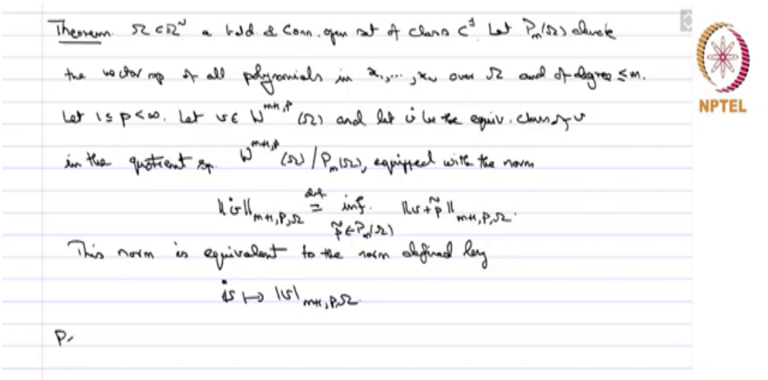
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Institute of Management Science
Compactness theorems – Part 3

So we were proving the various compactness theorems, namely the Rellich - Kondrachov theorem, which said that in a bounded open set the Sobelev embeddings almost all of them are compact embeddings, except one, namely when $p < N$, $W^{1,p} \to L^{p^*}$ is not compact. Apart from that all the rest we have compact and we proved it, for the case p bigger than n it comes directly from the Ascoli Arzela theorem.

And the fact that functions are in $W^{1,p}$ are Holder continuous. For $p = N$, We proved it using $p < N$ and for $p < N$ we use the Frechet Kolmogorov theorem, which characterizes relatively compact sets in L^{p^*}analogous to the Ascoli Arzela theorem. So, now we, these theorems are very useful when studying partial differential equations, especially nonlinear problems and Eigenvalue problems. And now we will see an application of this theorem.

(Refer Slide Time: 1:19)

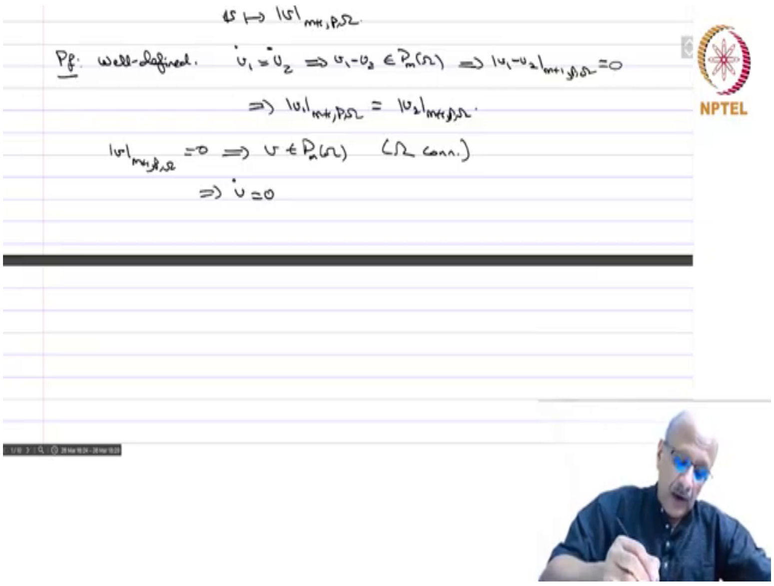


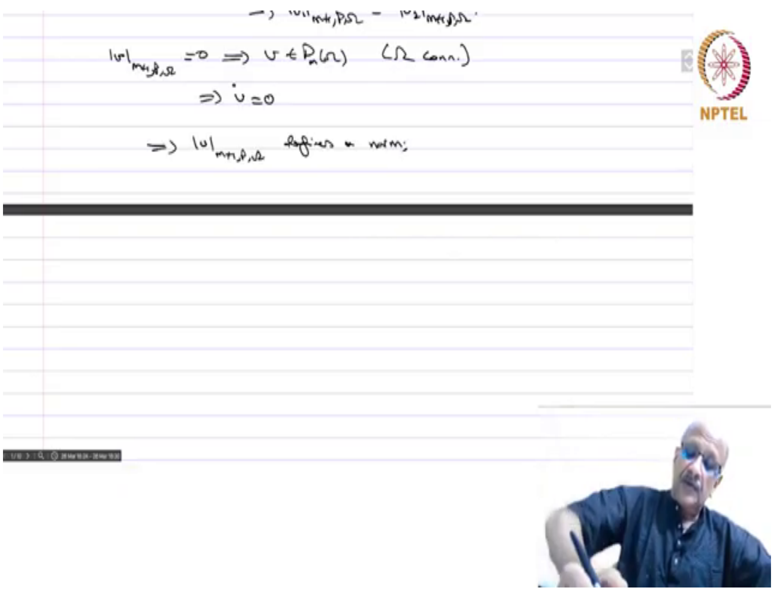
Theorem: $\Omega \subset \mathbb{R}^N$ -a bounded and connected open set of class C^1. Let $P_m(\Omega)$ denote the vector space of all polynomials in the variables $x_1, x_2,..., x_N$ over Ω and of degree less than or equal to m. Let $1 \leq p < \infty$, let $W^{m+1,p}(\Omega)$ and let $\dot{v}$ be the equivalence class of v, in the quotient space $W^{m+1,p}(\Omega)/P_m(\Omega)$, equipped with the norm

$$||\dot{v}||_{m+1,p,\Omega} = \inf_{\tilde{P}\in P_m(\Omega)} ||v + \tilde{P}||_{m+1,p,\Omega} \ .$$

So, this is the classical norm. This norm, so the theorem is this, is equivalent to the norm defined by $\dot{v} \rightarrow |v|_{m+1,p,\Omega}$.

(Refer Slide Time: 4:53)

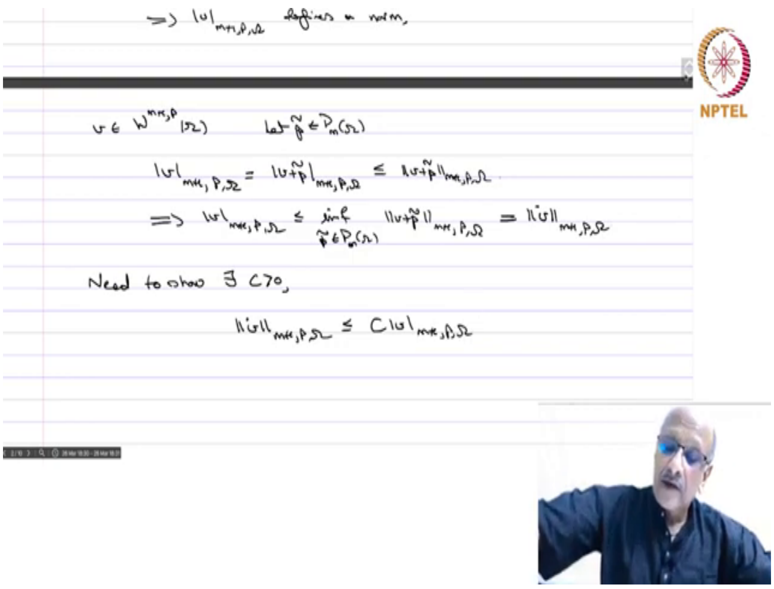


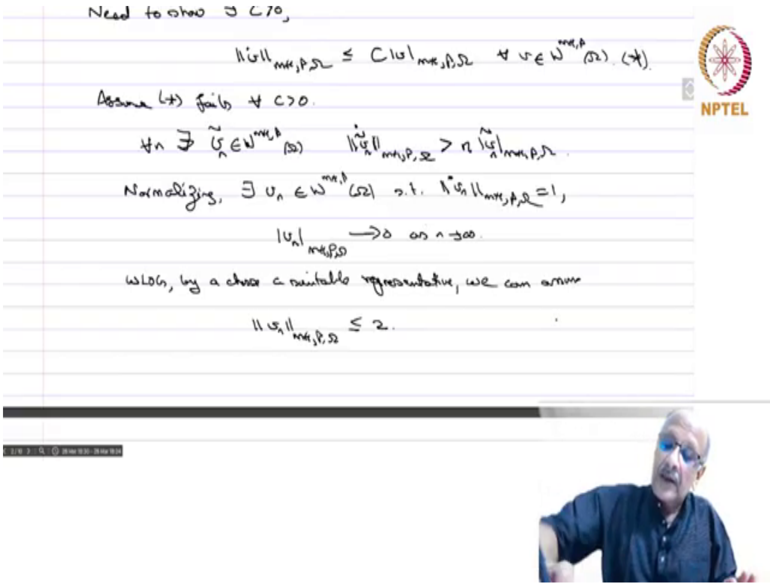


proof. First of all, we want to show that this is well defined. So, if $\dot{v}_1 = \dot{v}_2$, this implies that

$v_1 - v_2 \in P_m(\Omega) \Rightarrow |v_1 - v_2|_{m+1,p,\Omega} = 0 \Rightarrow |v_1|_{m+1,p,\Omega} = |v_2|_{m+1,p,\Omega}$ and therefore it is well defined. It does not depend on the representative which you are choosing in the quotient space. So, that is fine. So, the norm is well defined. And secondly if you have $|v|_{m+1,p,\Omega} = 0$, so this means that $v \in P_m(\Omega)$ because Ω is connected, so if the first derivative vanishes it is a constant, if the second derivative vanishes is the polynomial of degree 1, if the third derivative vanishes the polynomial degree 2 etcetera, if the m plus 1th order derivative vanishes then it has to be a polynomial of degree m.

That means v dot is equal to the 0 element in the quotient space, so therefore, and the converse of course is true, and therefore you have that this defines a norm. So, this $|v|_{m+1,p,\Omega}$ is in fact a norm on the quotient space, so there is no problem about that. (Refer Slide Time: 6:52)





So, now let us assume that $v \in W^{m+1,p}(\Omega)$, then you have $\widetilde{P} \in P_m(\Omega)$, it is a polynomial of degree less than equal to m, then

$$|v\ |_{m+1,p,\Omega} = |v + \widetilde{P}|_{m+1,p,\Omega} \leq ||v + \widetilde{P}||_{m+1,p,\Omega}$$

and this is true for all p. And therefore this implies that

$$|v + \widetilde{P}|_{m+1,p,\Omega} \leq \inf_{\widetilde{P} \in P_m(\Omega)} ||v + \widetilde{P}||_{m+1,p,\Omega} = ||\dot{v}\ ||_{m+1,p,\Omega}$$

So, one-way inequality we already have. So we want to show that the reverse inequality, so to need to show there exists a constant C positive such that the reverse is true,

$$||\dot{v}\ ||_{m+1,p,\Omega} \leq C|\dot{v}\ |_{m+1,p,\Omega},\ \ v \in W^{m+1,p}(\Omega) \text{ ----------- } (*)$$

So, we will do it by contradiction, so this is the standard method of doing it and see how compactness will come into play. So, assume the star fails. So, assume that this fails for every c positive. So, what does it mean? That means even if I take, however large C I take there will always be one for which the inequality will fail. So, I take C equals n for every n and therefore for every n there exists $\widetilde{v}_n \in W^{m+1,p}(\Omega)$ such that

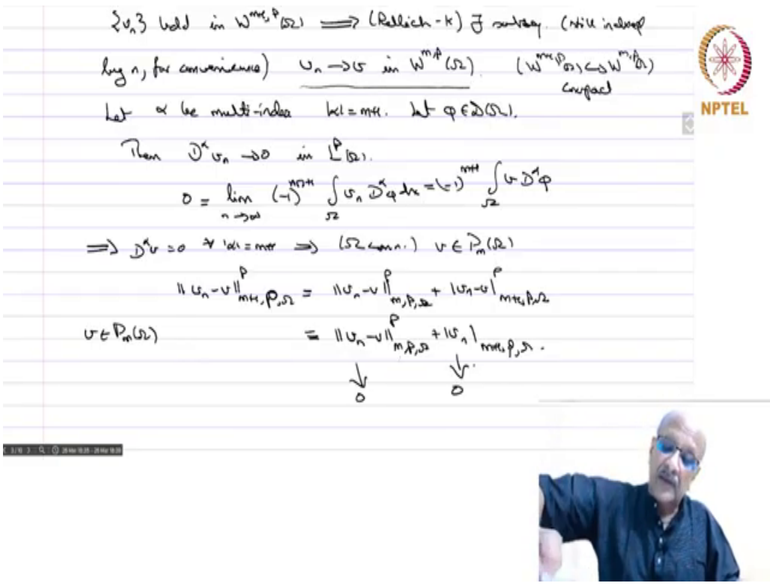
$$||\dot{v}_n||_{m+1,p,\Omega} > n|\dot{v}_n|_{m+1,p,\Omega}$$

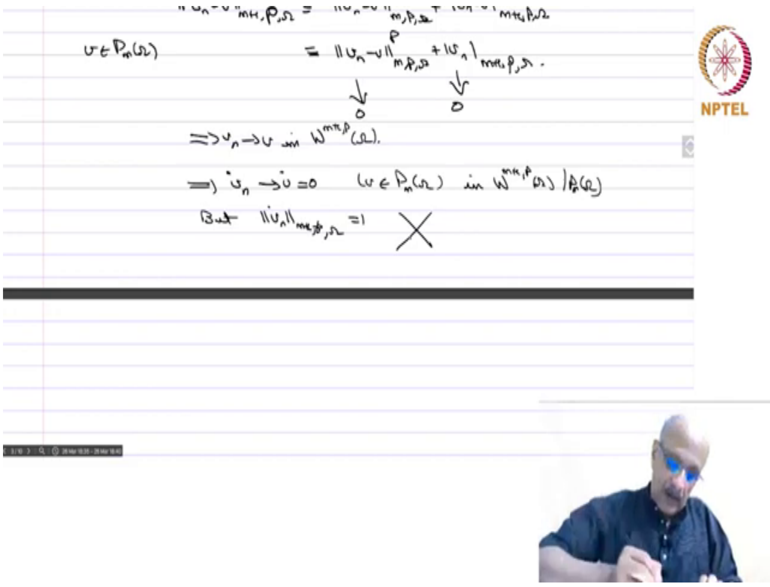
So, we can normalize this, so normalizing there exists $v_n \in W^{m+1,p}(\Omega)$ such that

$||v_n||_{m+1,p,\Omega} = 1$ *and* $|v_n|_{m+1,p,\Omega} \to 0$ as $n \to \infty$.

So, now we have to, so we assume the inequality fails for every constant and we have got a sequence. Now, we have to get a contradiction. So, without loss of generality by choosing a suitable representative we can assume $|v_n||_{m+1,p,\Omega} \leq 2$. So, we can choose a p tilde suitably such that it is also the infimum is 1, so there must be 1 p tilde for which it will be less than 2, so if we add that to this nothing changes in all these inequalities, so we can assume that norm vn is less than equal to 2. Just by the definition of the norm in the quotient space.

(Refer Slide Time: 11:35)





So, now vn is bounded in $W^{m+1,p}(\Omega)$ and so by Rellich - Kondrachov or Rellich you have, so we will work with the subsequences, there exists a subsequence, still indexed by n for convenience because and I will work with that subsequence, so I do not want the original sequence anymore, so there is no need to write a n, k and all that, so we will just, such that a bounded sequence is as convergent subsequence, so $v_n \rightarrow v \; in \; W^{m+1,p}(\Omega)$.

So, we proved the last corollary, which we proved is $W^{m+1,p}(\Omega) \rightarrow W^{m,p}(\Omega)$ is compact and we iterated it subsequently and then we have that this is compact. So, if you have a bounded sequence there will have to be a convergence of sequence in the image and therefore since it is just the inclusion map so we have $v_n \rightarrow v \; in \; W^{m,p}(\Omega)$. So, let alpha be a multi index such that $|\alpha| = m + 1$. So, let $\phi \in D(\Omega)$, then $D^{\alpha} v_n \rightarrow 0 \; in \; L^{p}(\Omega)$, because that we know, because vn in m plus 1 p omega goes to 0 that means for all the m plus 1th order derivatives they have to go to 0 in $L^{p}(\Omega)$, so you have

$$0 = \lim_{n \rightarrow \infty} (-1)^{m+1} \int_{\Omega} v_n D^{\alpha}\phi \, dx = (-1)^{m+1} \int_{\Omega} v \; D^{\alpha}\phi \, dx \; .$$

So, this implies that $D^{\alpha} v = 0, \; \forall \, |\alpha| = m + 1$.

and that therefore Ω - connected again implies that $v \in P_m(\Omega)$. So,

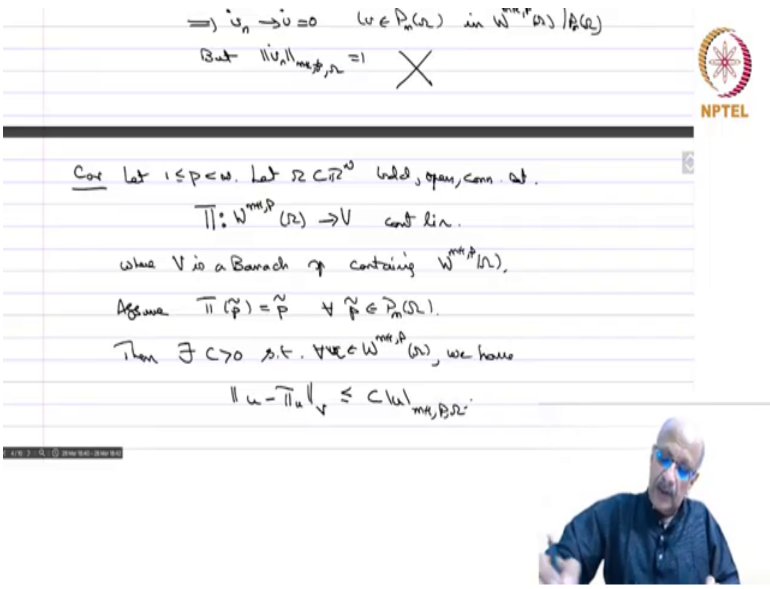
$$||v_n - v||^p_{m+1,p,\Omega} = ||v_n - v||^p_{m,p,\Omega} + |v_n - v|^p_{m+1,p,\Omega}$$

$$= ||v_n - v||^p_{m,p,\Omega} + |v_n|^p_{m+1,p,\Omega} \to 0$$

as $v \in P_m(\Omega)$, (so this v n minus v m plus 1th order derivatives of v will all disappear). Now, this goes to 0 since vn converges to v in Wmp omega, so you have that and therefore this goes to 0 and this goes to 0 by the assumption on the subsequence here and therefore both of these go to 0 and therefore this means that $v_n \to v \; in \; W^{m+1,p}(\Omega)$ and then

$\dot{v}_n \to \dot{v}$, but $\dot{v} = 0$ since $v \in P_m(\Omega)$ but in $W^{m+1,p}(\Omega)/P_m(\Omega)$. But $||v_n||_{m+1,p,\Omega} = 1$, that was our definition, assumption. So, and therefore we have a contradiction and this completes the proof.

(Refer Slide Time: 17:11)



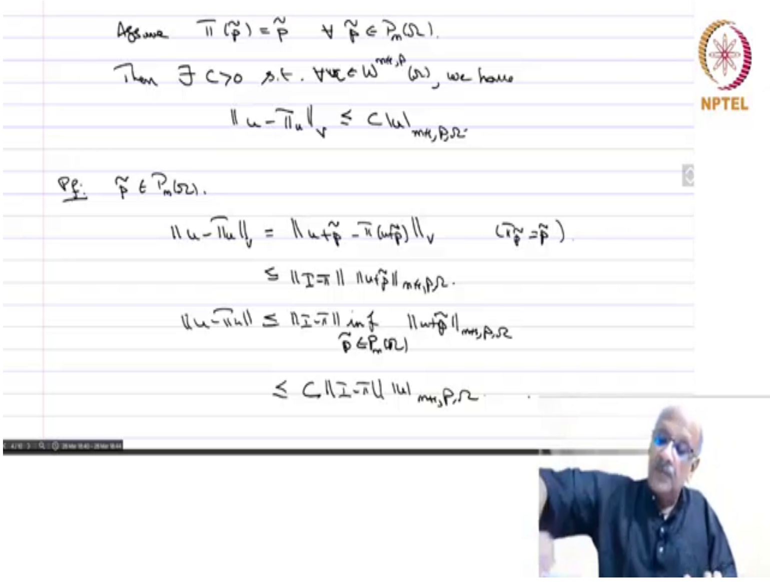
corollary: so let $1 \le p < \infty$, let $\Omega \subset \mathbb{R}^N$ −bounded open connected set and

$\Pi: W^{m+1,p}(\Omega) \to V,$ continuous linear where V is a Banach space containing $W^{m+1,p}(\Omega)$.

Assume $\Pi(\widetilde{P}) = \widetilde{P}$, for every $\widetilde{P} \in P_m(\Omega)$. So, it preserves all the polynomials. Then there exists a $C > 0$ such that for every $u \in W^{m+1,p}(\Omega)$, we have

$$||u - \Pi u||_v \le C|u|_{m+1,p,\Omega} .$$

(Refer Slide Time: 19:12)



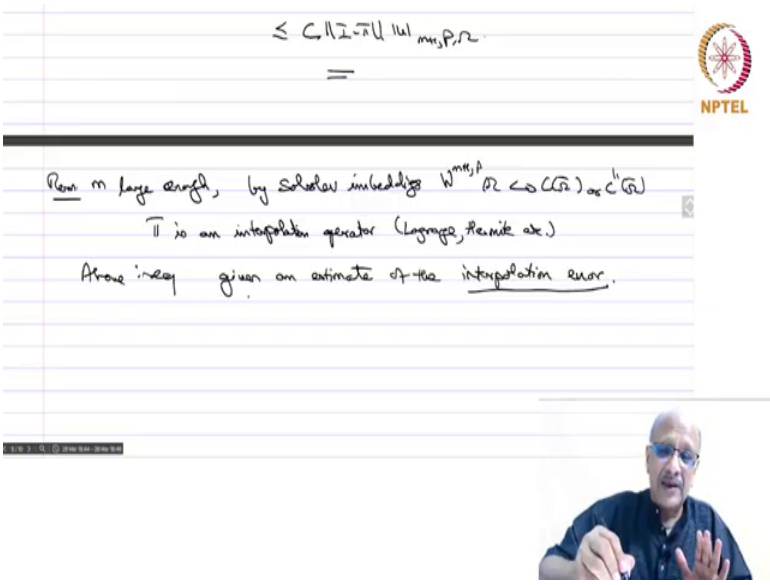
Proof: So, let $\tilde{p} \in P_m(\Omega)$, so

$$||u - \Pi u||_v = ||u + \tilde{P} - \Pi(u + \tilde{P})||_v \leq ||I - \Pi||\,||u + \tilde{P}||_{m+1,p,\Omega}.$$

$$\Rightarrow ||u - \Pi u||_v \leq ||I - \Pi|| \inf_{\tilde{P} \in P_m(\Omega)} ||u + \tilde{P}||_{m+1,p,\Omega} \leq C||I - \Pi||\,|u|_{m+1,p,\Omega}.$$

So, that tells you, that completes the proof.

(Refer Slide Time: 20:41)



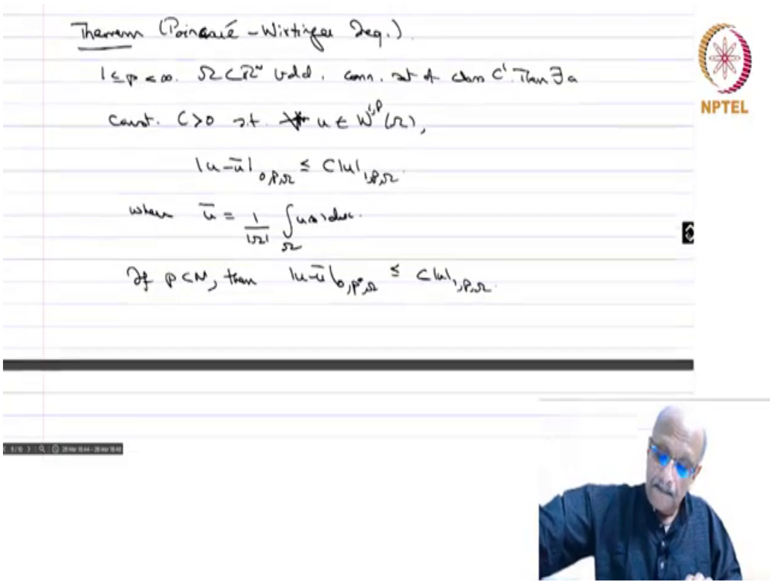
Now, how is this theorem useful?

remark: if m large enough, then by Sobolev embeddings, we have

$W^{m+1,p}(\Omega) \to C(\overline{\Omega})$ *or* $C^{k}(\overline{\Omega})$ it depends for some k. So, we can assume Π is an interpolation operator Lagrange, Hermite, etcetera, so that, so there will be, so these interpolation operators means that for some degree they will reproduce polynomials as it is and any other function they will give you another function here.

So, the above estimate, gives an estimate; above inequality gives an estimate of the interpolation error that means u by u is the interpolated polynomial operator and then you get the u minus pi u is the interpolation error, so this is very useful in numerical analysis, especially in the error estimate error analysis of methods like the finite element method.

(Refer Slide Time: 22:45)

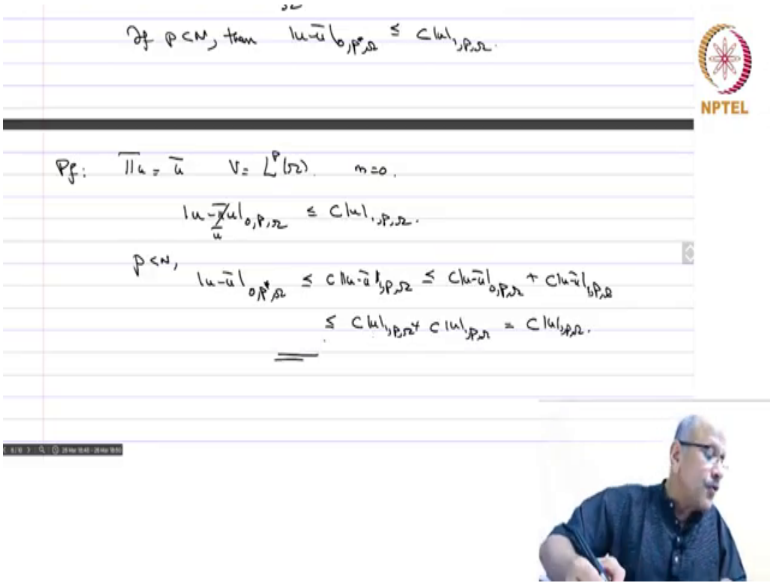


Now, we have another consequence of this theorem which is a very important inequalities extension of the Poincare inequality.

Theorem: (Poincare-Wirtinger inequality). So, $1 \le p < \infty$, let $\Omega \subset \mathbb{R}^{N}$ −bounded open connected set of class C^{1} . Then there exists a constant $C > 0$ such that for every $u \in W^{1,p}(\Omega)$, we have $|u - \overline{u}|_{0,p,\Omega} \le C|u|_{1,p,\Omega}$, *where* $\overline{u} = \frac{1}{|\Omega|}\int_{\Omega} u(y)dy$.

If $p < N$, then $|u - \overline{u}|_{0,p^*,\Omega} \le C|u|_{1,p,\Omega}$.

(Refer Slide Time: 24:32)



Proof: So, we are going to take $\Pi u = \bar{u}$, so this is like an interpolation operator. So, we have $V = L^p(\Omega)$, it contains $W^{m+1,p}(\Omega)$ and m equal to 0. We have m equal to 0 and so we have

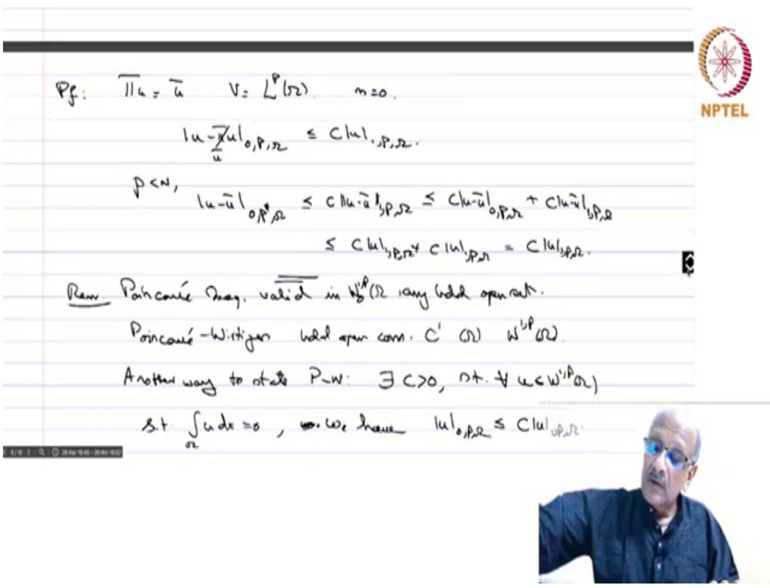
$$|u - \Pi u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega}.$$

If $p < N$, then you have that

$$|u - \bar{u}|_{0,p^*,\Omega} \leq C||u - \bar{u}||_{1,p,\Omega} \leq C|u - \bar{u}|_{0,p,\Omega} + C|u - \bar{u}|_{1,p,\Omega}$$

$$\leq C|u|_{1,p,\Omega} + C|u|_{1,p,\Omega} = C|u|_{1,p,\Omega}.$$

(Refer Slide Time: 26:49)



remark: Poincare inequality is valid in ${W^{1,p}}_0(\Omega)$ for any bounded open set Ω Poincare-Wirtinger is bounded, open , connected $C^1(\Omega)$ and, but it is valid in $W^{1,p}$ (Ω). So, there is a price we have to pay with this. So, another way to state Poincare-Wirtinger is that there exists a constant C positive, such that for all $u \in W^{1,p}$ (Ω), such that $\int_{\Omega} udx = 0$,we have $|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega}$. So, this is the analog of the Poincare inequality, L^p norm bounded by H^1, but on a condition namely that the average is 0. So, then we have this.