Sobolev Spaces and Partial Differential Equations
Professor S Kesavan.
Department of Mathematics, IMSc
Indian Institute of Technology, Madras
Lecture 4
Examples – Part 1

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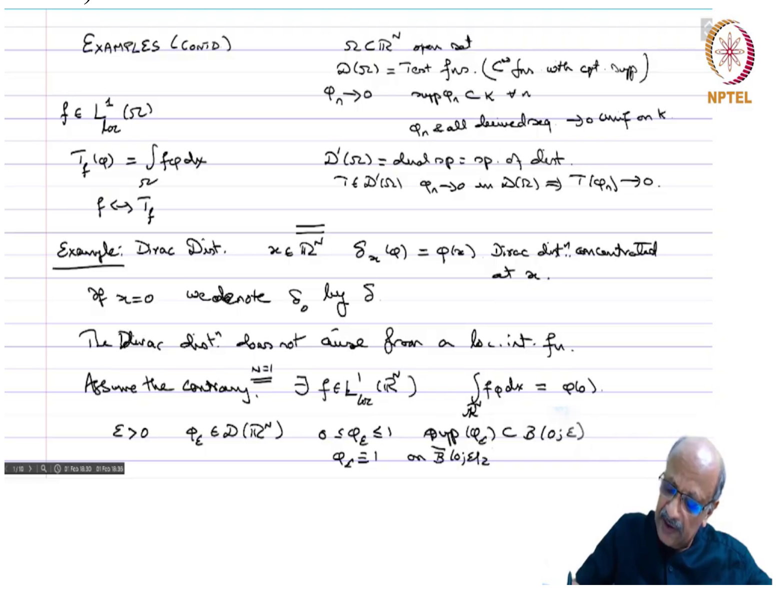

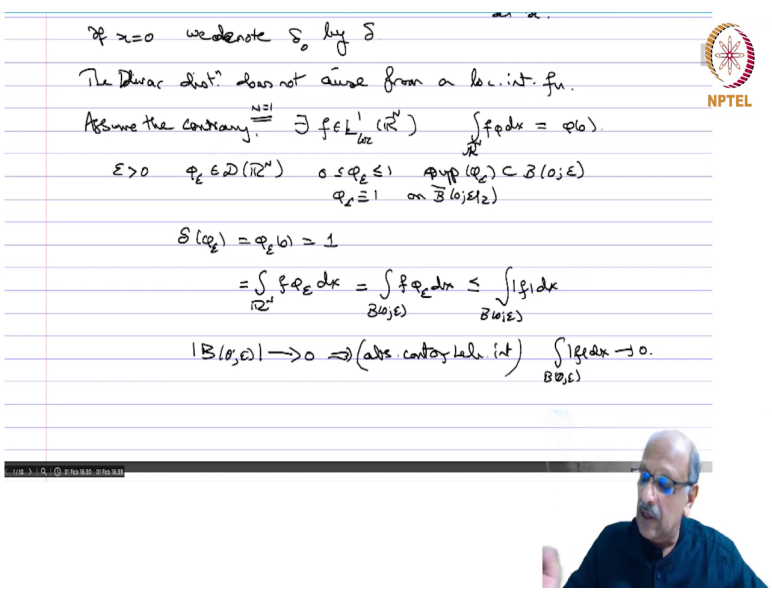

So, we will continue examples of distributions.

Examples (Contd.): So, let us briefly recall what we have been doing up to now. So, $\Omega \subset \mathbb{R}^N$ is an open set, $D(\Omega)$ is the space of test functions, that means C^{∞}- infinity functions with compact support in Ω. And the topology on this is dictated by the sequences. So, $\phi_n \to 0$ in $D(\Omega)$ means: there exists compact K such that $\text{supp}(\phi_n) \subset K$ and ϕ_n and all its derived sequences converge to 0 uniformly on K. So, this is the, this thing and dual space. So, the dual space $D'(\Omega)$: which is a space of distributions. If $T \in D'(\Omega)$ and $\phi_n \to 0$ in $D(\Omega)$, this implies $T(\phi_n) \to 0$. So, this is what we have to check when something is to be called a distribution.

So, the first example which we saw was: if $f \in L^1_{loc}(\Omega)$, then you have that f defines the distribution namely: $T_f(\phi) = \int_\Omega f\phi\, dx$, for all $\phi \in D(\Omega)$.

So, this defined distribution and we also saw that if you have two locally integrable functions giving the same distribution then the two functions have to be equal almost everywhere.

So, between f and T_f, you have a one-to-one correspondence and therefore, very often we will just say function is a distribution, by that we mean that function generates this distribution which is defined in this fashion.

Example: (*Dirac distribution*): This is almost the starting point for the search for a proper theory people like Dirac and Heaviside they used some kind of symbolic calculus, which made no sense to the mathematicians and it was a attempt to clarify what is going on, but yet they were getting reasonable results out of this and so, an effort to clarify these things led to the discovery of the theory of distributions by Laurent Schwartz for which he got the Fields Medal.

Let $x \in \mathbb{R}^N$ and you define: $\delta_x(\phi) = \phi(x)$. So, this is called Dirac distribution concentrated at x. So, that this is a linear functional on ϕ; it is clear and also if $\phi_n \to 0$, then automatically $\phi_n(x) \to 0$. And therefore, we have that this is a distribution in a very trivial way.

If $x = 0$, the origin, then we denote δ_0 by just a symbol δ . So, δ is usually the famous Dirac delta function which you might have come across when studying differential equations for instance. So, now, this distribution does not, is not included in the previous example, namely the Dirac distribution does not arise from locally integrable function.

So, assume the contrary: let there exists $f \in L^1_{loc}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} f\phi\, dx = \phi(0)$.

For $\epsilon > 0$, consider $\phi_\epsilon \in D(\mathbb{R}^N)$ such that $0 \leq \phi_\epsilon \leq 1$, supp$(\phi_\epsilon) \subset B(0;\epsilon)$ and

$$\phi_\epsilon \equiv 1 \text{ on } \overline{B(0;\frac{\epsilon}{2})}.$$

Now, $\delta\ (\phi_\epsilon) = \phi_\epsilon(0) = 1 = \int_{\mathbb{R}^N} f\phi_\epsilon\, dx = \int_{B(0;\epsilon)} f\phi_\epsilon\, dx \leq \int_{B(0;\epsilon)} |f|\, dx.$

Since $|B(0;\epsilon)| \to 0 \Rightarrow$ (by absolute continuity of integral) $\int_{B(0;\epsilon)} |f|\, dx \to 0$, and this is not possible and therefore, that gives you a contradiction. So, the Dirac distribution is a new object, it does not come from the functions.

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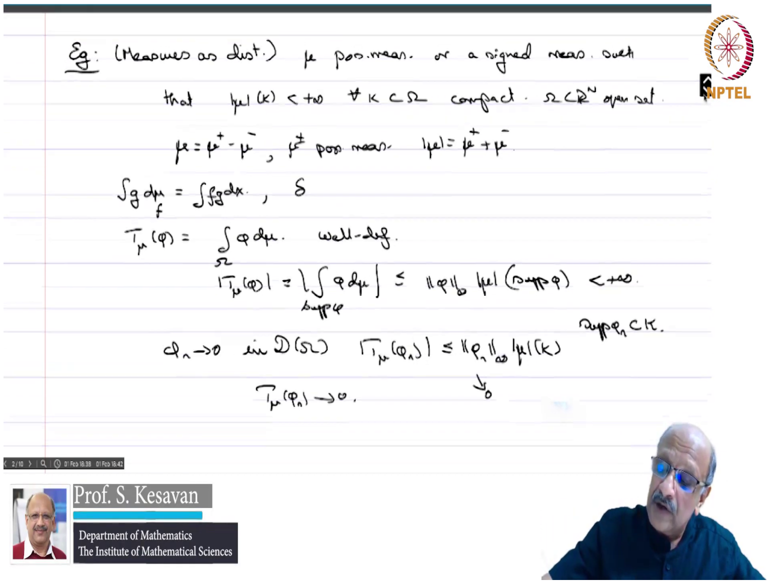

Example: (Measures as distributions) So, we will take μ either to be a positive measure, measure in the usual sense or a signed measure such that $|\mu|(K)$ is finite for all $K \subset \Omega$ − compact. So, again $\Omega \subset \mathbb{R}^N$ open set is a standing hypothesis. So, what is $|\mu|$? If μ is a positive measure, then $|\mu|$ is the same as μ ; if μ is a signed measure, then you can write

$$\mu = \mu^+ - \mu^-,\ \mu^+ \text{ positive measure; } |\mu| = \mu^+ + \mu^-$$

So, this comes from and you have studied the signed measures you would have come across this Jordan decomposition, Hahn decomposition etc. which tells you that every signed measure can be broken up as a difference of two positive measures one of them being finite and so, this has to be finite on every complex.

So, then, if you for instance the local integrable function gives you a signed measure. So, $\int g\, d\mu_f = \int fg\, dx$. So, this gives you how a locally integrable function can give you a signed measure and because of local integrability this means that this is finite on every compact set, because $|f|$ is integrable on any compact set.

Dirac measure is also a finite measure and therefore, automatically comes in this. So, both previous examples are covered by this particular example.

So, we now define

$$T_{\mu}(\phi) = \int_{\Omega} \phi \, d\mu ,$$

So, this is well defined because

$$|T_{\mu}(\phi)| = |\int_{supp(\phi)} \phi \, d\mu| \leq ||\phi||_{\infty} |\mu|(supp(\phi)) < \infty .$$

So, again if if $\phi_n \to 0$, you have $|T_{\mu}(\phi_n)| \leq ||\phi_n||_{\infty} |\mu|(K)$. So, $|T_{\mu}(\phi_n) \to 0$, because of the uniform convergence of ϕ_n and therefore T_{μ} defines a distribution.

So, this way we define a new distribution, all the measures, signed measures which are finite on compact sets will define another distribution. And this covers both the case of locally integrable functions and the Dirac distribution, the Dirac distribution comes from the Dirac measure. So, this way we have this, now, we will define another distribution example.

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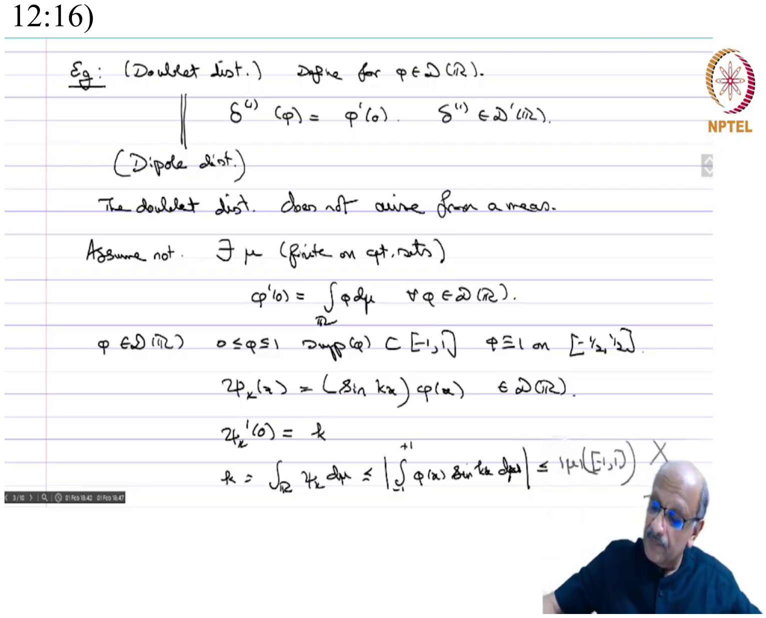

Doublet distribution: Define for $\phi \in D(\mathbb{R})$,

$$\delta^{(1)}(\phi) = \phi'(0).$$

Again, this is linear and it is continuous because if $\phi_n \to 0$, then ϕ_n and all its derivatives go to 0 uniformly on the supports and therefore, ϕ'_n will also go to 0. So, clearly this defines a distribution.

So, this is called the doublet or dipole, so, doublet is also called the dipole distribution. So, now and this is familiar to people who have done some mathematical physics. So, now this the doublet distribution does not arise from a measure.

So, this is a totally new object so up to now we have been gradually using familiar objects to define the distribution, we use locally integrable functions, we use measures and we saw that the signed measure covered everything up to them and now we are having a new object called the dipole distribution and they claim that this does not come from any other any measure or anything.

So, assume the contrary: there exists a measure μ (which is finite on compact sets as in the previous example), such that either μ or |μ| is a signed measure. And so, we have

$$\phi'(0) = \int_{\mathbb{R}} \phi \, d\mu, \text{ for all } \phi \in D(\mathbb{R}),$$

So, we now take a function $\phi \in D(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\mathrm{supp}(\phi_\epsilon) \subset [-1, 1]$, and $\phi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. And we are going to define,

$$\psi_k(x) = \sin(kx)\phi(x) \in D(\mathbb{R})$$

Then $$\psi_k'(0) = k$$

$$k = \int_{\mathbb{R}} \psi_k \, dx \leq \left| \int_{-1}^{1} \sin(kx)\phi(x) \, d\mu \right| \leq |\mu|([-1, 1])$$

Now, this is a bounded finite fixed quantity and it is bigger than or equal to k for every k positive integral; that is not possible, this goes to infinity, this remains fixed and therefore, you do not have any, so this gives you a contradiction. So, we have now seen as I am repeatedly saying functions have been locally integrable functions have been regarded as distributions, measures which are finite on compact sets are also regarded as distributions.

And then we have also produced some new objects, which are called, like the doublet distribution, which is not covered by any of these things, and therefore, that gives you a new set of distributions. So, we have plenty of interesting examples of distributions. So, before proceeding further so, I just wish to state the theorem, I will not prove it, because I will tell you what.

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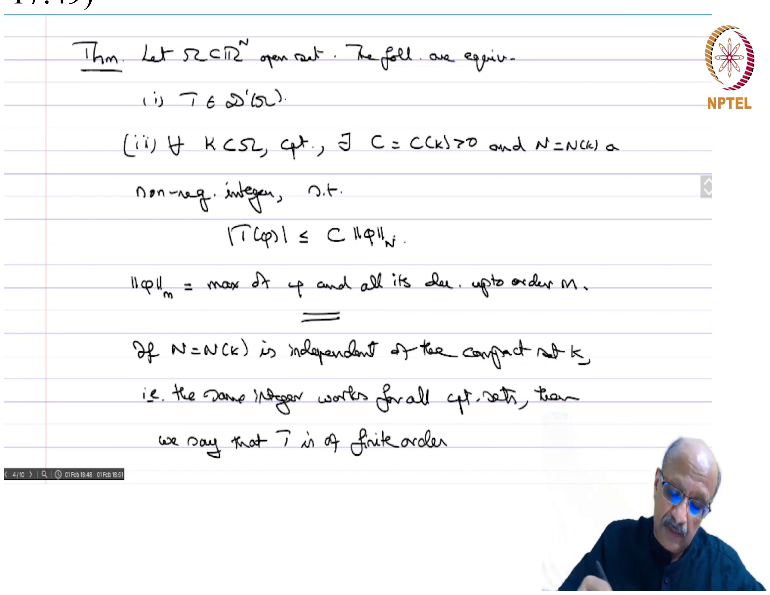

NPTEL

Thm. Let $\Omega \subset \mathbb{R}^N$ open set. The foll. are equiv.

(i) $T \in \mathcal{D}'(\Omega)$.

(ii) $\forall\ K \subset \Omega$, cpt., $\exists\ C = C(K) > 0$ and $N = N(K)$ a non-neg. integer, s.t.

$$|T(\varphi)| \leq C \|\varphi\|_N .$$

$$\|\varphi\|_m = \max_{x \in \Omega} |D^\alpha \varphi|$$

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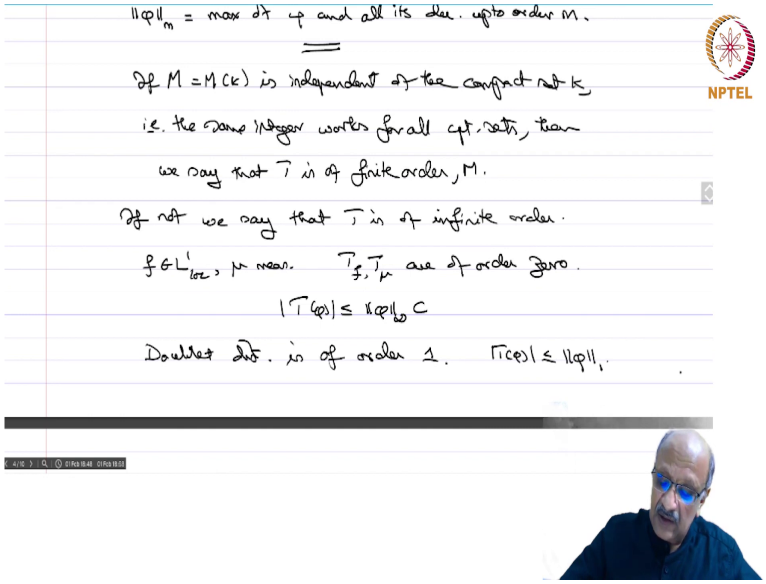
$$|T(\varphi)| \leq C \|\varphi\|_M .$$

$\|\varphi\|_m$ = max of φ and all its der. upto order M.

If $M = M(K)$ is independent of the compact set K, i.e. the same integer works for all cpt. sets, then we say that T is of finite order, M.



Theorem: Let $\Omega \subset \mathbb{R}^N$ be an open set. Then the following are equivalent:

(i) $T \in D'(\Omega)$.

(ii) $\forall K \subset \Omega -$ compact, $\exists\, C = C(K) > 0$ and $M = M(K) -$ a positive integer, such that $|T(\phi)| \leq C||\Phi||_M$ for all $\phi \in D(\Omega)$ with supp(ϕ)$\subset K$, where $||\phi||_m$= max of ϕ and all its derivative upto order m.

Now, this result we will not prove because it involves the nature of the topology of $D(\Omega)$. Which we have deferred for the moment and therefore, we will just accept it.

Now, here we have if $M = M(K)$ is independent of the compact set K, that is the same integer works for all compact sets, then we say that T is of finite order.

So, this is called the order of the distribution. If not, that is, if this M depends on the compact set and it changes for every K, we say that T is of infinite order.

So, if you look at $f \in L^1_{loc}(\Omega)$ or μ is the measure which is finite on compact sets then T_f, T_μ are of order zero.

So, here we saw because we saw that $|T(\phi)| \leq C||\Phi||_{\infty}$ in all these cases which was the measure of the compact set something like that. So, this is nothing but when the case, when M equals 0. So, you are only taking no derivatives and therefore the C.

So, the doublet distribution is of order 1. Because,

$$|T(\phi)| \leq ||\Phi||_{1}$$

So, now we will continue and define a calculus of distributions, we will define differentiation, multiplication by C^{∞}- functions etc., which we will look at later.