Sobolev Spaces and Partial Differential Equations
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Compactness theorems – Part 2

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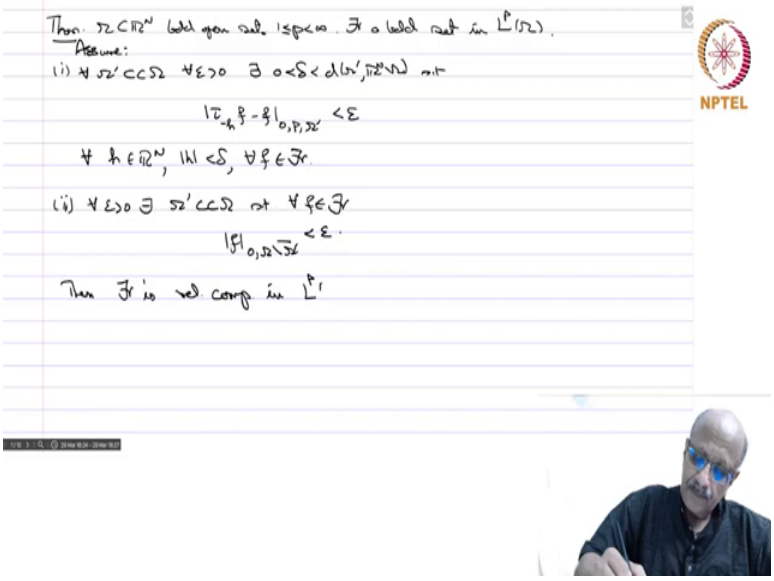

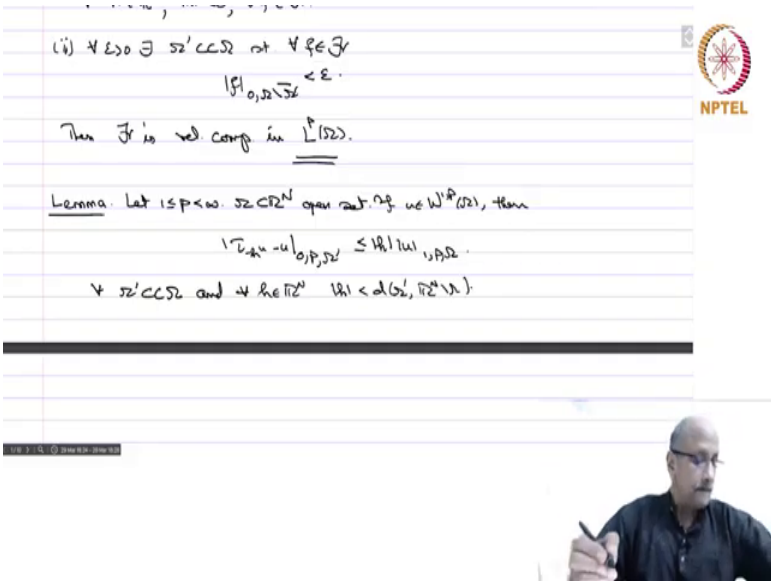

So we were studying about compactness in L^p spaces, so let me recall the theorem.

Theorem: $\Omega \subset \mathbb{R}^N$ bounded open set, F a bounded set in $L^p(\Omega)$, $1 \leq p < \infty$.

Assume:

(i) for every $\epsilon > 0$ and $\Omega' \subset\subset \Omega$, there exists a $\delta > 0$ such that $0 < \delta < d(\Omega', \mathbb{R}^N \backslash \Omega)$ and $|\tau_{-h} f - f|_{0,p,\Omega'} < \epsilon$, $\forall h \in \mathbb{R}^N$ $s.t.$ $|h| < \delta$, $\forall f \in F$,

(ii) for every $\epsilon > 0$, there exists $\Omega' \subset\subset \Omega$ s.t. $\forall f \in F$

$$|f|_{0,p,\Omega\backslash\overline{\Omega'}} < \epsilon .$$

Then F is relatively compact in $L^p(\Omega)$.

So, we want to verify these two conditions when we are studying the various Sobolev inclusions. So, for that we need to start with one more technical lemma.

Lemma: Let $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ open set. If $u \in W^{1,p}(\Omega)$,

$$|\tau_{-h} u - u|_{0,p,\Omega'} < |h||u|_{1,p,\Omega} ,$$

for all $\Omega' \subset\subset \Omega$, $h \in \mathbb{R}^N$, $|h| < d(\Omega', \mathbb{R}^N \backslash \Omega)$.

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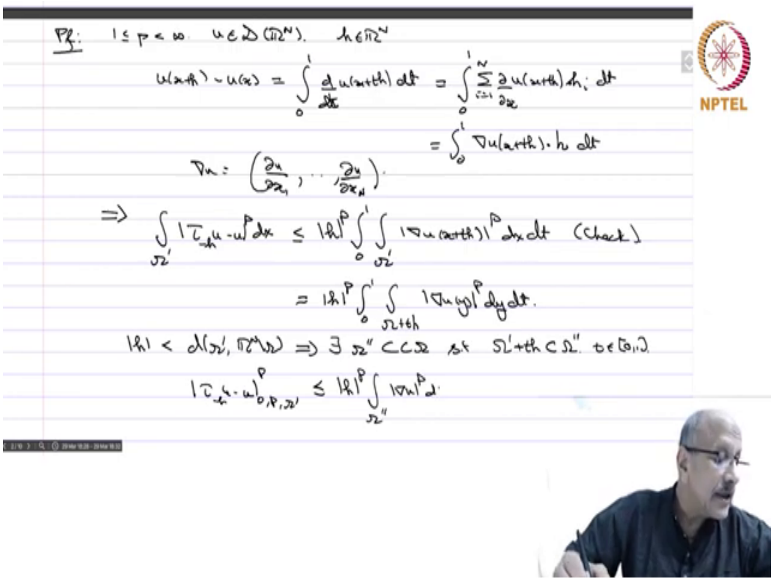

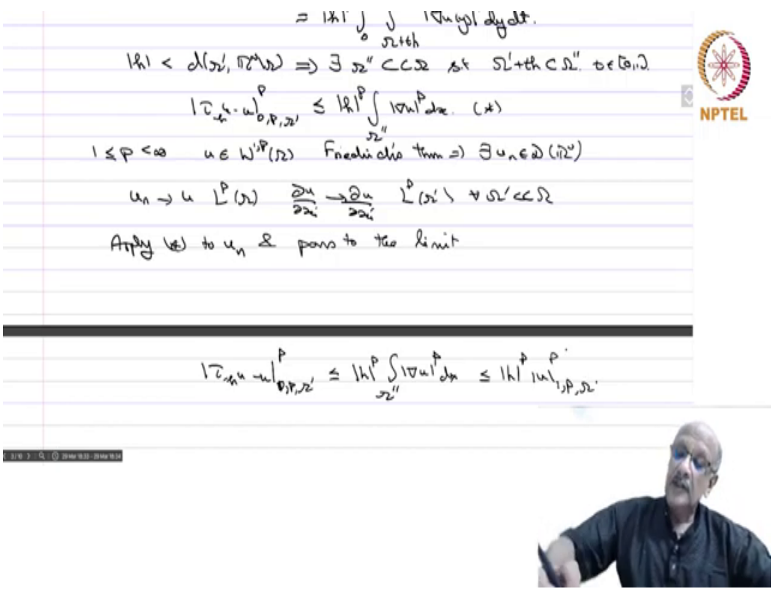

Proof: So, $1 \leq p < \infty$, $u \in D(\mathbb{R}^N)$, $\quad h \in \mathbb{R}^N$.

$$u(x + h) - u(x) = \int_0^1 \frac{d}{dt} u(x + th)dt = \int_0^1 \sum_{i=0}^{N} \frac{\partial}{\partial x_i} u(x + th_i)h_i dt = \int_0^1 \nabla u(x + th).hdt$$

$$\Rightarrow \int_{\Omega'} |\tau_{-h} u - u|^p dx \leq |h| \int_0^1 \int_{\Omega'} |\nabla u(x + th)|^p dx dt \qquad \text{[Check !]}$$

$$= |h| \int_0^1 \int_{\Omega + th} |\nabla u(y)|^p dy\, dt$$

So, now if you have $|h| < d(\Omega', \mathbb{R}^N \backslash \Omega)$. So this will imply that there exists $\Omega'' \subset\subset \Omega$

$\Omega + th \subset \Omega''$, $t \in (0, 1)$. So, then

$$|\tau_{-h} u - u|^p_{0,p,\Omega'} \leq |h| \int_{\Omega''} |\nabla u(y)|^p\, dy \qquad ------ (*)$$

So, now we have $1 \leq p < \infty$ and therefore if $u \in W^{1,p}(\Omega)$, then by Freidrich's theorem implies there exists $u_n \in D(\mathbb{R}^N)$ such that $u_n \rightarrow u \text{ in } L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i}$, for every $\Omega' \subset\subset \Omega$, that is Freidrich's theorem. So, apply (*) to u_n because that is in d of omega d of Rn and then pass to the limit because you are going to pass the limit in LP of omega prime.

There is no problem in either of the two integrals and then and pass to the limit. Then you get

$$|\tau_{-h}u - u|^p_{0,p,\Omega'} \le |h| \int_{\Omega''} |\nabla u(y)|^p \, dy \le |h|^p |u|^p_{1,p,\Omega}.$$

and now you take the pth root, then you will get exactly what you wanted.

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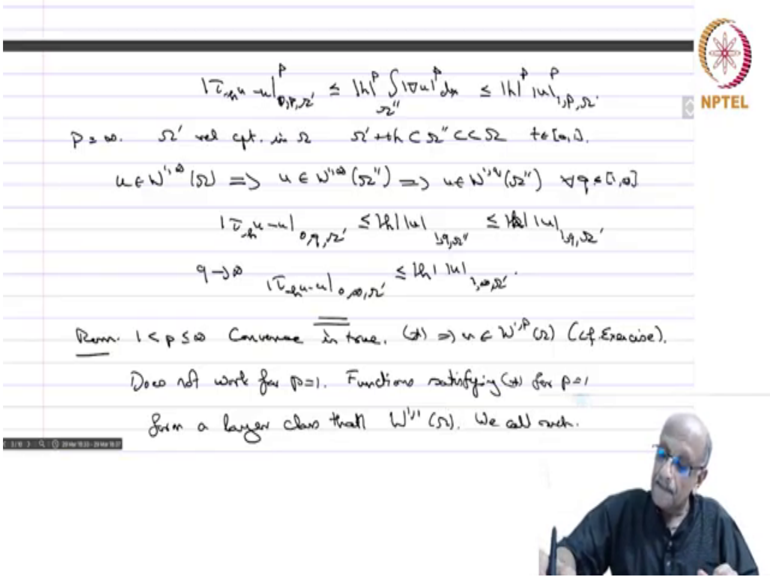

So now we take the case $p = \infty$, so Ω' relatively compact in omega and then you choose

$\Omega' + th \subset \Omega'' \subset \Omega$, $t \in (0, 1)$ as before. And then if you take

$u \in W^{1,\infty}(\Omega) \Rightarrow u \in W^{1,\infty}(\Omega'') \Rightarrow u \in W^{1,q}(\Omega)$, $q \in [1, \infty]$. And now you let q tend to infinity then you get $|\tau_{-h}u - u|_{0,\infty,\Omega'} \le |h| \; |u|_{1,\infty,\Omega'}$.

So that proves this particular expression. Now, remark.

remark: if $1 < p \le \infty$, the converse is true. Namely, if you have the any if that is (*) implies $W^{1,p}(\Omega)$ and in fact, we have seen this in the exercises.

We did this exercise. So, it does not work for p =1. So, functions satisfying (*) for p equals 1 form a larger class than $W^{1,1}(\Omega)$. $W^{1,1}(\Omega)$ anyway satisfies it, that is what we have proved just now. But if you take the converse, if you have a star is true then it is not necessary in W 1, 1 omega, it is in some bigger space we call that, we call such functions, functions of bounded variation and such spaces are called B. V. spaces.

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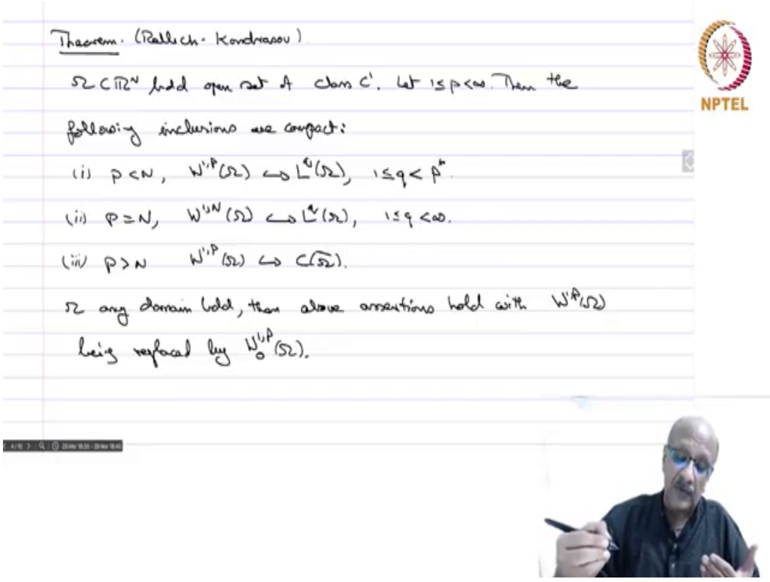

So, now we are ready to prove the important theorem of this section. Namely, Rellich - Kondrachov.

Theorem (Rellich - Kondrachov**):** Let $\Omega \subset \mathbb{R}^N$ bounded open set of class C^1 and $1 \le p < \infty$, then the following inclusions are compact.

(i) $p < N$, then $W^{1,p}(\Omega) \to L^q(\Omega)$, $1 \le q < p^*$.

(ii) $p = N$, then $W^{1,p}(\Omega) \to L^q(\Omega)$, $1 \le q < \infty$.

(ii) $p > N$, then $W^{1,p}(\Omega) \to C(\overline{\Omega})$.

Ω bounded domain, then above assertion holds true if $W^{1,p}(\Omega)$ is replaced by $W_0^{1,p}(\Omega)$.

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Pf: Already proved case (iii).

Assume (i) proved

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$$

$p \uparrow N \implies p^* \uparrow \infty$.

$q < \infty$, $\exists\ \varepsilon > 0$ suff small s.t. $(N-\varepsilon)^* > q$.

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Ω bdd, $W^{1,N}(\Omega) \hookrightarrow W^{1,N-\varepsilon}(\Omega) \hookrightarrow L^q(\Omega)$

↑ compact by (i).

$\implies W^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ comp $\forall\ 1 \le q < \infty$.

To prove (i). $p < N$, B closed unit ball in $W^{1,p}(\Omega)$.

To verify (i) & (ii) of Fréchet-Kolmogorov (1st thm. above)

with $\mathcal{F} = B$.

$1 \le q < p^*$. Then $\exists\ \alpha \in (0,1]$

$$\frac{1}{q} = \frac{\alpha}{1} + \frac{1-\alpha}{p^*}.$$

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$$\frac{1}{q} = \frac{\alpha}{1} + \frac{1-\alpha}{p^*}.$$

If $u \in B$, $\Omega' \subset\subset \Omega$

$h \in \mathbb{R}^N$ $|h| < d(\Omega', \mathbb{R}^N \setminus \Omega)$ Hölder $\implies$

$$|\tau_{-h}u - u|_{0,q,\Omega'} \le |\tau_{-h}u - u|_{0,1,\Omega'}^{\alpha}\ |\tau_{-h}u - u|_{0,p^*,\Omega'}^{1-\alpha}$$

proof: Already proved case (iii), if p is bigger than n we already saw that W 1 p omega where all the Holder continuous functions, so if you to the unit ball then you would get those functions are bounded and uniformly continuous and therefore you had by the Ascoli Arzela theorem the compactness.

So, assume (i) is proved, namely for $p < N,$ assume that we have proved all these things.

Then using that we can prove the $p > N$; so when p increases to N, so they call that 1 by p star is 1 by p minus 1 by n. This implies that the p star goes to infinity. So, if q is less than infinity there exists epsilon positive sufficiently small such that the n minus epsilon star is bigger than q. Now, we have W omega bounded, so $W^{1,N}(\Omega)$ is continuously embedded in W 1 n minus epsilon of omega because if you are in any L^p you are in any smaller LP for sets of finite measure and now this is in $L^q(\Omega)$.

And this is compact by 1, because you have n minus epsilon star is bigger than q and therefore the first assertion tells you that if you are less than the critical Sobolev, the p star is called the critical Sobolev exponent, you lose compactness there and therefore anything less than that it is compact and therefore this one is compact, so this implies that $W^{1,N}(\Omega)$ in $L^q(\Omega)$ compact for all 1 less than equal to q strictly less than infinity.

So, to prove one, we take p less than n and b closed unit ball in $W^{1,p}(\Omega)$. So, we need to verify the two conditions of the Frechet Kolmogorov theorem, which I restated in the beginning of this video, one and two, we have to show that for every omega prime relative compact in omega and for every epsilon you have tau minus f is less than 0, in 0 p omega prime is less than epsilon for h sufficiently small.

And then there is omega epsilon such that for all elements in f, so we have to verify it with the F equals B. So, to verify one and two of Frechet Kolmogorov, first theorem above with F equal to B, that is all that we have to do now. so, let us take one less than equal to q strictly less than p star; then there exists alpha which belongs to 0, 1, 0 is excluded such that 1 by q can, we have done this before, alpha by 1 plus 1 minus alpha by p star.

So, if you want 1, if you want p star then alpha must be 0, which you are excluding because we are saying less than p star, so then if $u \in B$, and $\Omega' \subset\subset \Omega$, $h \in \mathbb{R}^N$, $|h| < d(\Omega', \mathbb{R}^N \backslash \Omega)$, you have

$$|\tau_{-h}u - u|_{0,q,\Omega'} \leq |\tau_{-h}u - u|^{\alpha}_{0,1,\Omega'} \ |\tau_{-h}u - u|^{1-\alpha}_{0,p^*,\Omega'} .$$

We have seen this before when we proved that after the Sobolev inequality $W^{1,p}(\mathbb{R}^N)$ is in L^{p^*} and then we wanted to show that it is in every L^q for q in between p and p* and we made a similar expression for q and then we had this inequality just because of Holder inequality, so this is Holder. And mod tau minus h u minus u in 0 p star omega dash to the power of 1 minus alpha.

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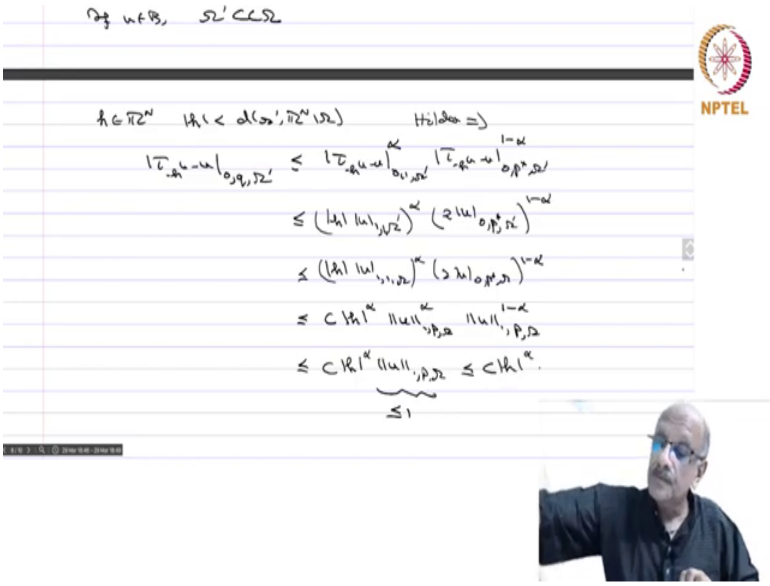

Now,

$$|\tau_{-h}u - u|_{0,q,\Omega'} \leq |\tau_{-h}u - u|^{\alpha}_{0,1,\Omega'} \ |\tau_{-h}u - u|^{1-\alpha}_{0,p^*,\Omega'}$$

$$\leq (|h||u|_{1,1,\Omega'})^{\alpha} \quad (2|u|_{0,p^*,\Omega'})^{1-\alpha}$$

$$\leq (|h||u|_{1,1,\Omega})^{\alpha} \quad (2|u|_{0,p^*,\Omega})^{1-\alpha}$$

$$\leq C|h|^{\alpha}||u||_{1,p,\Omega}^{\alpha} \quad ||u|^{1-\alpha}_{1,p,\Omega}$$

$$\leq C|h|^{\alpha}||u||_{1,p,\Omega} \quad \leq C|h|^{\alpha} .$$

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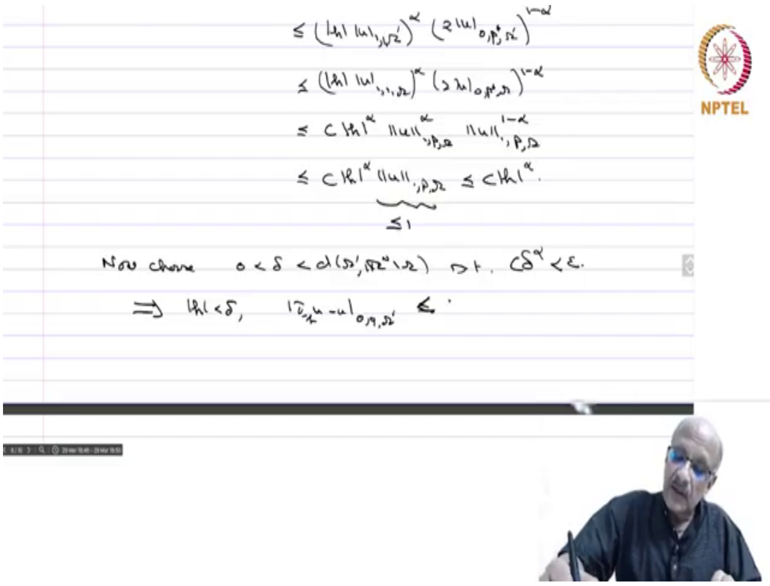

So, now choose $0 < \delta < d(\Omega', \mathbb{R}^N \backslash \Omega)$ such that $C\delta^{\alpha} < \epsilon$. So, then this implies that for all $|h| < \delta$, we have that

$$|\tau_{-h}u - u|_{0,q,\Omega'} < \epsilon .$$

So, this proves the first condition in the Frechet Kolmogorov theorem.

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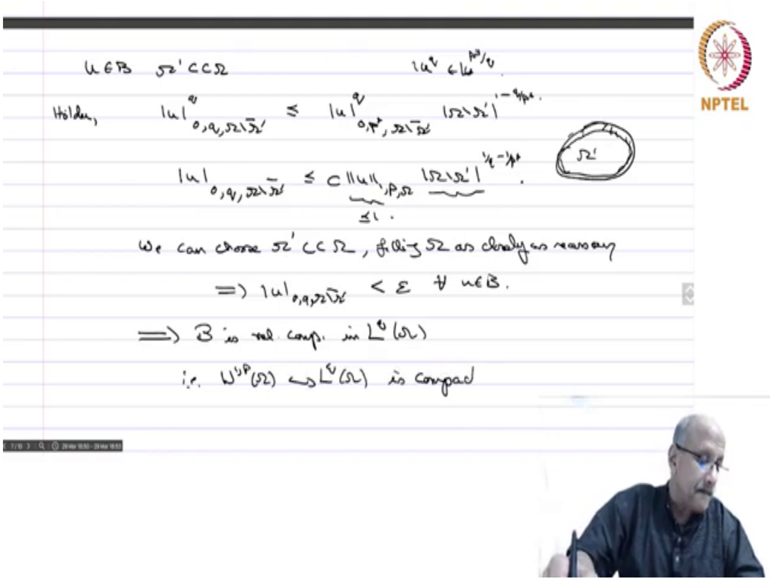

The second one is to show that you for there exists an Ω' that for every u in B you can make it the integral small. So, now if $u \in B$ and $\Omega' \subset\subset \Omega$, then you have by Holder inequality,

$$|u|^q_{0,q,\Omega\setminus\overline{\Omega'}} \leq |u|^q_{0,p^*,\Omega\setminus\overline{\Omega'}} |\Omega\setminus\overline{\Omega'}|^{1-\frac{q}{p^*}} .$$

So, you will get $|u|_{0,q,\Omega\setminus\overline{\Omega'}} \leq C||u||^q_{1,p,\Omega} |\Omega\setminus\overline{\Omega'}|^{\frac{1}{q}-\frac{1}{p^*}}$, because of the Sobolev inequality.

This one is less than the norm in l p star omega which is less than the norm of 1 p omega into mod omega minus omega dash of 1 by q minus 1 by p star. So, now this of course is less than or equal to 1, therefore we can choose $\Omega' \subset\subset \Omega$ filling as closely as possible, as necessary. So, you have that, you have Ω here, I can take Ω' very close to the boundary like this, so this will be Ω'.

And therefore what is left will be very-very small, so this measure can be made as small as you like and this implies that $|u|_{0,q,\Omega\setminus\overline{\Omega'}} < \epsilon, \forall u \in B$.

So, this is, so the two conditions are satisfied, therefore this implies that B is relatively compact in $L^q(\Omega)$, that is $W^{1,p}(\Omega) \rightarrow L^q(\Omega)$ is compact and that proves the Frechet Kolmogorov theorem, I mean, Rellich - Kondrachov theorem.

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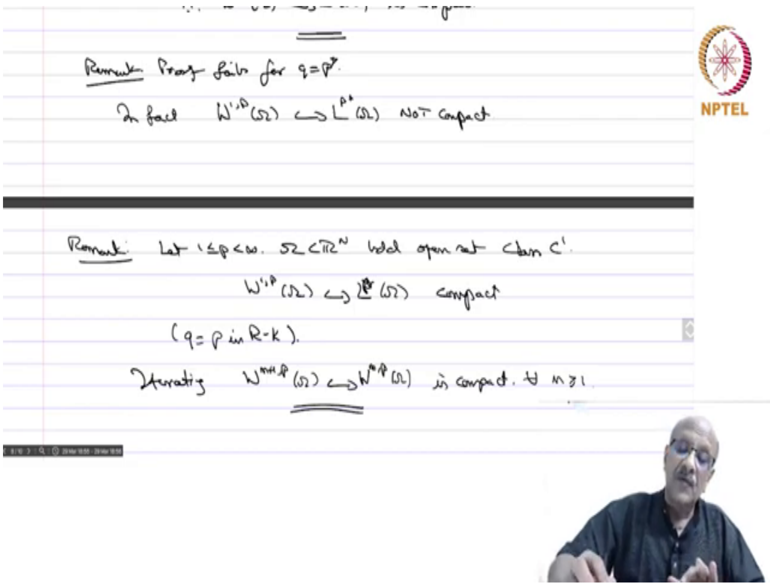

remark: Proof fails forn $q = p$ *. Why does it fail for$q = p$ *? It fails in both the verifications. The first verification you needed 1 by q, if you put this then you need alpha equal to 0, so if you get alpha equal to 0, then the c mod h to the alpha becomes just a constant and therefore you cannot make it small. You cannot make it less than epsilon by choosing delta small, therefore this step will fail.

This step will fail because you do not have this term here. Now, the second one will fail because this omega minus omega dash is again become power 0 here and therefore this term will fail, now this term will be lost and therefore you cannot make the mod u q omega minus omega prime closure as small as you like, so both cases it fails so the proof fails for q equals omega and in fact we can show that $W^{1,p}(\Omega) \rightarrow L^{p^*}(\Omega)$ not compact.

The fact that this proof failed is not proof that this is not compact, we can actually show that this n cannot be compact, we will see examples in the exercises. So, that shows. So, now another remark:

remark: Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^N$ bounded domain open set class C^1, so then $W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ is compact. So particular case, so $q = p$ in Rellich. So, this is a particular case. So, then iterating, we get the $W^{m+1,p}(\Omega) \rightarrow W^{m,p}(\Omega)$ is compact for all $m \geq 1$. So, you can easily check this. So, this is another remark. So, now we will see some applications of this next time.