Sobolev Spaces and Partial Differential Equations
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Imbedding theorems Case p greater than N - Part 3

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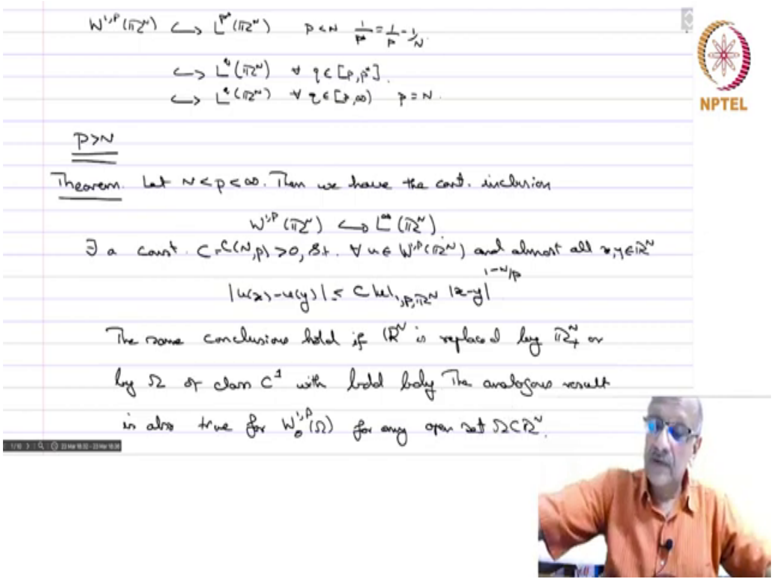


We are looking at Embedding theorems. So, we had that

$$W^{1,p}(\mathbb{R}^N) \to L^{p^*}(\mathbb{R}^N),\ \ p < N,\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}\ .$$

$$\to L^q(\mathbb{R}^N)\ ,\ \ q \in [p, p^*]\ .$$

$$\to L^q(\mathbb{R}^N)\ ,\ \ q \in [p, \infty],\ \ \ p = N\ .$$

So, now we want to look at the case when $p > N$. So, this is the third case which we want to do and here we state the theorem for this.

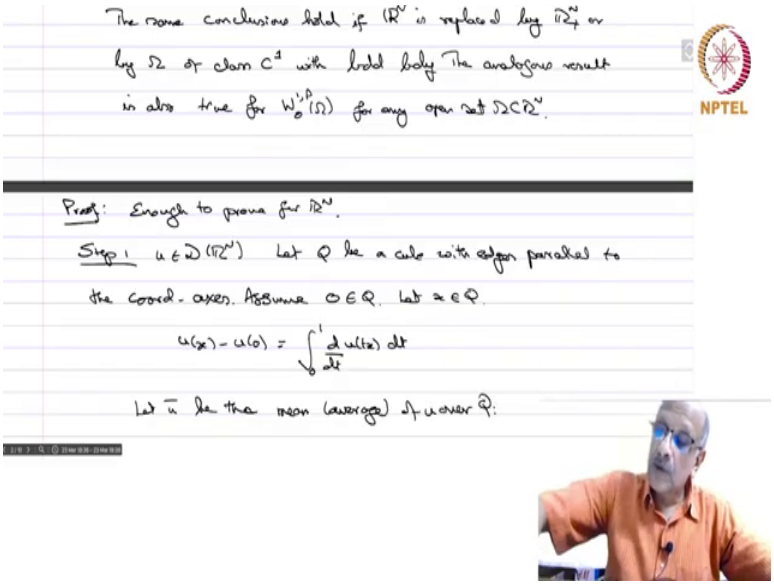
Theorem: So, let $N < p < \infty$. Then we have the continuous inclusion

$$W^{1,p}(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)\ .$$

There exists a constant $C = C(N, p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^N)$, we have and almost all $x, y \in \mathbb{R}^N$ that means except for a set of measures 0 all pairs of points you will have the following inequality,

$$|u(x) - u(y)| \leq C|u|_{1,p,\mathbb{R}^N}|x - y|^{1-\frac{N}{p}}.$$

The same conclusions hold if $\mathbb{R}^N$ is replaced by $\mathbb{R}_+^N$ or by Ω of class C^1 with a bounded boundary. The analogous result is also true for $W^{1,p}(\Omega)$ for any open set Ω. (Refer Slide Time: 4:34)



proof: Enough to prove for $\mathbb{R}^N$ as I explained just now extension operators will take care of the rest of the cases.

So, step 1: Let $D(\mathbb{R}^N)$. So, let Q be a cube with edges parallel to the coordinate axis. Assume 0 is in Q and let x belong to Q. So,

$$u(x) - u(0) = \int_0^1 \frac{d}{dt} u(tx)dt.$$

So, let $\overline{u}$ be the average mean or average of u over Q.

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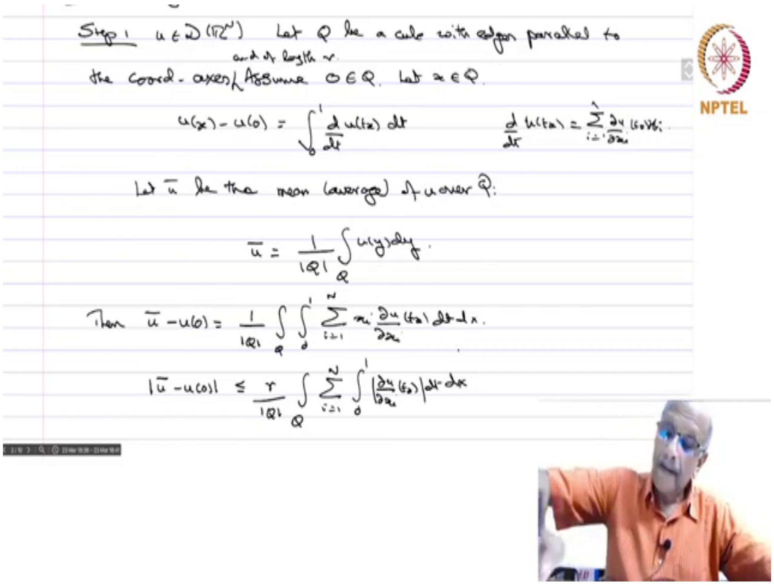

So,

$$\overline{u} = \frac{1}{|Q|}\int_Q u(y)dy.$$

So, then $\overline{u} - u(0) = \frac{1}{|Q|}\int_Q \int_0^1 \sum_{i=1}^{N} x_i \frac{\partial u}{\partial x_i}(tx)dtdx$

So, now $|\overline{u} - u(0)| \leq \frac{r}{|Q|}\int_Q \sum_{i=1}^{N} \int_0^1 |\frac{\partial u}{\partial x_i}(tx)|dtdx == \frac{1}{r^{N-1}}\int_0^1 \int_Q \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}(tx)|dtdx$

[as $|Q| = r^N$]

$$= \frac{1}{r^{N-1}}\int_0^1 \int_{tQ} \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}(y)|t^N dtdy \text{ [putting } tx = y]$$

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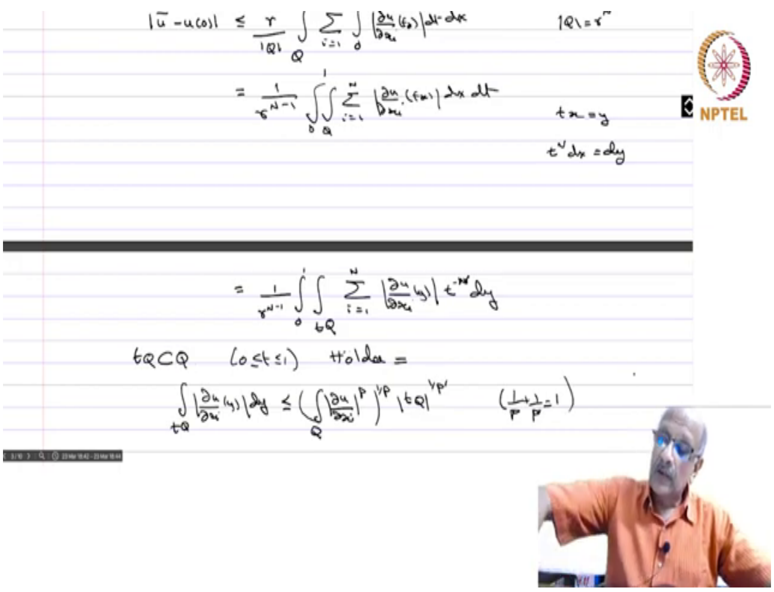

Now, $tQ \subset Q \;\; (0 \leq t \leq 1)$. And therefore, you have that and now you apply Holder to the inner integral.

$$\int_{tQ} |\frac{\partial u}{\partial x_i}(y)| dy \leq (\int_{Q} |\frac{\partial u}{\partial x_i}(y)|^p)^{\frac{1}{p}} \;\; |tQ|^{\frac{1}{p'}} \;\; , \quad \frac{1}{p} + \frac{1}{p'} = 1 \, .$$

so this is what we have.

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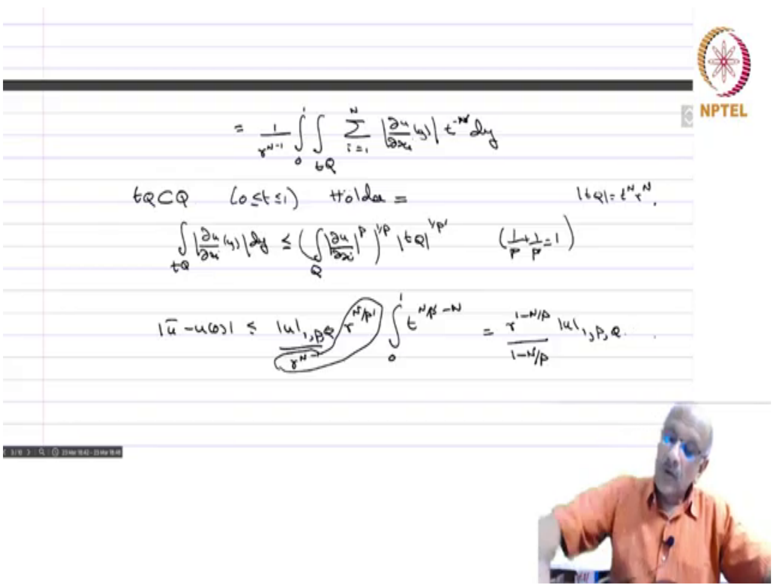

So, now if you substitute that in the previous inequality you get mod u bar minus u 0 is less than equal to, so you will get mod u 1, P Q because I have sigma i equals 1 to all of the powers will come there. And then by R power N minus 1 is already there and then I will get what is mod of t Q.

So, mod of t Q equals t power N R power N, so and then I am going to take 1 by P prime of that so R power N by P prime will come there and then integral 0 to 1 of dt will be t power N by P minus N because there is a t power minus N already. Now, I will get t power N by P prime, sorry, N by P prime minus N and that is equal to so if I R power N minus, so R power N by P prime minus N will give you N times 1 by P prime minus 1 which is minus 1 by P, so that is equal to R power 1 minus N by P.

If you simplify this expression here you get R power 1 minus N by P and then this integral if I integrate this will just give you 1 minus N by P again using the fact that 1 by P plus 1 by P prime equal to N and then mod u 1, P, Q. So, we have a nice relationship.

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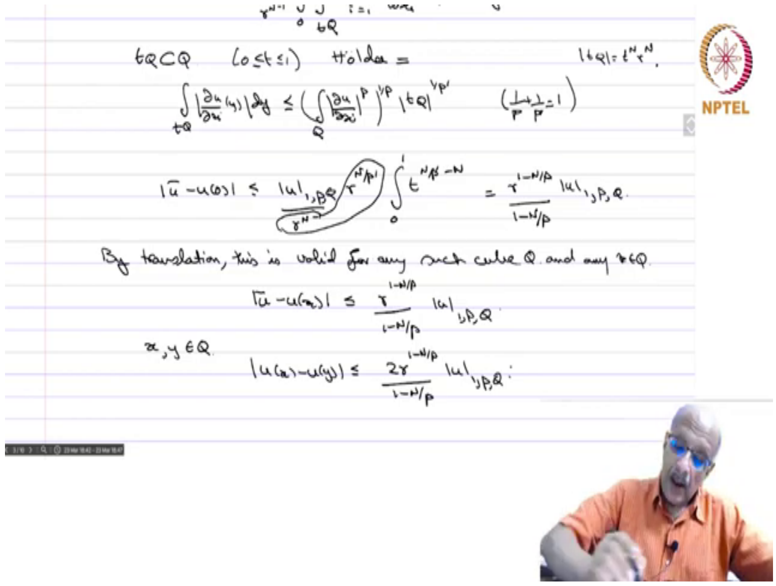

Now, by translation this is valid for any such cube Q and any x in Q. So, you have that for any such Q you have

$$|\overline{u} - u(x)| \leq \frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}} |u|_{1,p,Q} .$$

Now, if x and y belong to Q you apply it both to x and to y and then you get by the triangle inequality $|u(y) - u(x)| \leq \frac{2r^{1-\frac{N}{p}}}{1-\frac{N}{p}} |u|_{1,p,Q}$.

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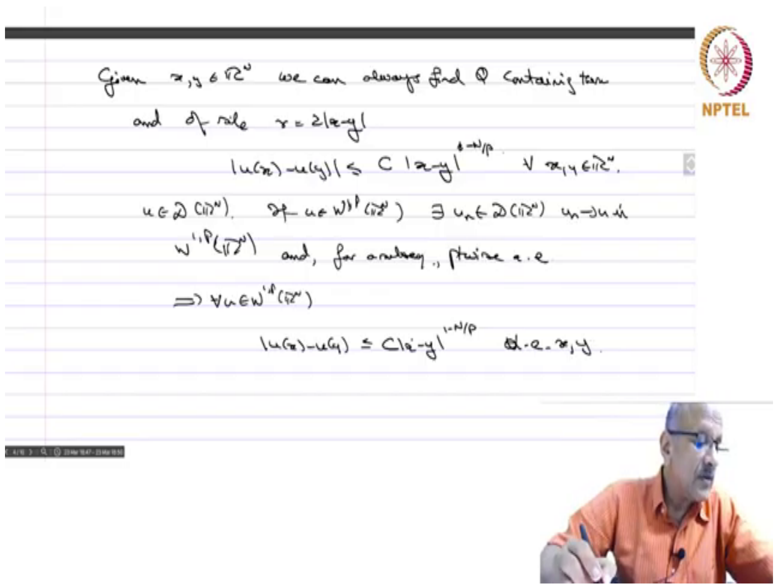

Now, given $x, y \in \mathbb{R}^N$, We can always find Q containing them and of side $r = 2|x - y|$. And therefore, you have that substituting that you get

$$|u(y) - u(x)| \leq C|x - y|^{1-\frac{N}{p}}, \ \forall x, y \in \mathbb{R}^N.$$

So, this is for almost for every, for all x, y in $\mathbb{R}^N$. So, this is for $u \in D(\mathbb{R}^N)$. So, if $u \in W^{1,p}(\mathbb{R}^N)$, then there exists $u_n \in D(\mathbb{R}^N)$ such that $u_n \to u \ in\, W^{1,p}(\mathbb{R}^N)$. And for a subsequence pointwise almost everywhere. So, then that will imply for all $u \in W^{1,p}(\mathbb{R}^N)$. So you apply it to u n and then you pass to the limit you get

$$|u(y) - u(x)| \leq C|x - y|^{1-\frac{N}{p}}, \ a.e.\ x, y \in \mathbb{R}^N.$$

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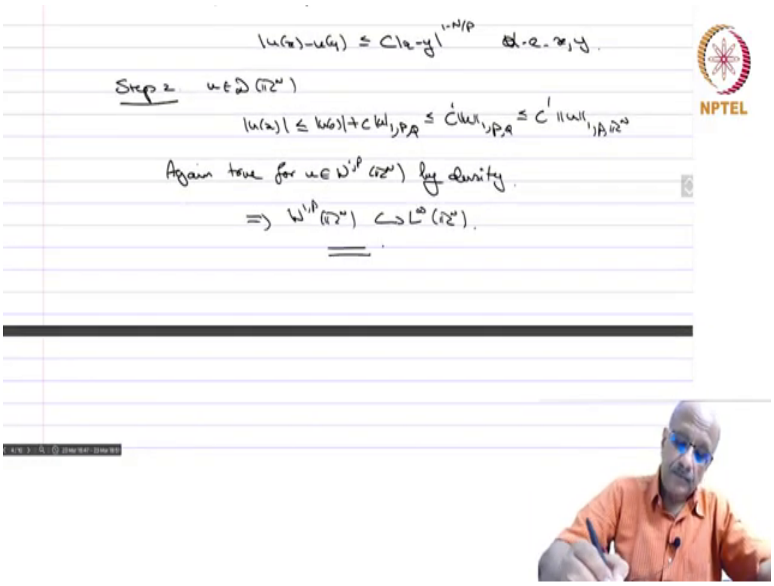

step 2: if $u \in D(\mathbb{R}^N)$, you have

$$|u(x)| \leq |u(0)| + C|u|_{1,p,Q} \leq C'||u||_{1,p,Q} \leq C'||u||_{1,p,\mathbb{R}^N}.$$

And now again true for $u \in W^{1,p}(\mathbb{R}^N)$ by density, $D(\Omega)$ is dense and therefore you have this. So, this implies that $W^{1,p}(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$.

So, this completely proves the theorem here.

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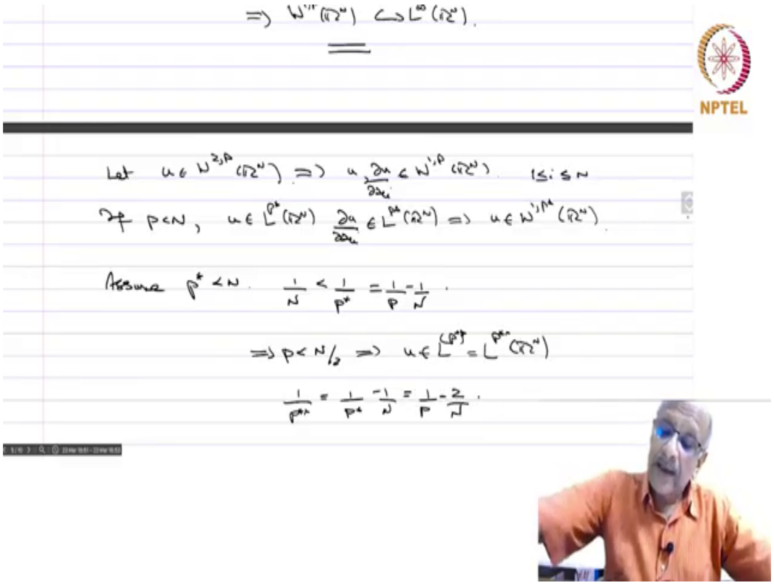

So, now you consider $u \in W^{2,p}(\mathbb{R}^N) \Rightarrow u, \frac{\partial u}{\partial x_i} \in W^{1,p}(\mathbb{R}^N)$, $1 \le i \le N$. Now, if $p < N$, then $u, \frac{\partial u}{\partial x_i} \in L^{p^*}(\mathbb{R}^N) \Rightarrow u \in W^{1,p^*}(\mathbb{R}^N)$.

Assume $p\ {*}{<}\ N$. So, when will this happen?

$$\frac{1}{N} < \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} \Rightarrow p < \frac{N}{2} \Rightarrow L^{(p^*)^*} = L^{p^{**}}(\mathbb{R}^N).$$

$$\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{N}.$$

So, we can now iterate this to any positive integer and then we can iterate all the theorems.

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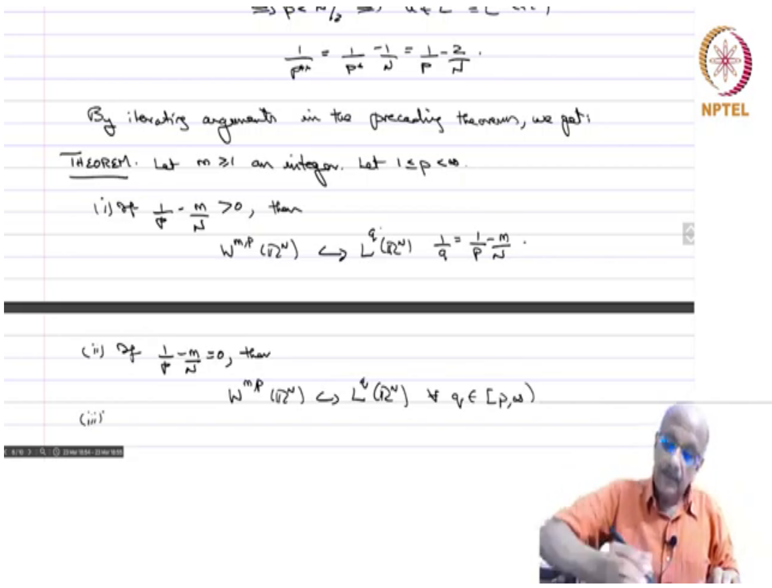

So, by iterating arguments in the previous preceding theorems we get the following.

Theorem: Let $m \geq 1$ an integer. Let $1 \leq p < \infty$.

(i) If $\frac{1}{p} - \frac{m}{N} > 0$, then $W^{m,p}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$, $\frac{1}{q} = \frac{1}{p} - \frac{m}{N}$.

(ii) If $\frac{1}{p} - \frac{m}{N} = 0$, then $W^{m,p}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$, $\forall q \in [p, \infty)$.

(iii) If $\frac{1}{p} - \frac{m}{N} < 0$, then $W^{m,p}(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$.

In the last case, let k be the integral part and θ be the fractional part of $m - \frac{N}{p}$. Then there exists $C > 0,\ s.t.\ \forall u \in W^{m,p}(\mathbb{R}^N)$, *we have*

$$|D^\alpha u|_{0,\infty,\mathbb{R}^N} \leq C||u||_{m,p,\mathbb{R}^N}, \quad \forall\, |\alpha| \leq k .$$

For almost all $x, y \in \mathbb{R}^N$ *and* $|\alpha| = k$,

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C||u||_{m,p,\mathbb{R}^N}|x - y|^\theta .$$

In particular, $W^{m,p}(\mathbb{R}^N) \to C^k(\mathbb{R}^N)$ and the k-th derivatives are Holder continuous.

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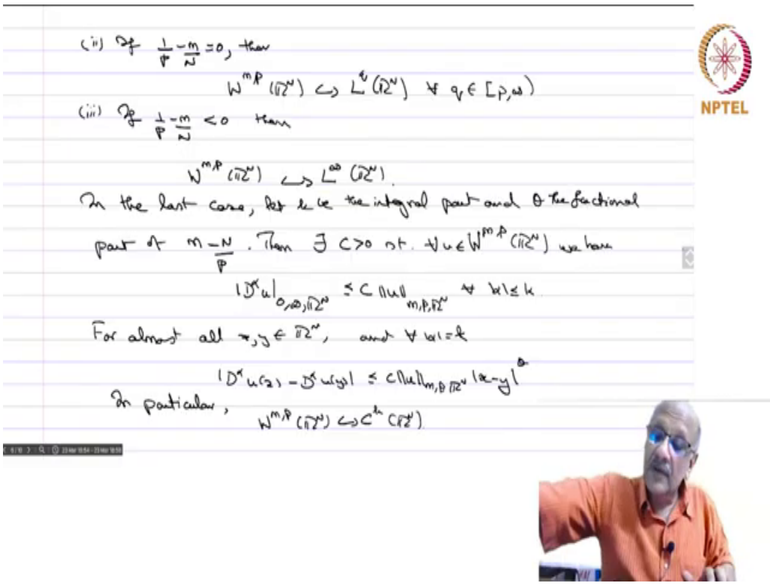

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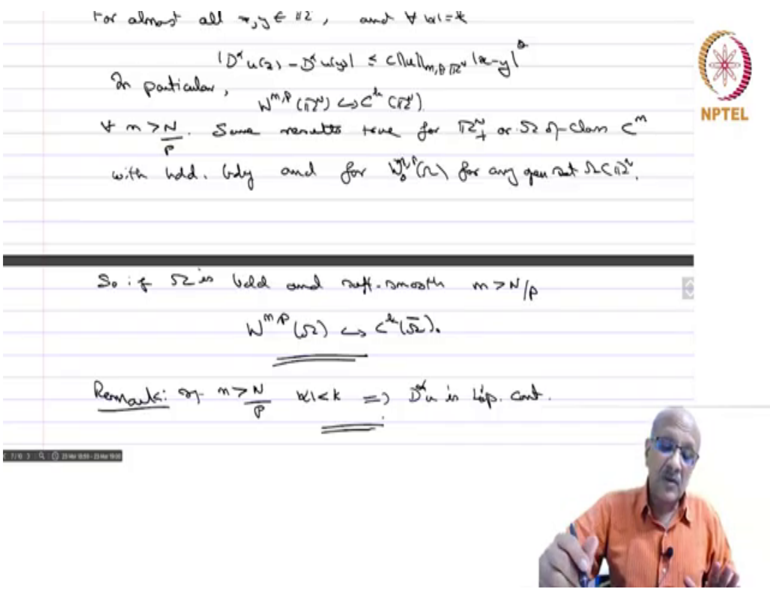

The previous derivatives will be actually Lipschitz continuous for all $m > \frac{N}{p}$. Same results true for $\mathbb{R}_+^N$ or Ω of class C^m, because you are iterating the condition for the first derivative you need C^1, so now for successive derivatives you have to include a bounded boundary and for $W_0^{m,p}(\Omega)$for any open set omega in $\mathbb{R}^N$.

remark: if $m > \frac{N}{p}$ and $|\alpha| < k$, then it implies that $D^\alpha u$ is Lipschitz continuous because any higher, the first derivative of $D^\alpha u$ is in L^∞, it is bounded. And therefore, $D^\alpha u$ will be automatically Lipschitz continuous. It is only when you come to alpha equals mod k you cannot say anything about the next derivative and therefore you only have Holder continuous, so that is just from the mean value theorem which says if the derivative is bounded then the function is Lipschitz continuous. So, this completes the study of the embedding theorems. Our next aim is to see which of these embeddings are going to be compact.