Sobolev Spaces and Partial Differential Equations

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The spaces W^{s, p}

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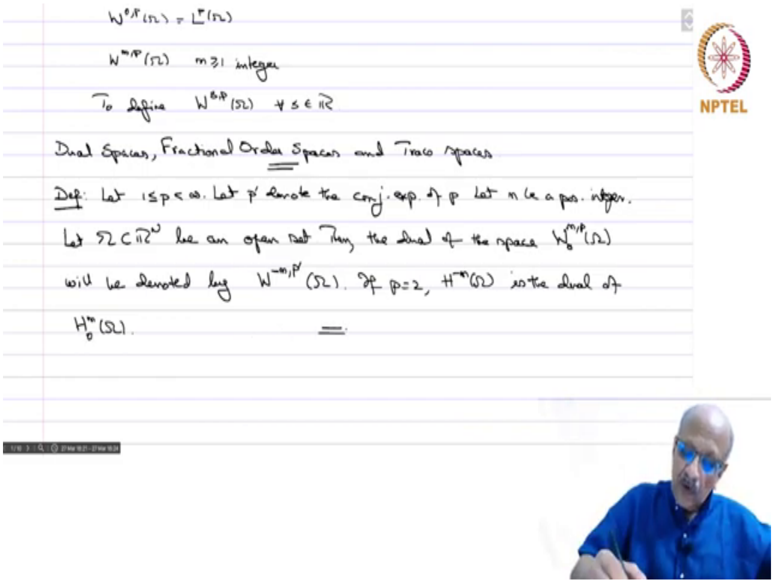

So, we defined $W^{0,p}(\Omega) = L^p(\Omega)$. Then we define $W^{m,p}(\Omega)$, so this is for all $m \geq 1$ integer. Now, we want to define $W^{s,p}(\Omega)$ for all $s \in \mathbb{R}$, so this is our aim, so this is what we are going to do now. So, we are going to study dual spaces, fractional order spaces and trace spaces. Trace spaces are the Sobolev spaces defined on the boundary, so this is our program for the moment.

So, we start with the negative integers. So, we have the following definition:

Definition: Let $1 \leq p < \infty$, let p' denote the conjugate exponent of p. Let m be a positive integer. Let $\Omega \subset \mathbb{R}^N$ be an open set. Then the dual of the space $W_0^{m,p}(\Omega)$ will be denoted by $W^{-m,p'}(\Omega)$. If $p = 2$, $H^{-m}(\Omega)$ is the dual of $H^m_0(\Omega)$. So, this is the definition of the Sobolev spaces for the negative integers.

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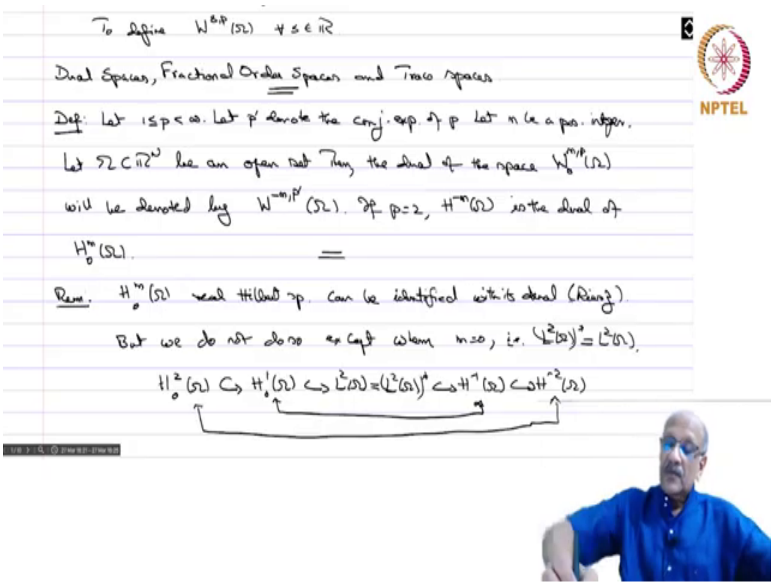

So, remark now.

Remark: $H^{m}_{0}(\Omega)$ real Hilbert space, so in principle by the Riesz representation theorem can be identified with its own dual. This is the Riesz representation theorem, but we do not do so. But we do not do so except when m equals 0, that is $L^{2}(\Omega)^{*} = L^{2}(\Omega)$.

So, we have, when we have a tower of Hilbert spaces, we will have

$$H^{2}_{0}(\Omega) \rightarrow H^{1}_{0}(\Omega) \rightarrow L^{2}(\Omega) = L^{2}(\Omega)^{*} \rightarrow H^{-1}(\Omega) \rightarrow H^{-2}(\Omega).$$

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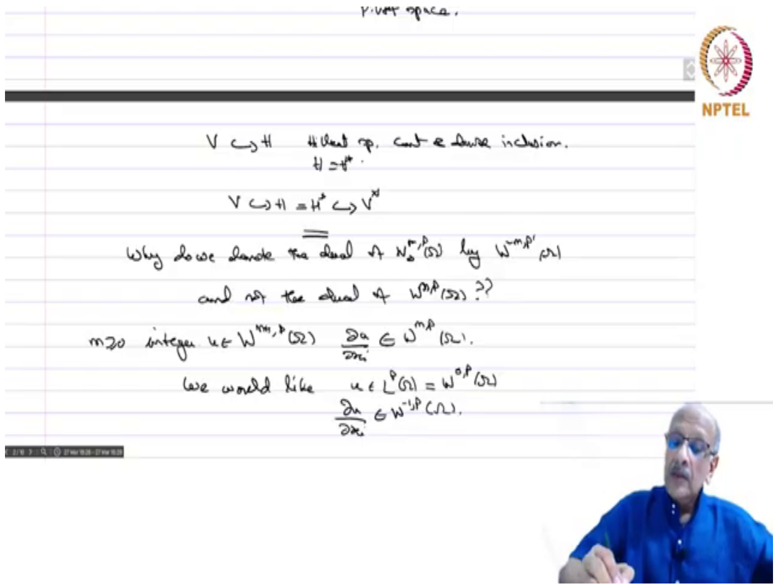

Now, why do we have these inclusions? Because in general if we have V contained in H in Hilbert spaces continuous and dense inclusion H identified with H*, then we have V is contained in H which is equal to H star, which can be identified as a subspace of V*. So, this standard thing in Hilbert space theory.

So, whenever you have a dense and continuous inclusion then the dual will be included in the opposite direction; that is true and since we identify the two duals so we have this dual so you have this inclusion. Now, it is obvious why we do not want to simultaneously identify v with v star because then all of them will become equal which is absurd and therefore we always keep them separate.

So, when you have a tower of Hilbert spaces, so you have one space which is called the Pivot space, which is identified with its own dual and all other duals are separated and you have because of the continuous and dense inclusions, you have the dense inclusions and continuous inclusions in the opposite direction. So, now more thing, the presence of p dash in the dual is clear because we are dealing with LP spaces and therefore the conjugate exponent will naturally appear in the dual space.

So, but why do we denote the dual of $W_0^{m,p}(\Omega)$ by $W^{-m,p'}(\Omega)$ and not the dual of $W^{m,p}(\Omega)$ itself. Now, there is a reason for this. The reason for this is now if you have W m plus 1, so m is a positive integer, then if you take $u \in W^{m,p}(\Omega)$, then $\frac{\partial u}{\partial x_i} \in W^{m,p}(\Omega)$.

So, we would like this to continue, so we would like if $u \in L^p(\Omega)$ which is $W^{0,p}(\Omega)$ then $\frac{\partial u}{\partial x_i} \in W^{-1,p}(\Omega)$. We want this to happen. Now, this will happen only with the definition which we have given as the following proposition we will prove.

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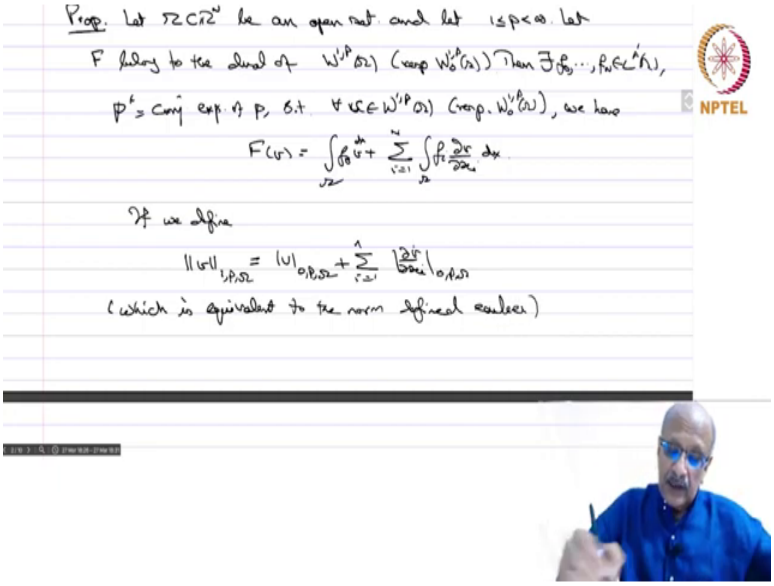

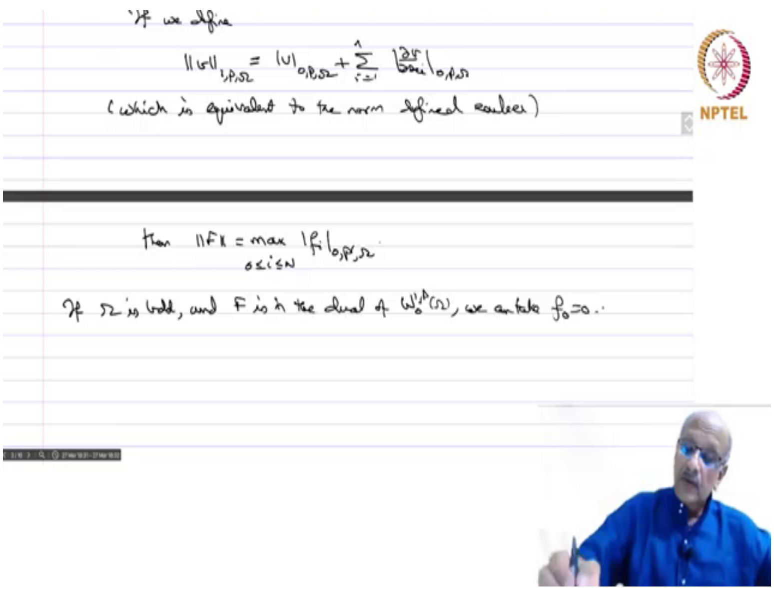

proposition: let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$, let capital F belong to the dual of $W^{1,p}(\Omega)$, (respectively $W_0^{1,p}(\Omega)$), then there exists $f_0, \ldots, f_N \in L^{p'}(\Omega)$, p dash equals conjugate exponent of p such that for all $u \in W^{1,p}(\Omega)$, (respectively $W_0^{1,p}(\Omega)$) we have

$$F(v) = \int_\Omega f_0 v \, dx + \sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} dx .$$

So, if we define $||v||_{1,p,\Omega} = |v|_{0,p,\Omega} + \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}|_{0,p,\Omega}$, which is equivalent to the norm defined earlier. If Ω is bounded and F is in the dual of $W^{1,p}{}_0(\Omega)$, we can take $f_0 = 0$.

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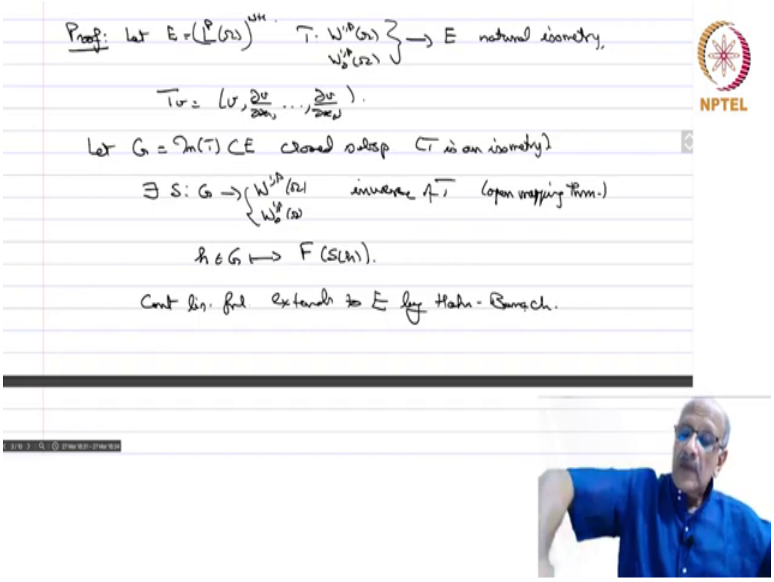

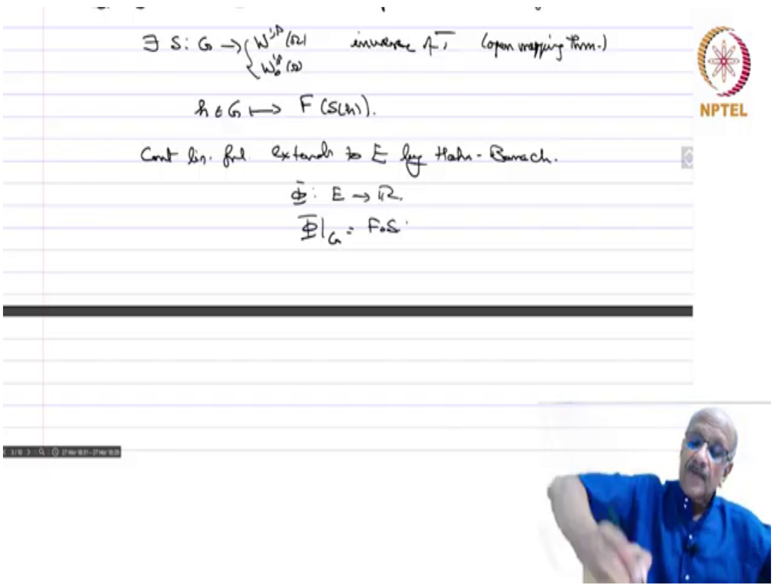

proof: So, let $E = L^p(\Omega)^{N+1}$ and let $T: W^{1,p}(\Omega)\ or\ W_0^{1,p}(\Omega) \rightarrow E$, the natural isometry, namely $T(v) = (v, \frac{\partial v}{\partial x_1},....,\frac{\partial v}{\partial x_N})$. Now, let $G = Im(T) \subset E$, a closed subspace because it is an isometry, T is an isometry, so close subspace of Banach space is also Banach and therefore by the open mapping theorem there exists an $S: G \rightarrow W^{1,p}(\Omega)\ or\ W_0^{1,p}(\Omega)$, inverse of T. So, this is the open mapping theorem. So, now you define $h \in G \rightarrow F(S(h))$, so this is a continuous linear functional and extends to E by Hahn - Banach, so we have a closed subspace on which we have (())(13:38) a function, thus preserving the norm, so this will extend it to by the Hahn Banach theorem. So, let us call that $\Phi: E \to \mathbb{R}$ and then $\Phi|_G = F \circ S$.

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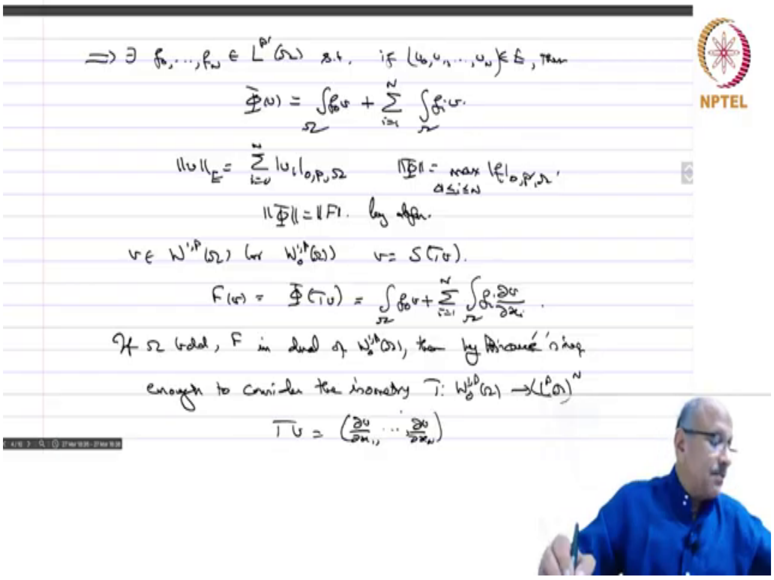

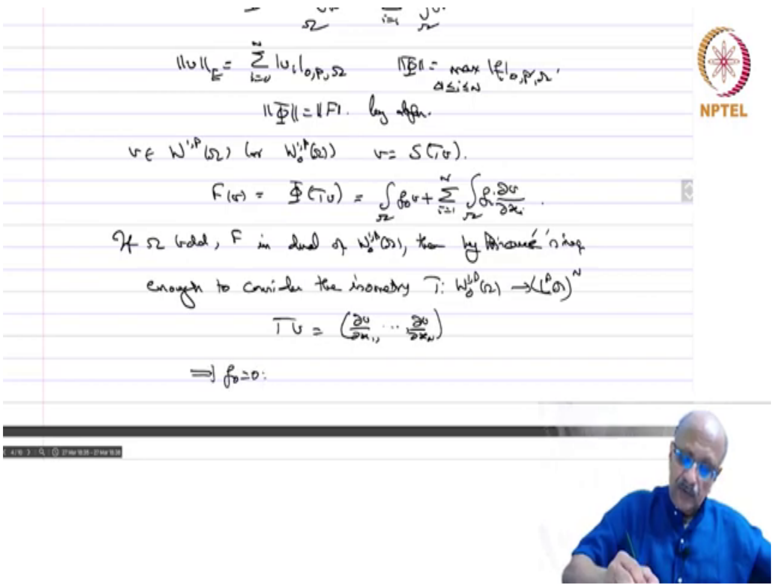

So, e is what LP (())(14:04), so this implies there exists $f_0,\ldots, f_N \in L^{p'}(\Omega)$, such that if $v_0, v_1,\ldots, v_N \in E$, then $\Phi(v) = \int_\Omega f_0 v\,dx + \sum_{i=1}^{N} \int_\Omega f_i v dx$, because it is just Riesz representation theorem is f naught v on omega plus sigma i equals 1 to n integral on omega fiv and if $||v||_k = \sum_{i=0}^{N} |v_i|_{0,p,\Omega}$, we have $||\Phi|| = \max_{0\le i\le N} |f|_{0,p',\Omega}$. This is just the product, dual space, dual of a product is given this way and of course, $||\Phi|| = ||F||$, by definition because it is a Hahn - Banach extension and therefore it is an extension. Now, for any $v \in W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, whichever we are considering, we have $v = S(T(v))$, therefore

$$F(v) = \Phi(Tv) = \int_\Omega f_0 v\, dx + \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial v}{\partial x_i} dx .$$

So, that proves one part of the theorem. If omega is bounded and F in dual of W1, p 0 of omega then by Poincare inequality enough to consider the isometry $T: W^{1,p}{}_0(\Omega) \to (L^p(\Omega))^N$, namely $T(v) = (\frac{\partial v}{\partial x_1},....,\frac{\partial v}{\partial x_N})$, because the norm here is given by just the LP norm the derivatives and therefore we can take $f_0 = 0$. So, this proves the proposition.

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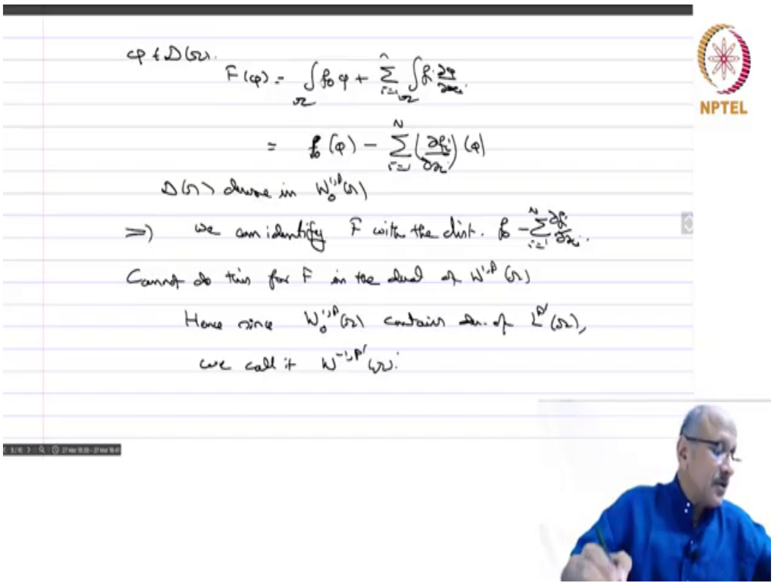

So, now what is the, how does this justify our notation? So, let us take $\phi \in D(\Omega)$. So, by definition,

$$F(\phi) = \int_\Omega f_0 \phi\, dx + \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial \phi}{\partial x_i} dx = f_0(\phi) - \sum_{i=1}^{N} \int_\Omega \frac{\partial f_i}{\partial x_i}(\phi) dx$$

So, these are the distribution derivatives and therefore, so you can write this, or let us leave it like that. So, then $D(\Omega)$ is dense in $W^{1,p}{}_0(\Omega)$ and therefore you can say we can identify f with the distribution, $f_0 - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$ and therefore, whereas if you are in the dual of $W^{1,p}$ (Ω), we cannot make, cannot do this for f in the dual of $W^{1,p}$ (Ω).

Because if you have a dense subspace then the definition of the functional is completely determined if you define it on the dense subspace, whereas if you are in this, then to the whole space there can be many possible extensions and therefore you cannot identify it with that distribution derivative there and therefore if you wanted…

So, that is why we say that the dual, so hence, since $W^{1,p}$ (Ω) contains derivatives of $L^{p'}(\Omega)$, we call it $W^{-1,p}$ (Ω). So, that is the reason which we are doing. So, now, that justifies it.

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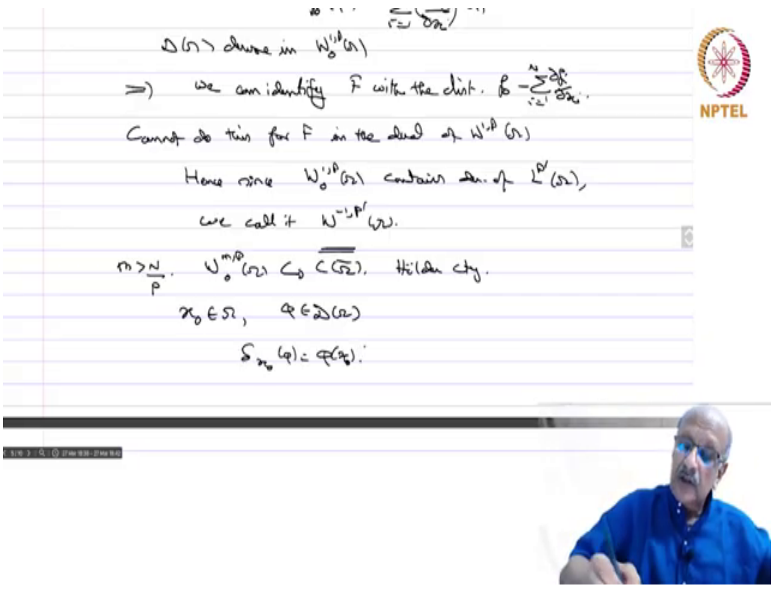

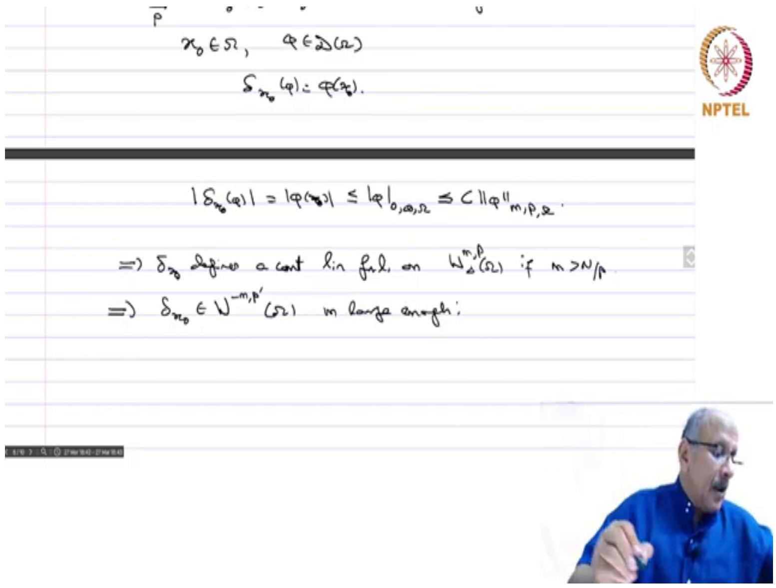

Now, let us take m bigger than N by P, then $W_0^{m,p}$ (Ω) is contained in $C(\overline{\Omega})$ and you have Holder continuous, Holder continuity is also there, therefore, if $x_0 \in \Omega$ and $\phi \in D(\Omega)$, we write $\delta_{x_0}(\phi) = \phi(x_0)$, the evaluation function, the Dirac distribution concentrated at this and you have $|\delta_{x_0}(\phi)| = |\phi(x_0)| \leq |\phi|_{0,\infty,\Omega} \leq C||\phi||_{m,p,\Omega}$. So, this implies that delta x naught defines a continuous linear functional on $W^{m,p}$ (Ω) if m is bigger than N over P. Therefore, this is for any p, therefore delta x naught belongs to Wm minus 1 p dash omega for m large enough.

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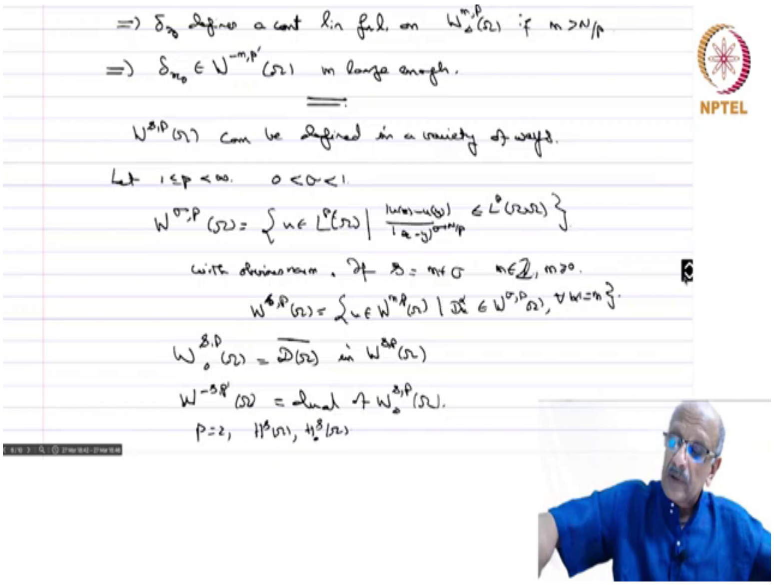

Now, you can define, so having defined the negative things, now we can define $W^{s,p}(\Omega)$ can be defined in a variety of ways and this gives rise to various types of Sobolev spaces, they sometimes called Beppo Levi's spaces and so on and so forth and but the thing, good thing is if omega is a C infinity set smooth open set, then all the definitions will agree, so that is the nice thing about it but depending on the singularities of the boundary you may get different spaces, different functions, the spaces may not be identical.

And so we can, there are various ways of defining depending on your necessity how to define it. So, one way of defining it is for the following; let $1 \leq p < \infty$ and you take $0 < \sigma < 1$, then you define

$$W^{\sigma,p}(\Omega) = \{u \in L^p(\Omega): \frac{|u(x)-u(y)|}{|x+y|^{\sigma+\frac{N}{p}}} \in L^p(\Omega \times \Omega)\}$$

So, you define this as $W^{\sigma,p}(\Omega)$ and with the obvious norm, namely the LP norm of this function, so that is the obvious norm. So, now if $s = \sigma + m, m \in \mathbb{Z}, m \geq 0$, you define

$$W^{s,p}(\Omega) = \{u \in W^{m,p}: D^{\alpha}u \in W^{\sigma,p}(\Omega)\ \forall\ |\sigma| = m\}.$$

So, then this is how you define the spaces.

Now, $W_0^{s,p}(\Omega) = \overline{D(\Omega)}$ in $W^{s,p}(\Omega)$ and $W^{-s,p}(\Omega)$ = dual of $W_0^{s,p}(\Omega)$. So, if p equals 2 we say $H^s(\Omega)$ and $H_0^s(\Omega)$, this is one way of defining Sobolev spaces for all the things.

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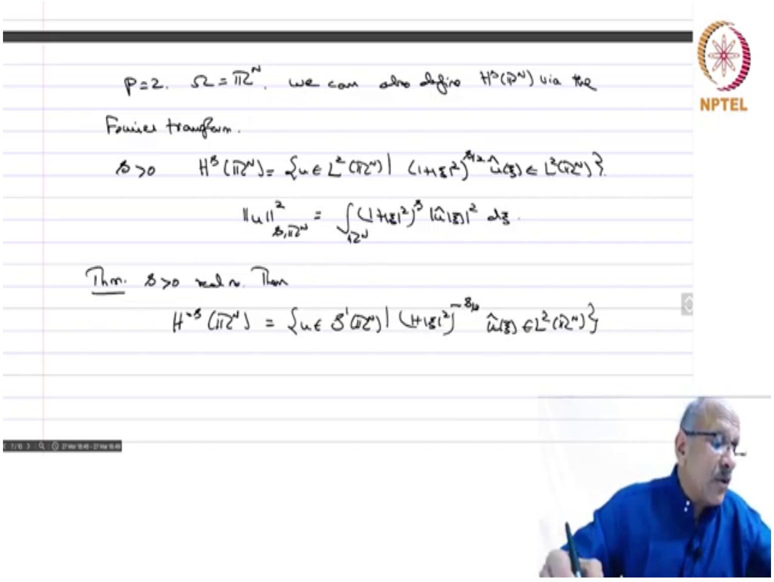


So, if $p = 2$, $\Omega = \mathbb{R}^N$, we can also define $H^s(\mathbb{R}^N)$ via the Fourier transform, so for s positive, we define

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}$$

and the associated norm,

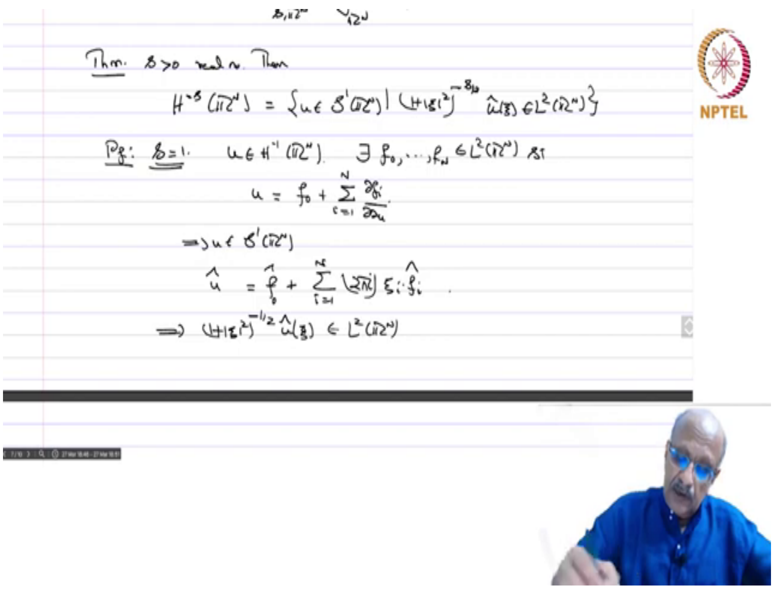
$$||u||^2_{s,\mathbb{R}^N} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| d\xi .$$

So, this when s equals m, we have already seen this when it is an integer, so we just simply do it the same way for all s, which is positive. So, now we show that this works for negative indices as well. So the theorem.

Theorem: $s > 0$ real number, then

$$H^{-s}(\mathbb{R}^N) = \{u \in E : (1 + |\xi|^2)^{-\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)\} .$$

where E is the set of tempered distributions in $\mathbb{R}^N$. (Refer Slide Time: 28:29)



proof: We will do it only in the case s equals 1. So, $u \in H^{-1}(\mathbb{R}^N)$, then we saw there exists f naught fn in L2 of Rn, this we saw in the previous proposition such that

$$u = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \Rightarrow u \in E'(\mathbb{R}^N).$$

Why is it so? Because L^2 functions are in the R tempered distribution, derivative of a tempered distribution is again a tempered distribution and therefore you have that u is in s dash of Rn and therefore you have u cap, we have seen the this this is equal to

$$\hat{u} = \hat{f}_0 + \sum_{i=1}^{N} (2\pi i)\xi_i \cdot \hat{f}_i \Rightarrow (1 + |\xi|^2)^{-\frac{1}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)$$

Therefore, you have that, so this proves one way, namely if it is in H minus 1 then all these things are true and now we want to prove the converse.

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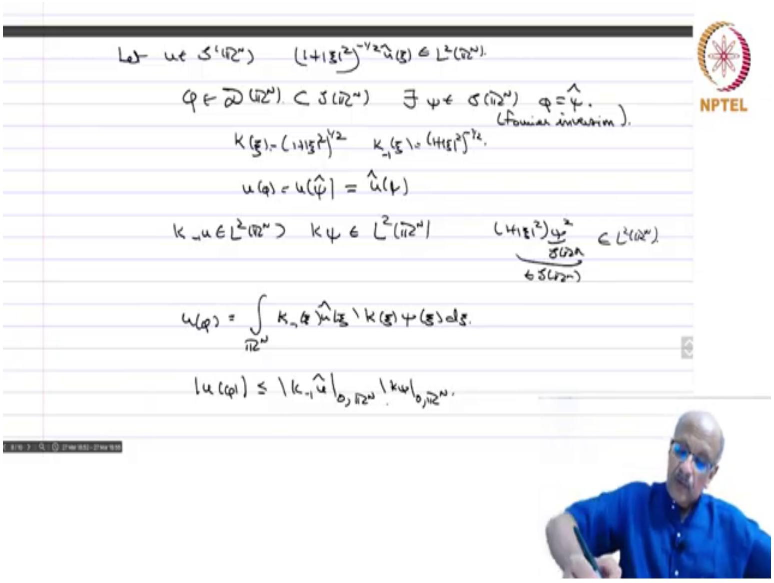

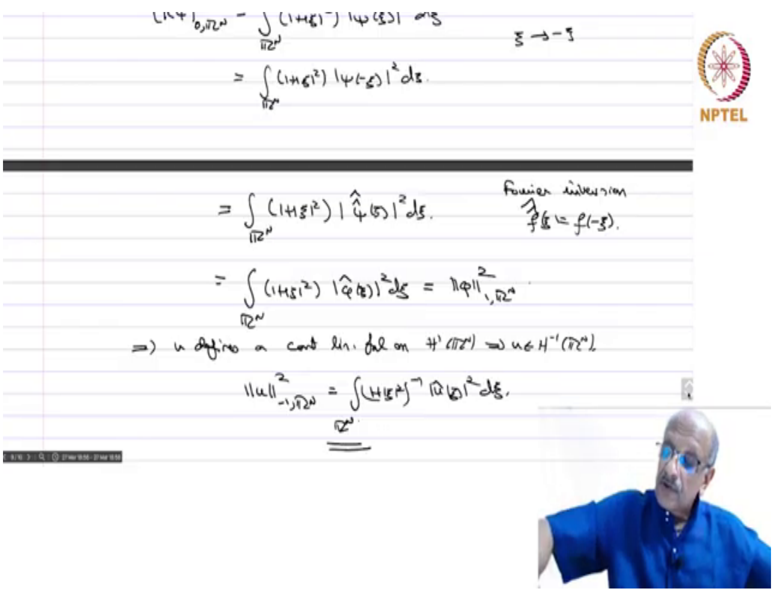

So, let $u \in S'(\mathbb{R}^N)$ and $(1 + |\xi|^2)^{-\frac{1}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^N)$. So, let $\phi \in D(\mathbb{R}^N) \subset S(\mathbb{R}^N)$, then the Fourier inversion formula, this is of course contained in s of Rn and therefore by the Fourier inversion formula there exists a $\psi \in S(\mathbb{R}^N)$ such that $\phi = \hat{\psi}$. This is Fourier inversion. So, now you take $k(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$ and $k_{-1}(\xi) = (1 + |\xi|^2)^{-\frac{1}{2}}$.

Now, $u(\phi) = u(\hat{\psi}) = \hat{u}(\psi)$. Now, $k_1 u \in L^2(\mathbb{R}^N)$and $k\psi \in S(\mathbb{R}^N)$ because that is the definition of $S(\mathbb{R}^N)$, you have, when you multiply by any such polynomial you will have that it is in, $k\psi \in L^2(\mathbb{R}^N)$. So, therefore you can write

$$u(\phi) = \int_{\mathbb{R}^N} k_1(\xi)\, \hat{u}(\xi) k(\xi)\, \hat{\psi}(\xi) d\xi$$

So, by the Cauchy Schwarz inequality, you have $|u(\phi)| \leq |k_{-1}\hat{u}|_{0,\mathbb{R}^N}\, |k\hat{\psi}|_{0,\mathbb{R}^N}$.

mod u phi is less than equal to k minus 1 u cap in L2, so 0 Rn into mod k times psi in 0 Rn.

This is some constant which we do not have to worry about, so let us look at what happens to k psi. So, k psi square 0 Rn L2 norm is equal to integral on Rn 1 plus mod z square into mod psi square d xi. Now, if I change the variable xi to minus xi, this will be integral over Rn, I will again get 1 plus mod xi square into mod psi of minus xi square d xi, which is equal to integral over Rn 1 plus mod xi square into mod psi hat hat xi square d xi.

Because by the Fourier, again by Fourier inversion of double hat xi is nothing but f of minus xi, so you can check that very easily just by writing out the formulae. So, this is equal to this and that is equal to integral over Rn 1 plus mod xi square into mod phi hat xi square d xi and that is nothing but norm phi in 1 Rn.

So, this implies that u defines continue, so you have, this one is a constant and this one is less than equal to constant times is nothing but norm phi square, norm phi in a 1 of Rn and therefore u defines a continuous linear functional on $H^1(\mathbb{R}^N)$, implies $H^{-1}(\mathbb{R}^N)$. So, we have proved completely that

$$||u||^2_{-1,\mathbb{R}^N} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-1} |\hat{u}(\xi)|^2 d\xi .$$

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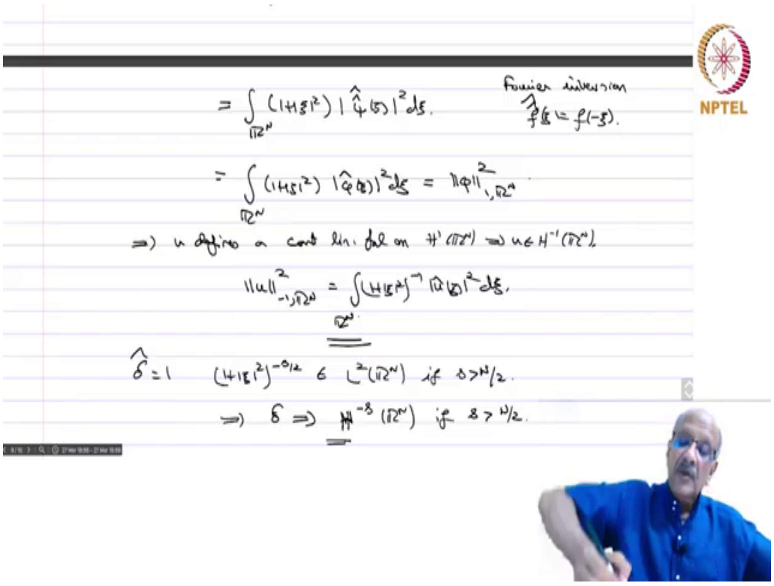

So, now we can recover for instance, we said that if m is sufficiently large then you have a delta is in $W^{-m,p}$ and so on. So, if you have $\hat{\delta} = 1$ and therefore you have $(1 + |\xi|^2)^{-\frac{s}{2}} \in L^2(\mathbb{R}^N)$, if $s > \frac{N}{2}$. So, this implies that $\delta \in H^{-s}(\mathbb{R}^N)$, if $s > \frac{N}{2}$.

So, whenever you, the same thing we already proved, $m > \frac{N}{p}$, we said $\delta \in W^{-m,p}$ and therefore the same way, the same result we are recovering in the case $p = 2$. So, we will continue with this later.