Sobolev Spaces and Partial Differential Equations
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Spaces W^{s, p} and Trace spaces

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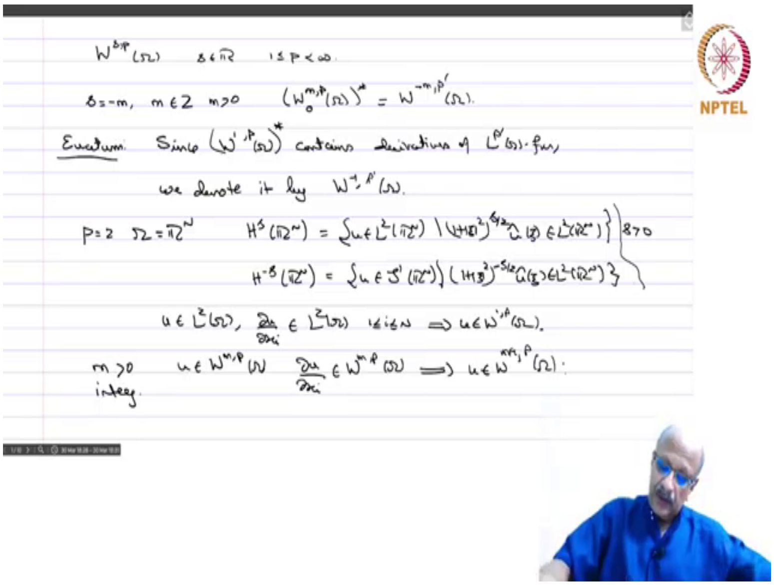

So, we are looking at the different definition of $W^{s,p}(\Omega)$, $s \in \mathbb{R}$, and $1 \leq p < \infty$. So, if $s = -m$, $m \in \mathbb{Z}$, $m \geq 0$, then we define it as the $(W_0^{m,p}(\Omega))^* = W^{-m,p'}(\Omega)$. So, this is how we define it. So, in this case there are some, I want to correct, something which I wrote last time.

Erratum: So, I wrote the following. Since $(W^{1,p}(\Omega))^*$ contains derivatives of $L^{p'}(\Omega)$ functions, we denote it by $W^{-1,p'}(\Omega)$, so maybe 1 p omega minus 1 p dash. So you might have noticed the mistake. So, this is, it should be the dual in this place and that is, please correct that mistake which we have. So, we said this.

Then in case of $p = 2$, $\Omega = \mathbb{R}^N$, then we define that

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \colon (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}.$$

$$H^{-s}(\mathbb{R}^N) = \{u \in S'(\mathbb{R}^N) \colon (1 + |\xi|^2)^{-\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}.$$

So, this is both of them for $s \geq 0$ and this is how we define it.

And then we showed that in fact when s equals 1 this is indeed the case. So, now let us by definition if $u \in L^2(\Omega)$ and $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, $1 \leq i \leq N$, then by definition this means $u \in W^{1,p}(\Omega)$. Similarly, if $u \in W^{m,p}(\Omega)$, m positive integer and $\frac{\partial u}{\partial x_i} \in W^{m,p}(\Omega)$, again by definition this implies that $u \in W^{m+1,p}(\Omega)$. So, this just comes by definition.

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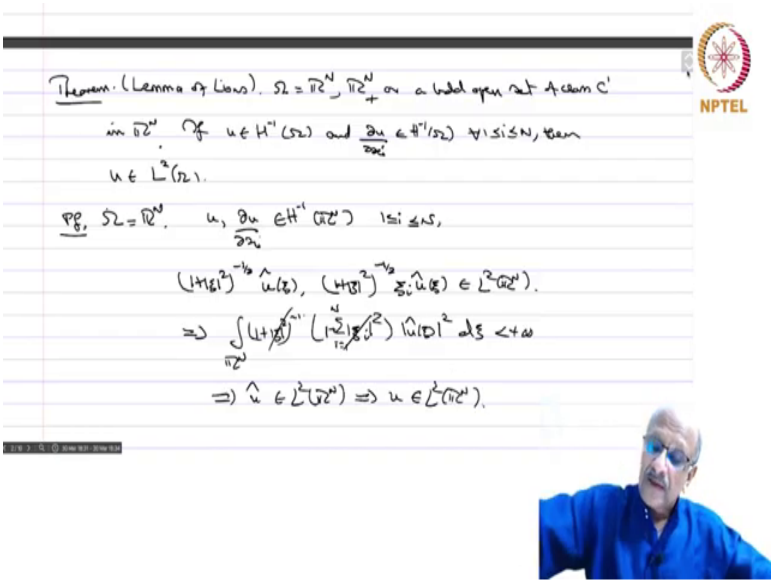

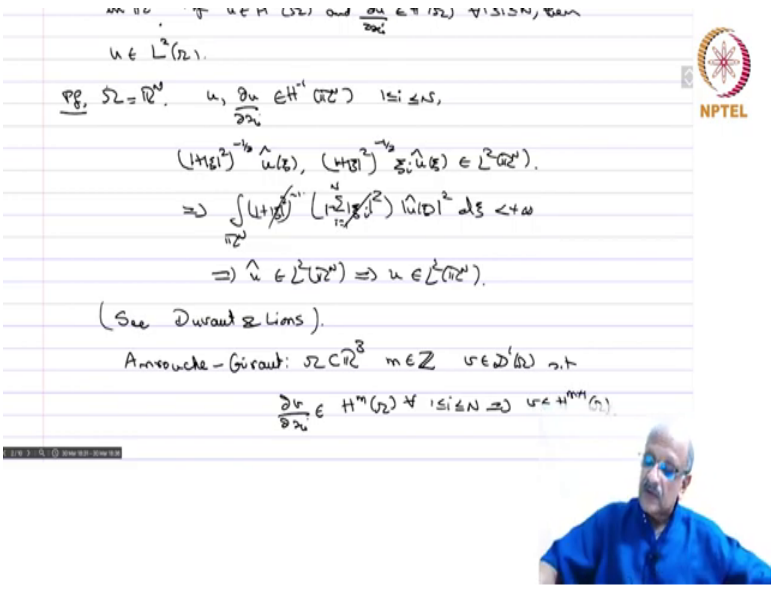

So, now we want to extend this to the case of negative indices also and so we have a very nice lemma, so this is a theorem and it goes by the name sometimes of lemma of Lions.

Theorem: So, $\Omega = \mathbb{R}^N$, $\mathbb{R}_+^N$, or a bounded open set of class C^1 $in\, \mathbb{R}^N$. If $u \in H^{-1}(\Omega)$, $\frac{\partial u}{\partial x_i} \in H^{-1}(\Omega)\ \forall\ 1 \le i \le N$, then $u \in L^2(\Omega)$.

proof: We will do it in the case $\Omega = \mathbb{R}^N$ where it is very easy. So,

$u \in H^{-1}(\Omega)$, $\frac{\partial u}{\partial x_i} \in H^{-1}(\Omega)\ \forall\ 1 \le i \le N$, then

$$(1 + |\xi|^2)^{\frac{1}{2}}\hat{u}(\xi),\ (1 + |\xi|^2)^{-\frac{1}{2}}\xi_i\,\hat{u}(\xi) \in L^2(\mathbb{R}^N)$$

because I am taking the Fourier transform of $\frac{\partial u}{\partial x_i}$ with a 2 pi i coming in but that is just a constant, so all these belong to $L^2(\mathbb{R}^N)$. And this means that

$$\int_{\mathbb{R}^N} (1 + |\xi|^2)^{-1}(1 + \sum_{i=1}^{N} |\xi_i|^2)|\hat{u}(\xi)|^2 d\xi < \infty .$$

$$\Rightarrow \hat{u} \in L^2(\mathbb{R}^N) \Rightarrow u \in L^2(\mathbb{R}^N).$$

So, for a full proof, see the book of Duraut and Lions, where you can have a proof of this. It is not very easy. Several generalizations of this result are available and one of the most comprehensive results is due to Amrouche and Giraut, which says that: if $\Omega \subset \mathbb{R}^3$ and m is any integer in Z positive or negative and if $v \in D'(\Omega)$ such that $\frac{\partial v}{\partial x_i} \in H^m(\Omega)$ for all $1 \le i \le N$, then this implies that $v \in H^{m+1}(\Omega)$.

So, v is just a distribution in this case. So, in fact, we have done one exercise long ago in the, where we said that the v is in distribution such that $\frac{\partial v}{\partial x_i}$ is in L^2, then $v \in H^1$, so we have seen such things here. So, now we will use this Lion's lemma in a very nice way to prove a very important inequality which is fundamental in the theory of elasticity which we will see in the next chapter. So, for that we need some notation.

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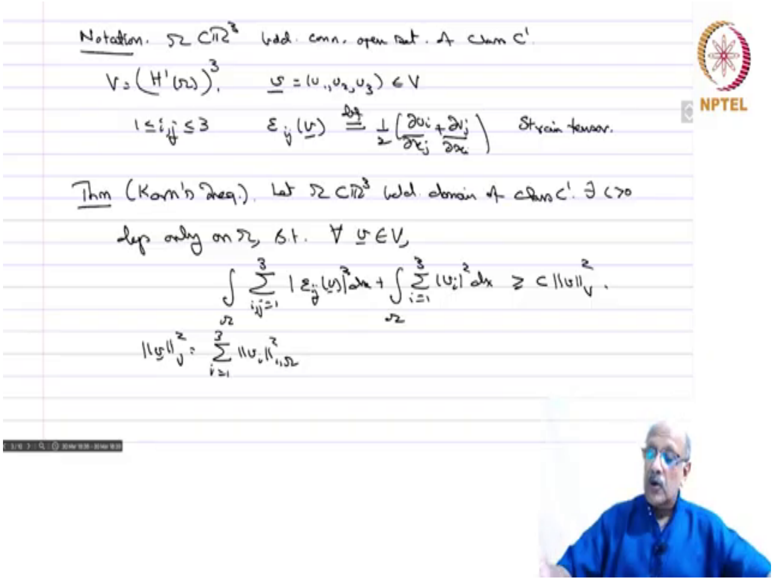

Notation: $\Omega \subset \mathbb{R}^3$ bounded open set, bounded connected open set, class C^1, $V = (H^1(\Omega))^3$, so each component is in $H^1(\Omega)$ and you have the product norm. So, if $\underline{v} = (v_1, v_2, v_3) \in V$, so then we define for $1 \leq i, j \leq 3$,

$\epsilon_{ij}(v) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ ------ in elasticity theory this is called the strain tensor. It is a symmetric thing, i j is the same as j i, so this is called a symmetric thing.

So, norm v is the usual product norm which is got from $H^2(\Omega)$.

Theorem: (Korn's inequality). Let $\Omega \subset \mathbb{R}^3$ bounded domain which means connected open set of class C^1. There exists a $C > 0$ depending only on omega such that for every $\underline{v} \in V$ we have

$$\int_\Omega \sum_{i,j=1}^{3} |\epsilon_{ij}(v)|^2 dx + \int_\Omega \sum_{i=1}^{3} |v_i|^2 dx \geq C\, ||v||_V^2 .$$

$$||\underline{v}||_V^2 = \sum_{i=1}^{3} ||v_i||_{1,\Omega}^2 .$$

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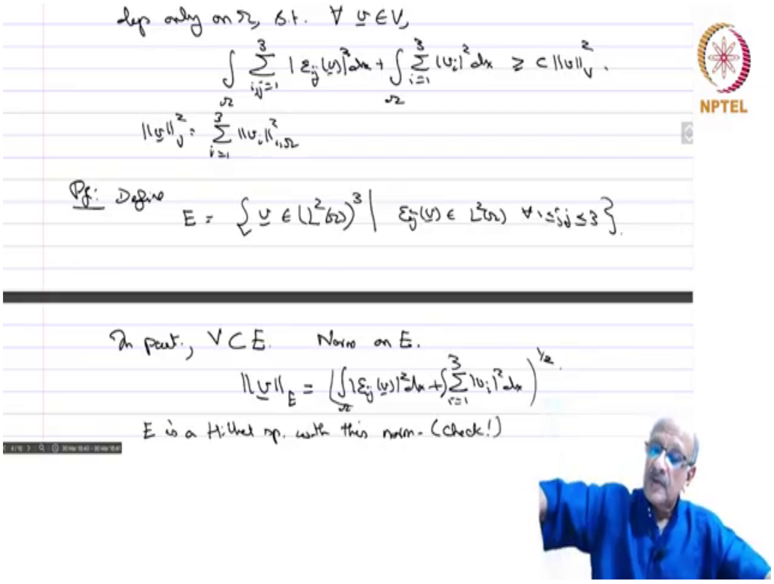

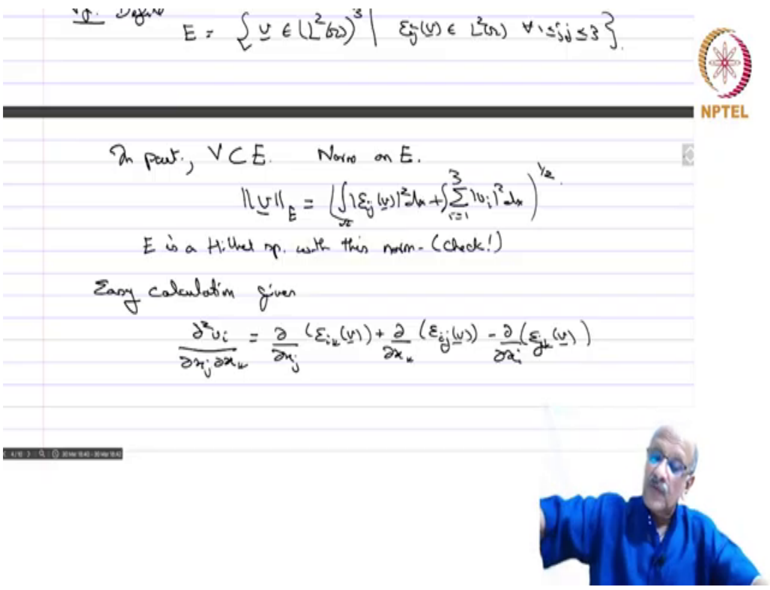

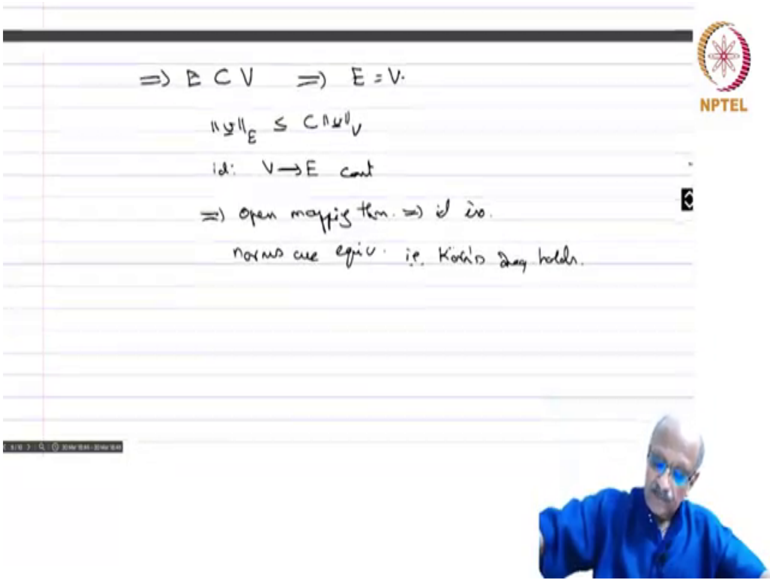

proof: Define

$$E = \{\underline{v} \in (L^2(\Omega))^3 : \epsilon_{ij}(\underline{v}) \in L^2(\Omega),\ 1 \le i, j \le 3\}.$$

So, in particular V is contained in E because if v is in H^1 all the first derivatives are in L^2 and therefore epsilon ij of v is also in L2, so automatically we have V is contained in E.

Then you equip norm on E is given by

$$||\underline{v}||_E = \left(\int_\Omega |\epsilon_{ij}(\underline{v})|^2 dx \ + \ \int_\Omega \sum_{i=1}^{3} |v_i|^2 dx\right)^{\frac{1}{2}}.$$

So, then E is a Hilbert space with this norm. I leave you to check this, so this is some simple checking which you have to do. So, you just have to show, well it is a norm, it comes from an inner product is obvious because you have all L2 terms here and then all you have to do is to check it is complete, so if you take a Cauchy sequence, already it will be Cauchy in L2, so you will have a convergent subsequence, then you have to show that this will also have epsilon ij of v will also convert to some vij. Now, you must show the limit is precisely epsilon ij of the limit of the base.

So, that is the standard checking term. Now, easy calculation gives the following

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}(\epsilon_{ik}(\underline{v})) + \frac{\partial}{\partial x_k}(\epsilon_{ij}(\underline{v})) - \frac{\partial}{\partial x_i}(\epsilon_{jk}(\underline{v})).$$

So, this is just algebra, you just have to write out and expand. so this means that E is contained in V. And the converse we already show, so E=V. So, you have that the two spaces are vector spaces. Now, we know that $||\underline{v}||_E \leq C||\underline{v}||_V$.

So, if you take the square, take the L^2 norm, etcetera that will be less than some constant times the norms of the first derivatives and therefore you will get that this is nothing but less than equal to constant times this. That means the $id: V \to E$ is continuous, which means by the open mapping theorem one, one on two continuous implies identity map is an isomorphism, therefore norms are equivalent and that is precisely Korn's inequality.

Because what is the Korn's inequality, it is precisely saying norm v is less than C times the other norm, so that is exactly the Korn's inequality. So, this very nice application of the open mapping theorem and it is nice, it is an important inequality in the theory of PDEs.

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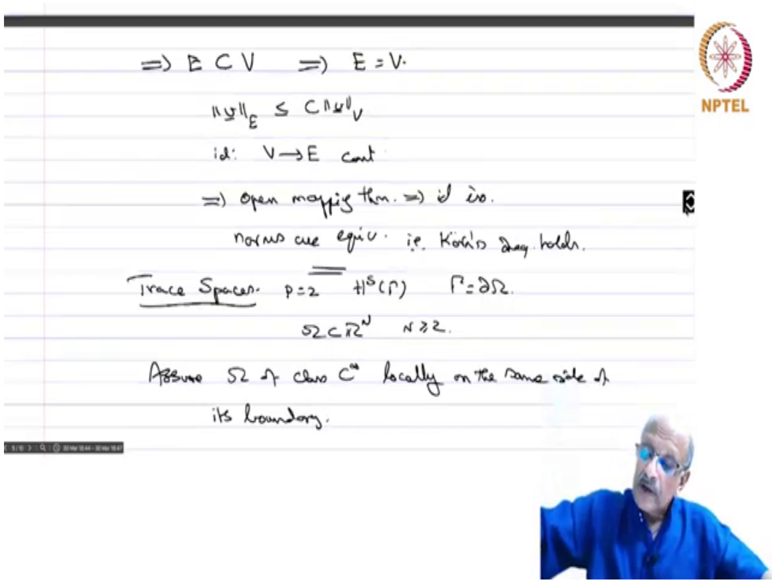

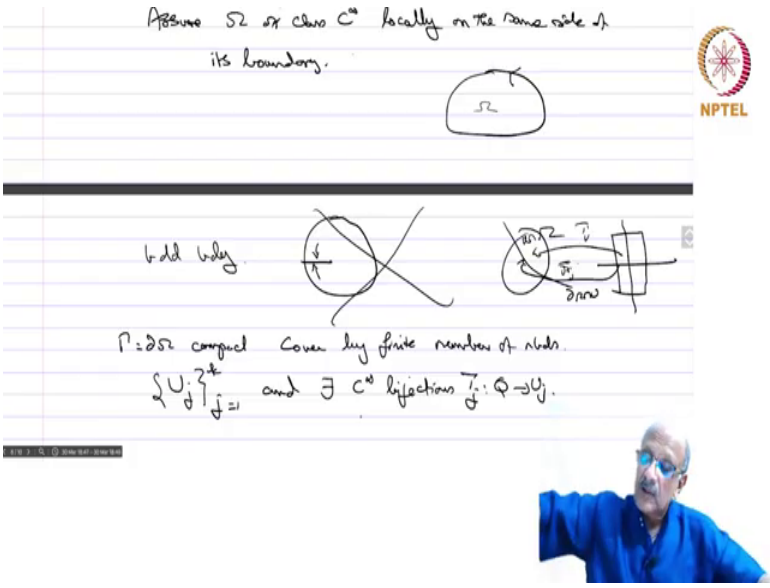

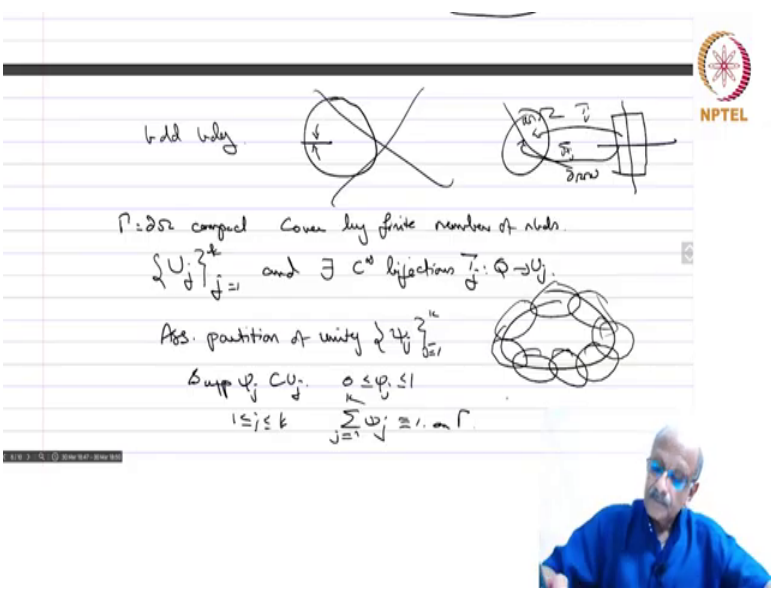

So, now we conclude with what we call the trace spaces.

Trace spaces: So, we are going to deal with p=2, though we can deal with other things, so we want to define $H^{s}(\Gamma)$ where $\Gamma = \partial\Omega$. So, $\Omega \subset \mathbb{R}^{N}$, $N \geq 2$, opens set, so assume Ω of class C^{∞}, so this much is not necessary.

But we do not want to fuss around, so we will assume maximum smoothness, so that we do not have to make any qualifications anywhere but you of course you can reduce the necessary smoothness certainly, and locally on the same side of its boundary. So, when you move along the boundary the domain is always on one side of it depending whether you go clockwise or anti-clockwise.

So, you have domains like this, so if you go like this in an anti-clockwise direction the domain will always be on the left of it. So, what we are not allowing is that a domain like this, suppose I have a domain like this and this is also taken away. Now, this domain, so you have a domain on both sides of this boundary and therefore this is not allowed such domains, so now gamma equals $\partial\Omega$ is compact.

Ω class of C^{∞} and bounded, so you can cover by finite number, and its same side of the boundary and since you say gamma is, when you say class C^{∞} we mean bounded boundary and therefore gamma will be compact, so we can cover by a finite number of neighborhoods

$\{U_j\}_{j=1}^{k}$, and there exists C^{∞} bijections T_j: $Q \to U_j$.

If you recall what is, so you had Q here and then you had the boundary, so every neighborhood you have you, so Q_+ will be mapped to U intersection omega and then Q_0 will be mapped to d omega intersection u and all these Tj and Tj inverse are, these are Tj, Tj inverse are all C^{∞}.

So, associated partition of unity $\{\psi_j\}_{j=1}^{k}$, , so we are only covering the boundary now, so we have a finite number of sets for the boundary, so that is an open collection of open sets, union is open, so you take a partition of unity corresponding to that. So, $\mathrm{supp}(\psi_j) \subset U_j$,

$1 \leq i \leq k,\ 0 \leq \psi_j \leq 1\ and\ \sum_{j=1}^{k} \psi_j = 1\ on\ \Gamma.$

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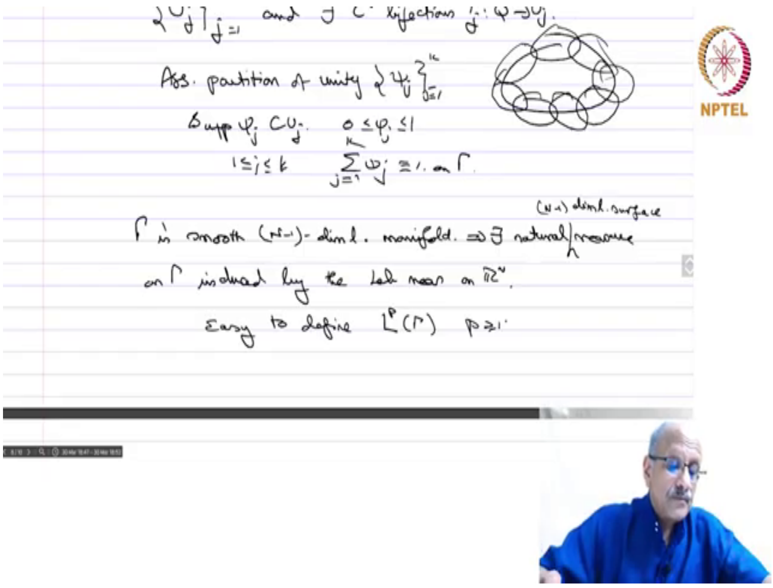

Now, Γ is a smooth $(N - 1)$ dimensional manifold, which implies there exists a natural measure on Γ induced by the Lebesgue measure on $\mathbb{R}^N$. You can do this in many ways, you can use an induced Lebesgue measure, you can also call the $(N - 1)$ dimensional, Housdoff measure, Minkowski, the various ways of defining the measure.

However, since the domain is smooth all these things will agree with each other and therefore you have some natural measure n minus 1 dimensional surface measure, natural, so n minus 1 dimensional surface measure on gamma induced by Lebesgue measure in $\mathbb{R}^N$. Therefore, we can define an $L^p(\Gamma),\ for\ p \geq 1$.

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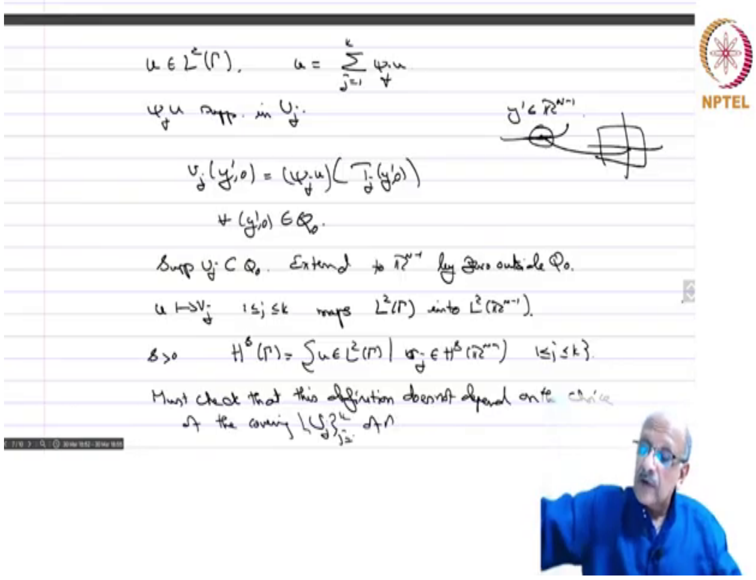

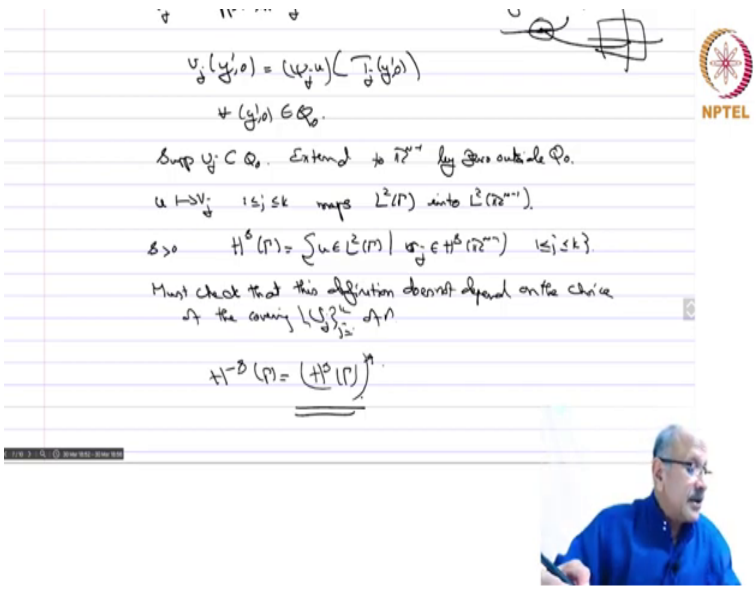

So, now let $u \in L^2(\Gamma)$, so then you can write $u = \sum_{j=1}^{k} \psi_j u$. Now $\psi_j u$ is supported in U_j. Now, you consider

$$v_j(y', 0) = (\psi_j u)(T_j(y', 0)) \ , \ \forall (y', 0) \in Q_0 .$$

So, what are you doing? You are taking a point on the axis in Q, from there you are going to u j to the boundary here and then you are taking the value of u psi ju which is supported in the set and then, so for all y dash 0 in q 0.

Now, $\text{supp}(U_j) \subset Q_0$, so extend to $\mathbb{R}^{N-1}$ by 0 outside Q_0. Now, $u \to v_j$, $1 \le j \le k$, maps $L^2(\Gamma) \to L^2(\mathbb{R}^{N-1})$, because you have taken the image using some C infinity maps. This is in L2, and then therefore this will be in $L^2(Q_0)$ outside you extend it by 0 and therefore they are all in $L^2(\mathbb{R}^{N-1})$. So, if s is positive then you define

$$H^s(\Gamma) = \{u \in L^2(\Gamma) \colon v_j \in H^s(\mathbb{R}^{N-1}), 1 \le j \le k\}.$$

Now, we must check, we will not do this, must check that this definition does not depend on the choice of the covering $\{U_j\}_{j=1}^{k}$ of Γ. So, whatever cover you take, finally the spaces which you get will always be the same. And if we define $H^{-s}(\Gamma) = (H^s(\Gamma))^*$, the dual space.

So, this is how we define H^s spaces on the boundary. You can do it for the other p also, we have done it for p=2. There is a similar way you can easily do it for other p as well. Now, these are called the trace spaces which we will use when studying boundary value problems later on.