Sobolev Spaces and Partial Differential Equations
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Compactness Theorems – Part 1

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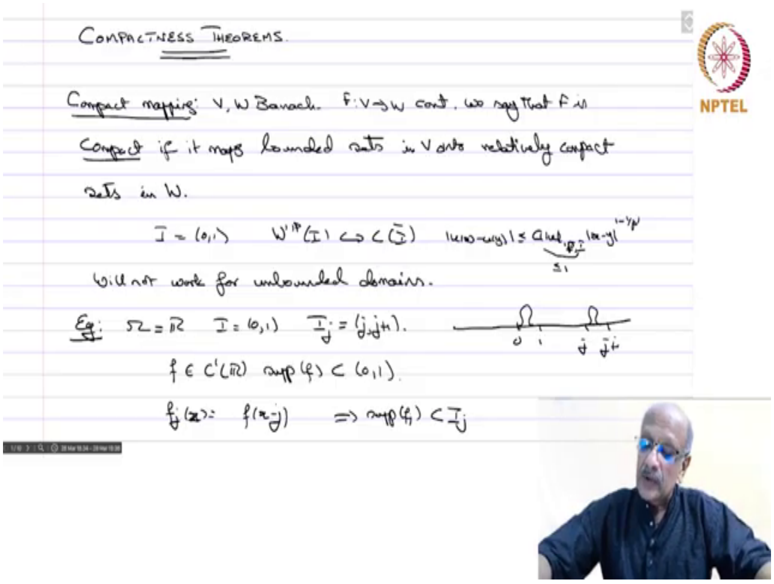

We will now discuss **Compactness Theorems**. So, we have been seeing several Sobolev embeddings, so we have Sobolev spaces embedded in Lq spaces or spaces of continuous or differentiable functions. We want to know if these mapping, inclusion mappings are compact. So, what is a compact mapping?

compact mapping: So, V, W Banach and $F: V \to W$ continuous, and we say that F is compact if it maps bounded sets in V onto relatively compact sets in W.

That means the closure should be compact, then you say it's relatively compact. So, what is the use of compact mapping? So, if you for instance have a bounded sequence in V then the image f of V N will be a relatively, in a relatively compact set so it will have a convergence subsequence in W. Therefore, that is the use of a compact mapping so you can have convergent subsequences in the image.

Now, all the mappings which we are looking at are linear and therefore it is enough to look at the unit ball and see if the image is going to be relatively compact, which is enough for us in most of the cases.

So, we already saw for instance if $I = (0, 1)$, then you have $W^{1,p}(I) \to C(\bar{I})$ and this also had that $|u(x) - u(y)| \leq C|u|_{1,p,I}|x - y|^{1-\frac{1}{p'}}$.

And therefore, if you have the unit ball so norm u 1 P I is less than or equal to 1 and therefore for all u x minus u y you will have Holder continuity and this will show that these are equi continuous. And then of course because it is a bounded linear map its set will also be bounded, bounded and equi continuous by Ascoli-Arzela theorem will say that this is compact. So, this inclusion was compact and that is how we proved it.

Now, we want to see which of the maps which we have now proved in the Sobolev embedding theorems are compact. Now, a quick short introspection will tell you that this will not work, so will not work for unbounded domains. So, let us take an example:

example: so you take $\Omega = \mathbb{R}$, $I = (0, 1)$ and you take $I_j = (j, j + 1)$. And now you take $f \in C^1(\mathbb{R})$, $supp(f) \subset (0, 1)$. And you define $f_j(x) = f(x - j)$ and therefore this means that $supp(f_j) \subset I_j$. In fact, so you just take, so this is 0, this is 1, this is j, this is j plus 1. So, if you took a function f whose graph is like this then the graph of f j will be just you move this function forward and or backward depending where the interval is and then you will get the function, so this is f and this is f j.

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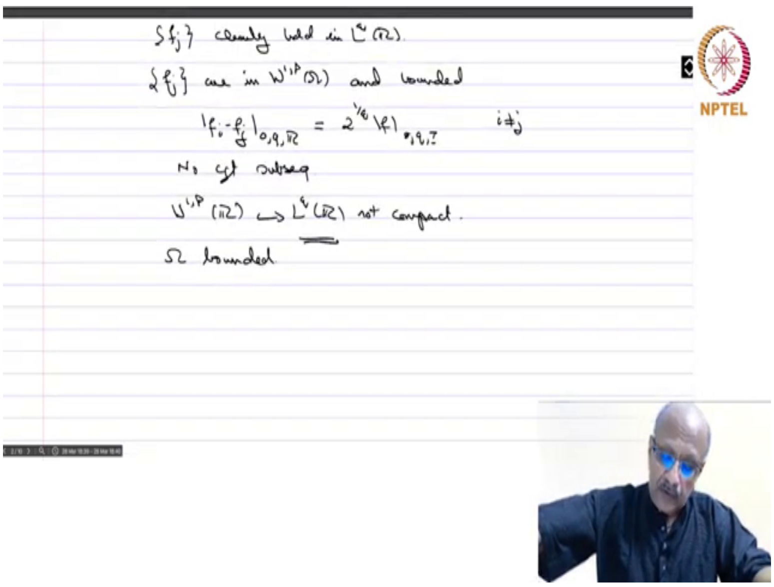

So, because of the translation invariance of the Lebesgue measure, now $\{f_j\}$ is clearly bounded in $L^q(\mathbb{R})$ for any q and therefore and all the $f_j \in W^{1,p}(\Omega)$ and bounded there. In fact, the norms are all the same because you have just translated them. However, if you take

$$|f_i - f_j|_{0,q,\mathbb{R}} \leq 2^{\frac{1}{q}} |f|_{0,q,I} \ , \ \ i \neq j \, .$$

And therefore, no convergent subsequence because all the functions when I is not equal to j they are at the same distance apart so you cannot have a Cauchy subsequence and consequently no convergent subsequence. Therefore, $W^{1,p}(\mathbb{R}) \to L^q(\mathbb{R})$ not compact. So, henceforth we will always assume that omega is bounded. So, this is what we are going to look at.

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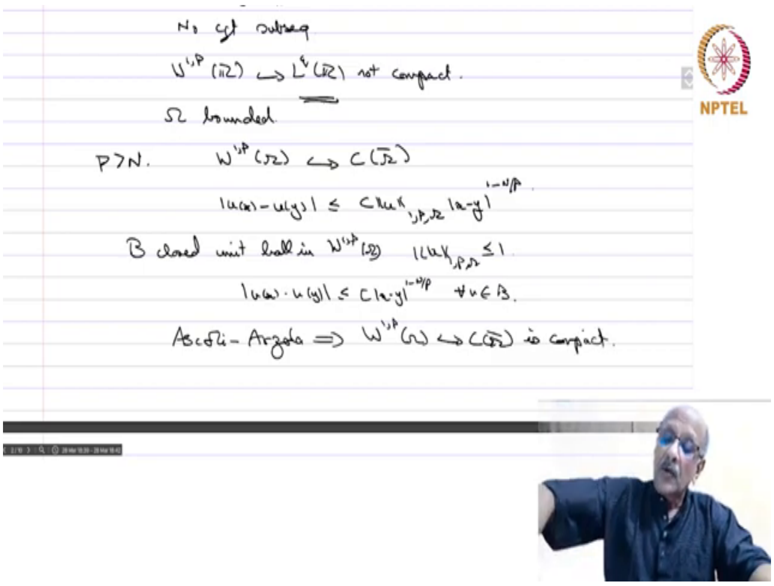

Now, first of all, let us take the case $p > N$, when P is bigger than N then we know that if $W^{1,p}(\Omega) \to C(\overline{\Omega})$. Again you have

$$|u(x) - u(y)| \leq C|u|_{1,p,\Omega} |x - y|^{1-\frac{N}{p}} \ .$$

So, the same argument we can say, so if B is the closed unit ball in $W^{1,p}(\Omega)$. So, then $||u||_{1,p,\Omega} \leq 1$ and therefore

$$|u(x) - u(y)| \leq C\,|x - y|^{1-\frac{N}{p}} \ , \ \ \forall u \in B\,,$$

and therefore it is again equi-continuous and by the continuous inclusion of W 1 P in C omega bar it is also uniformly bounded. So, uniformly bounded equi continuous, continuous functions on a compact set, omega is bounded, so omega closure this compact. Therefore, again by Ascoli-Arzela, $W^{1,p}(\Omega) \to C(\overline{\Omega})$ is compact.

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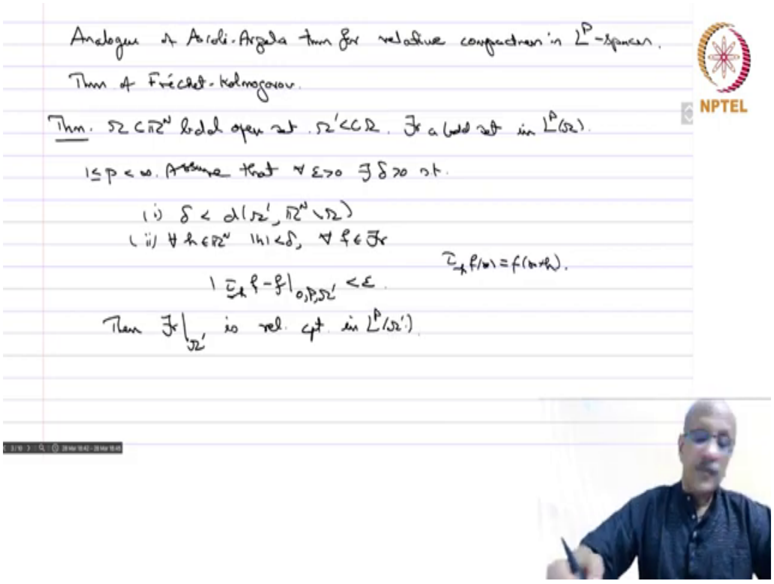

So, analog of Ascoli-Arzela theorem for compactness, for relative compactness in L^p spaces. And this is the Frechet-Kolmogorov theorem which we will now prove.

Theorem: $\Omega \subset \mathbb{R}^N$ bounded open set, $\Omega' \subset\subset \Omega$, F a bounded set in LP omega $1 \leq p < \infty$.

Assume that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

(i) $\delta < d(\Omega', \mathbb{R}^N \backslash \Omega)$.

(ii) $\forall h \in \mathbb{R}^N$ $s.t.$ $|h| < \delta$, $\forall f \in F$,

$$|\tau_{-h}f - f|_{0,p,\Omega'} < \epsilon, \quad where\ \tau_{-h}f(x) = \tau(x + h) .$$

Then $F|_{\Omega'}$ is relatively compact in $L^p(\Omega')$.

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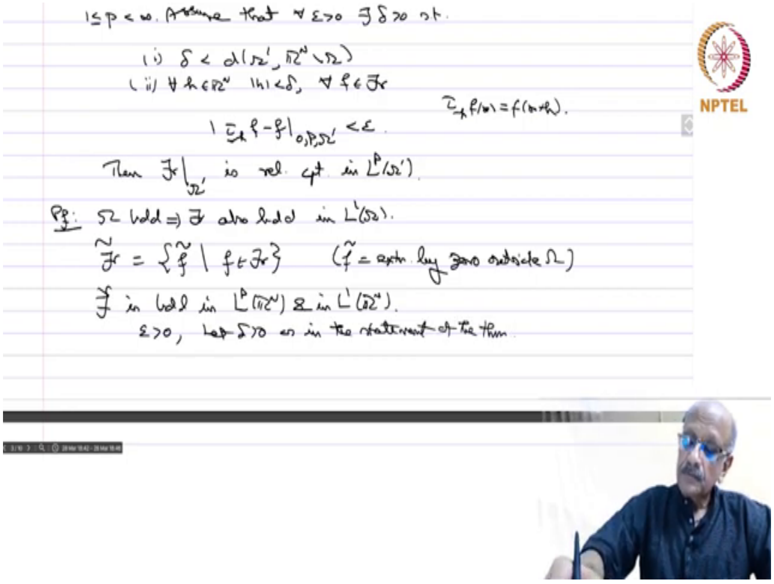

proof: So, Ω bounded, so it has finite measure and therefore F is also bounded in $L^1(\Omega)$**.** So if you take

$$\widetilde{F} = \{\widetilde{f} : f \in F\} \quad , \quad (\widetilde{f} \; is \; extension \; by \; zero \; outside \; \Omega)$$

Then $\widetilde{F}$ is bounded in $L^p(\mathbb{R}^N)$ and in $L^1(\mathbb{R}^N)$. So, let $\epsilon > 0, \delta > 0$ as in the statement of the theorem.

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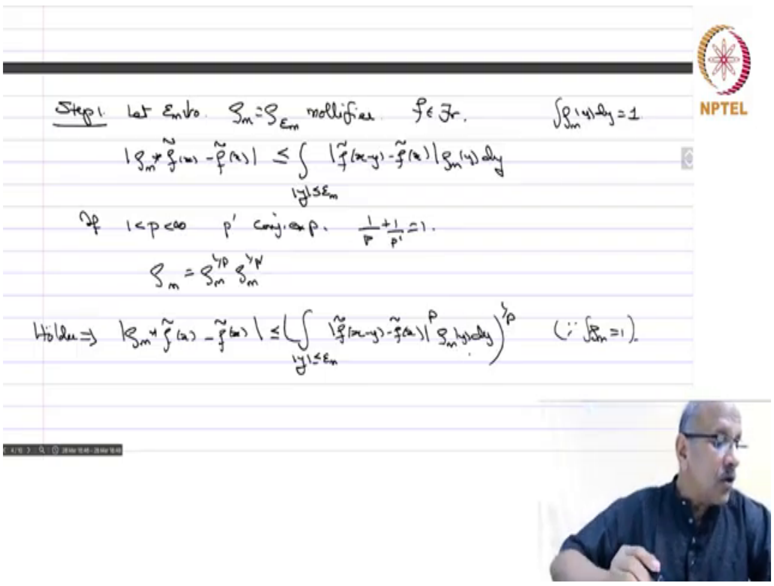

Step 1. So, let $\epsilon_m \downarrow 0$, $\rho_m = \rho_{\epsilon_m}$ mollifiers. So f is in F and therefore

$$|\rho_m * \widetilde{f}(x) - \widetilde{f}(x)| \leq \int_{|y|\leq\epsilon_m} |\widetilde{f}(x-y) - \widetilde{f}(x)|\rho_m(y)dy$$

We have used the fact that integral rho m y dy equal to 1, because this f tilde x is a constant in this integral it would have come out and rho integral rho m d of y is 1 and now I have taken the modulus therefore I can write this. This is fine if 1 is less, P equals 1. I am happy with this if P is bigger than 1, 1 less than P less than infinity then P dash is the conjugate exponent. So, 1 by P plus 1 by P dash equal to 1. So, then I can write $\rho_m = \rho_m^{\frac{1}{p}}\rho_m^{\frac{1}{p'}}$ and then substitute here.

So, now I am going to use the Holder inequality. So,

$$|\rho_m * \widetilde{f}(x) - \widetilde{f}(x)| \leq \left(\int_{|y|\leq\epsilon_m} |\widetilde{f}(x-y) - \widetilde{f}(x)|^p\rho_m(y)dy\right)^{\frac{1}{p}}$$

Since, again since, integral rho m equal to 1, so this is Holder. So, this inequality is true for all p. So, p equals 1 is here and for other p's we have proved this here.

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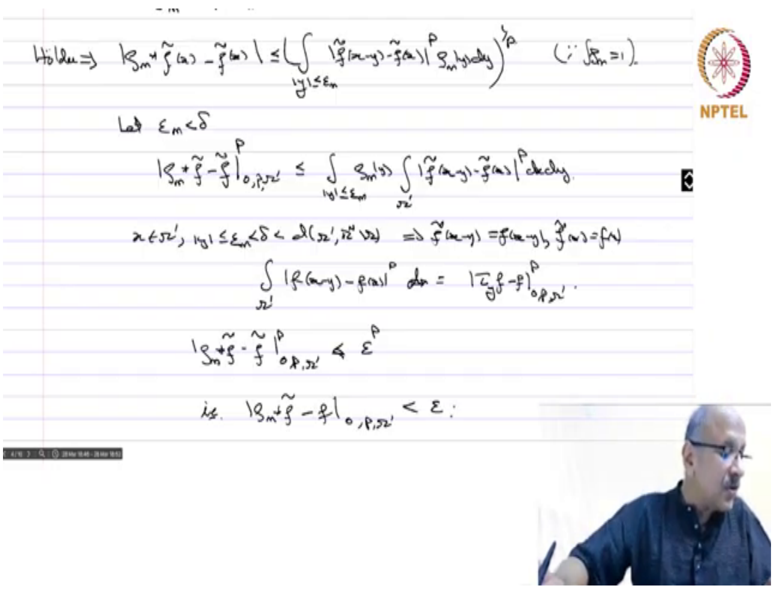

So, we have now let $\epsilon_m < \delta$. So,

$$|\rho_m * \widetilde{f} - \widetilde{f}|^p_{0,p,\Omega'} \leq \int_{|y|\leq\epsilon_m} \rho_m(y)\int_{\Omega'} |\widetilde{f}(x-y) - \widetilde{f}(x)|^p dxdy$$

So, if $x \in \Omega'$, and $|y| \leq \epsilon_m < \delta < d(\Omega', \mathbb{R}^N \backslash \Omega) \Rightarrow \tilde{f}(x - y) = f(x - y)$ and

$\tilde{f}(x) = f(x)$ and then

$$\int_{\Omega'} |\tilde{f}(x - y) - \tilde{f}(x)|^p dx = |\tau_y f - f|^p_{0,p,\Omega'}$$

So, and that we know is in epsilon power P. So, this $|\rho_m * \tilde{f} - \tilde{f}|^p_{0,p,\Omega'} < \epsilon^p$. Therefore

$$|\rho_m * \tilde{f} - \tilde{f}|_{0,p,\Omega'} < \epsilon \ .$$

So, this is what we have.

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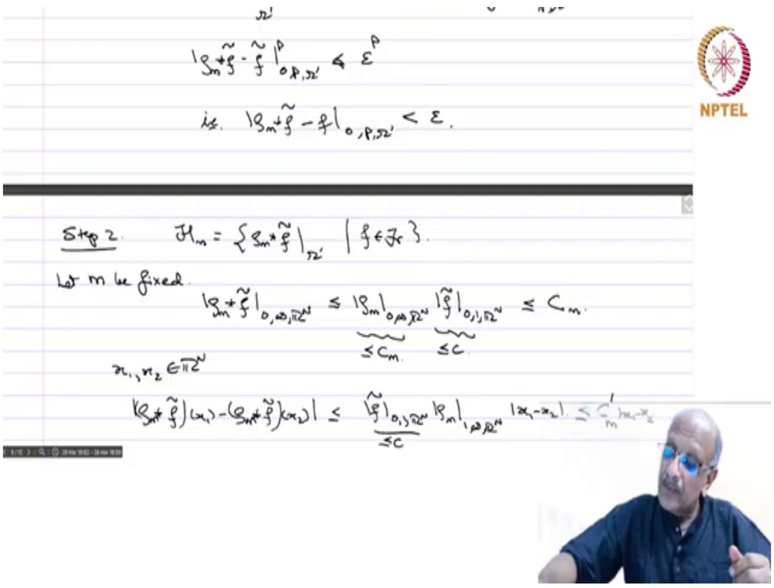

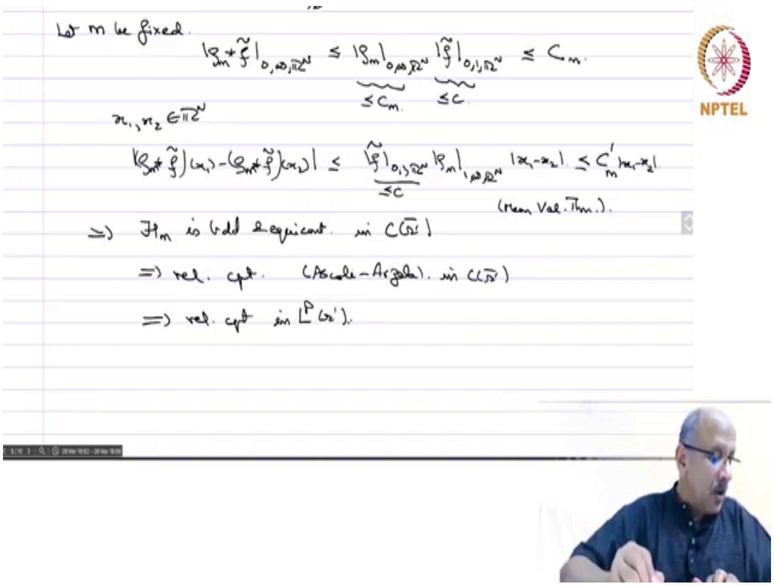

Step 2: let us take $H_m = \{\rho_m * \widetilde{f}|_{\Omega'} \,:\, f \in F\}$. Let m be fixed, then

$$|\rho_m * \widetilde{f}|_{0,\infty,\mathbb{R}^N} \leq |\rho_m|_{0,\infty,\mathbb{R}^N} \, |\widetilde{f}|_{0,1,\mathbb{R}^N} \leq C_m$$

Now, if $x_1, x_2 \in \mathbb{R}^N$, then you have

$$|(\rho_m * \widetilde{f})(x_1) - (\rho_m * \widetilde{f})(x_2)| \leq |\rho_m|_{0,1,\mathbb{R}^N} \, |\widetilde{f}|_{1,\infty,\mathbb{R}^N} \, |x_1 - x_2| \leq C'_m |x_1 - x_2|$$

So, this is nothing but the mean value theorem I am applying rho m of x 1 minus rho m of x 2 and therefore you have rho, we will have rho m of x 1 minus y minus rho 2 of x minus y and that is less than or equal to mod x 1 minus x 2 times the maximum of the derivatives.

Again, so this is less than equal to some, this is again bounded in L 1 of RN and therefore this is again some C m dash mod x 1 minus x 2. Again, C m and C m dash may be very large for each m but m is fixed and so we do not care, because these functions become very steep so this comes from the mean value theorem.

So, it follows from these two relations that H_m is bounded equicontinuous in $C(\overline{\Omega'})$ and therefore you have that it is relatively compact, again just Ascoli-Arzela. And therefore, this implies relatively compact in $C(\overline{\Omega'})$ and implies relatively compact in $L^p(\Omega')$, because if something is compact with respect to L^∞ norm it is automatically compact with respect to the omega prime L^p norm also, since Ω' is a finite measure and therefore we have no problems.

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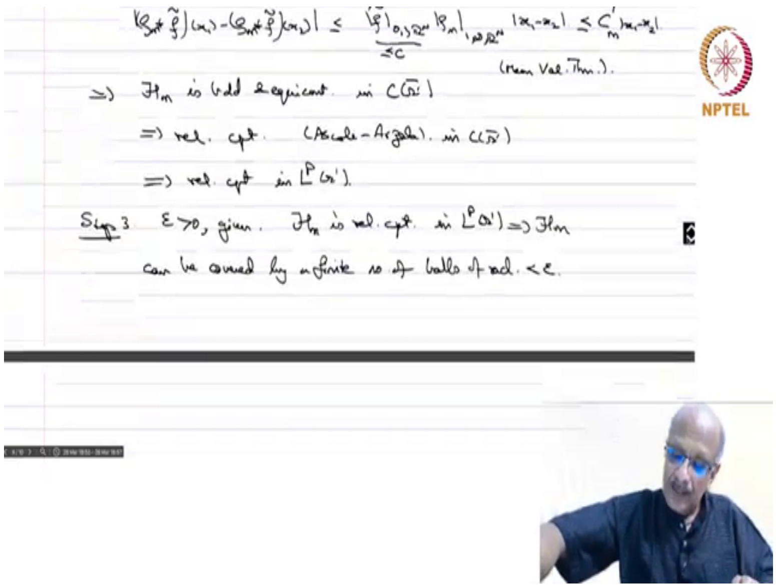

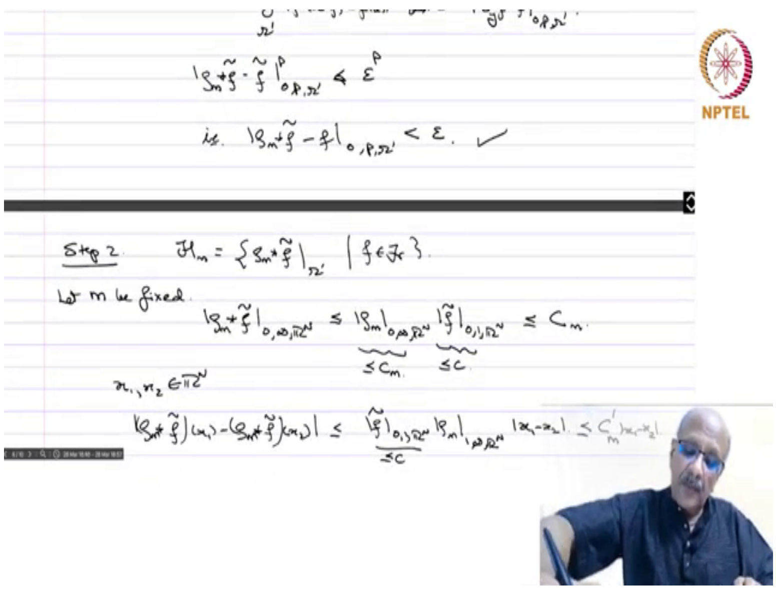

Step 3: so let any $\epsilon > 0$ be given, H_m is relatively compact in $L^p(\Omega')$ $\Rightarrow H_m$ can be covered by a finite number of balls of radius less than ϵ. Now, given any f you have an element rho epsilon star element in f m such that f minus rho m f tilde rho m, rho m star f tilde minus f is less than epsilon in norm, in the $L^p(\Omega')$ norm.

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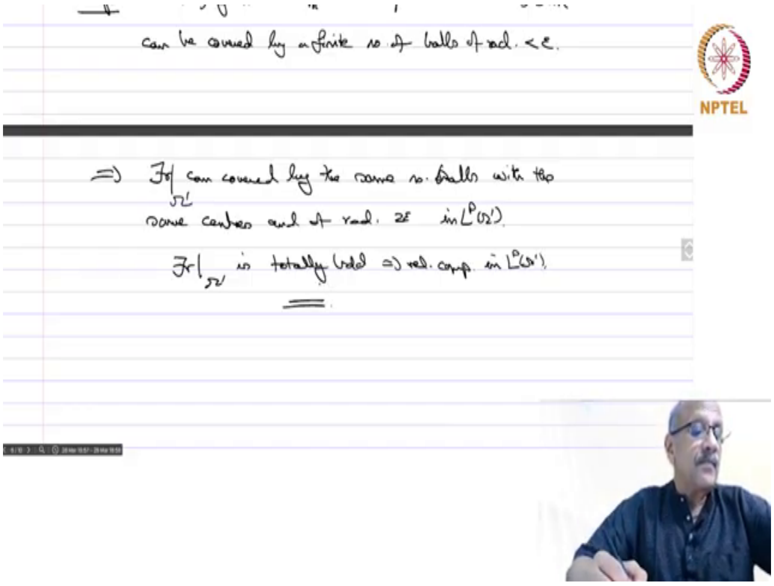

And therefore, this implies that F can be covered by the same number of balls with the same centers and of radius 2ϵ in $L^p(\Omega')$. And therefore $F|_{\Omega'}$ is totally bounded $\Rightarrow$ relatively compact in $L^p(\Omega')$. So, this proves this theorem completely.

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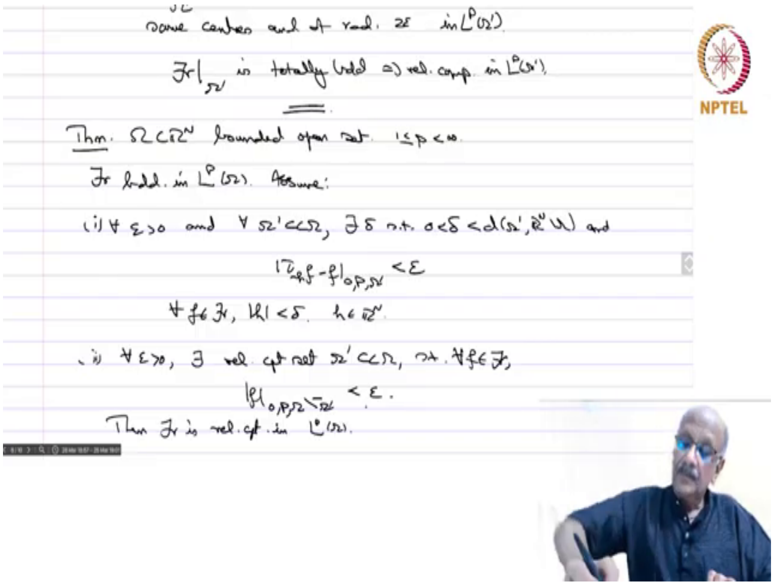

But we are not interested in $L^p(\Omega')$, we want compactness condition in $L^p(\Omega)$. So, we want to use this theorem to prove the next theorem.

Theorem: $\Omega \subset \mathbb{R}^N$ bounded open set, F a bounded set in $L^p(\Omega)$, $1 \leq p < \infty$.

Assume:

(i) for every $\epsilon > 0$ and $\Omega' \subset\subset \Omega$, there exists a $\delta > 0$ such that $0 < \delta < d(\Omega', \mathbb{R}^N \backslash \Omega)$ and $|\tau_{-h} f - f|_{0,p,\Omega'} < \epsilon, \forall h \in \mathbb{R}^N$ $s.t.$ $|h| < \delta, \forall f \in F,$

(ii) for every $\epsilon > 0$, there exists $\Omega' \subset\subset \Omega$ s.t. $\forall f \in F$

$$|f|_{0,p,\Omega\backslash\overline{\Omega'}} < \epsilon .$$

Then F is relatively compact in $L^p(\Omega)$.

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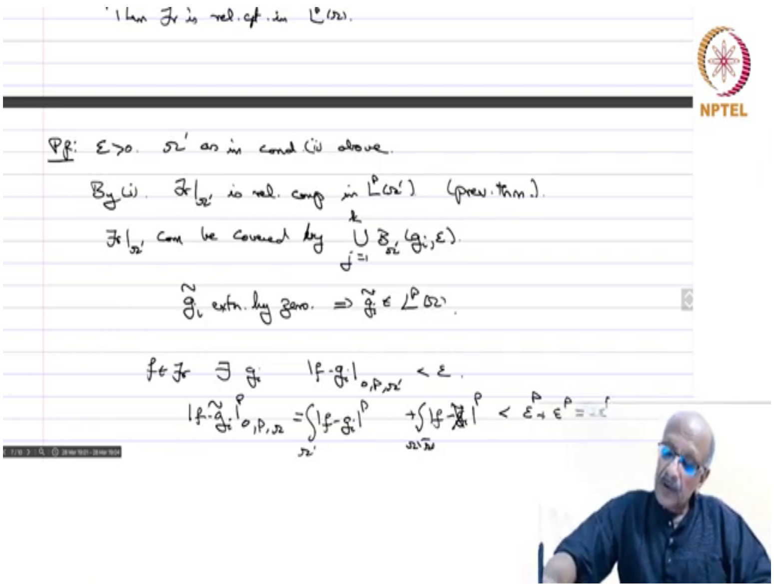

proof: so let $\epsilon > 0$ *and* Ω' as in condition (ii) above. So, by (i), $F|_{\Omega'}$ is relatively compact in $L^p(\Omega')$, that is the previous theorem. So, $F|_{\Omega'}$ can be covered by $\cup_{j=1}^{k} B_{\Omega'}(g_i, \epsilon)$. Now, you consider $\tilde{g}_i$ extension by 0 $\Rightarrow$ $\tilde{g}_i \in L^p(\Omega)$. Now, so if you take $f \in F$, there exists a gi such that $|f - g_i|_{0,p,\Omega'} < \epsilon$, this is what we have from this thing.

So, now if you take

$$|f - \tilde{g}_i|^p_{0,p,\Omega} = \int_{\Omega'} |f - g_i|^p + \int_{\Omega\backslash\Omega'} |f - \tilde{g}_i|^p < \epsilon^p + \epsilon^p = 2\epsilon^p$$

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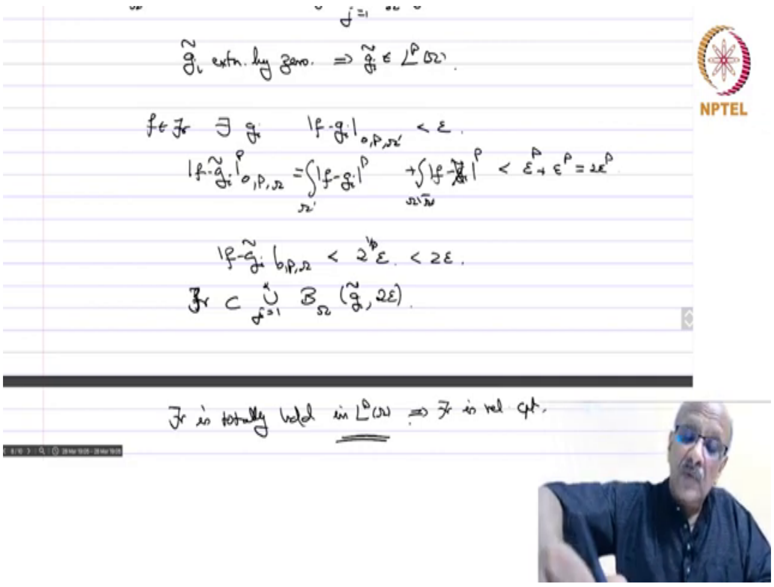

So, you have that $|f - \tilde{g}_i|_{0,p,\Omega} < 2^{\frac{1}{p}}\epsilon < 2\epsilon$.

So, we have that $F \subset \cup_{j=1}^{k} B_{\Omega'}(\tilde{g}_i, 2\epsilon)$. And therefore, again, this shows that capital F is totally bounded in $L^p(\Omega) \Rightarrow F$ is relatively compact.

So, this proves this. So, our next step is to use this lemma and try to show the compactness of various inclusions in the Sobolev embeddings. So, we have established what is necessary.