial}{\partial x_N} v(x', x_N)|^2 dx_N$$

So, now if you integrate with respect to x', so on the left hand side you will get integral So, in other words we have

$$\int_{\mathbb{R}^{N-1}} (|v(x', 0)|^2 dx' \leq \int_{\mathbb{R}^N_+} |\frac{\partial}{\partial x_N} v(x', x_N)|^2 dx_N$$

|v| restricted to $\mathbb{R}^{N-1} - \{0\}$,$\mathbb{R}^{N-1}$, the L^2 norm in Rs minus 1 is less than equal to norm v in 1 $\mathbb{R}^N_+$. So, v going to V of x dash 0 gives you a continuous linear map of $D(\mathbb{R}^N)$ restricted to $\mathbb{R}^N_+$ with norm 1 $\mathbb{R}^N_+$ into L^2 of $\mathbb{R}^{N-1}$ and therefore extends, but you know that $D(\mathbb{R}^N)$ restrict to $\mathbb{R}^N_+$ is dense in $H^{1`}(\mathbb{R}^N_+)$, one of the earliest theorems we proved in for any p in fact, and therefore in particular for p equals 2.

(Refer Slide Time: 11:16)

Extends uniquely to a cont. lin map

$$\gamma_0 : H^1(\mathbb{R}^N_+) \to L^2(\mathbb{R}^{N-1}).$$

Let $v \in H^1(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$.

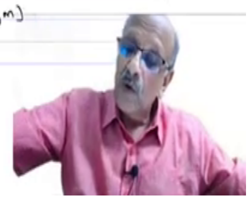
Extend v to $\mathbb{R}^N$ by reflection. $v \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$.

$\varepsilon_m \downarrow 0$ $\quad \rho_m = \rho_{\varepsilon_m}$ mollifier. $\quad \zeta \in \mathcal{D}(\mathbb{R}^N)$

supp$(\zeta) \subset \overline{B}(0;2)$ $\quad \zeta \equiv 1$ on $\overline{B}(0;1)$ $\quad 0 \le \zeta \le 1$.

$\zeta_m(x) = \zeta(x/m)$ $\quad$ supp$(\zeta_m) = \overline{B}(0;2m)$

$\zeta_m \equiv 1$ on $\overline{B}(0;m)$



$\zeta_m(x) = \zeta(x/m)$ $\quad$ supp$(\zeta_m) = \overline{B}(0;2m)$

$\zeta_m \equiv 1$ on $\overline{B}(0;m)$

$\rho_m * v(x) \to v(x)$ ptwise.

$\rho_m * v \to v$ in $H^1(\mathbb{R}^N)$

$v_m = \zeta_m(\rho_m * v) \to v$ in $H^1(\mathbb{R}^N)$.

$\Rightarrow v_m \to v$ in $H^1(\mathbb{R}^N_+)$

$\Rightarrow \gamma_0(v_m) \to \gamma_0(v)$ $\quad L^2(\mathbb{R}^{N-1})$.

$\rho_m * v \to v$ in $H^1(\mathbb{R}^N)$

$v_m = \zeta_m(\rho_m * v) \to v$ in $H^1(\mathbb{R}^N)$.

$\Rightarrow v_m \to v$ in $H^1(\mathbb{R}^N_+)$

$\Rightarrow \gamma_0(v_m) \to \gamma_0(v)$ $\quad L^2(\mathbb{R}^{N-1})$.

$v_m = \zeta_m(\rho_m * v) \to v$ ptwise.

$v_m|_{\mathbb{R}^{N-1}} = \gamma_0(v_m)$. $\quad v_m \in \mathcal{D}(\mathbb{R}^N)$

$\longrightarrow v|_{\mathbb{R}^{N-1}}$ ptwise.

$\Rightarrow \gamma_0(v) = v|_{\mathbb{R}^{N-1}}$.

And therefore extends uniquely to a continuous linear map which we call from

$$\gamma_0 : H^1(\mathbb{R}^N_+) \to L^2(\mathbb{R}^{N-1}).$$

So, now we have to show. So, now let v belong to $H^1(\mathbb{R}^N_+)$intersection continuous in $\mathbb{R}^N$ plus closure, so we want to show that gamma naught of v is nothing but the restriction. So, now extend v to $\mathbb{R}^N$ by reflection, then v belongs to $H^1(\mathbb{R}^N)$ this we know because the reflection is a prolongation operator and it is also because it is continuous on $\mathbb{R}^N$ plus and you are just reflecting it this also in C of $\mathbb{R}^N$

So, we have all these things. Now, you choose ϵ_m decreasing to 0, $\rho_m = \rho * \epsilon_m$ mollifier and then ς in D of $\mathbb{R}^N$ support of ς in B(0,2) and zeta identically 1 on B(0,1), $0 \leq \varsigma \leq 1$, and we put $\varsigma_m(x) = \varsigma(x/m)$ and therefore support of ς_m equals B(0,m) and ς_m is identically 1 on B (0, m).

So, all these we know before and what do you know? One of the first results which we have seen is that $\rho_m * v(x) \to v(x)$ pointwise, because you have a continuous function convolving with the mollifier. It just gives you converges to v of x, we have seen this when studying convolution of functions and also we have that v, m converges to v in $H^1(\mathbb{R}^N)$ and we have that ς_m, v_m.

All this we have seen in the very first theorem where we proved that the $D(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, $W^{1,p}(\mathbb{R}^N)$, so the same things we are doing here and therefore you have vm converges to v in $H^1(\mathbb{R}^N_+)$ as well and that implies that $\gamma_0(v_m) \to \gamma_0(v)$ in $L^2(\mathbb{R}^{N-1})$.

But what is ς_m? $\rho_m * v \to v$ point wise because zeta m is identically 1 on B 0 m, so you have bigger and bigger and so eventually every point will come under some big ball of radius m and consequently you will have that this becomes stationary point wise and consequently you will have that it also converges to 0 point wise and but what is v m restricted to Rn minus 1.

$\mathbb{R}^{N-1}$ is nothing but $\gamma_0(v_m)$ because that is how we proved, we extended the γ_0 was defined by this because $v_m \in D(\mathbb{R}^N)$ and therefore $\gamma_0(v_m)$ will converge it, converge and we know that $v_m \rightarrow \mathbb{R}^{N-1}$, converges to $v|_{\mathbb{R}^{N-1}}$. No, sorry, but because you know that it converges in $L^2(\gamma_0(v_m))$for some subsequence must converge point wise to $\gamma_0(v)$.

And therefore we have that $\gamma_0(v)$ is nothing but… so this implies that $\gamma_0(v) = v|_{\mathbb{R}^{N-1}}$. So, that is… this converges to $v|_{\mathbb{R}^{N-1}}$ point wise and since it also converges in L^2 therefore you have that these two are equal, so that proves this theorem.