Sobolev Spaces and Partial Differential Equations
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Imbedding theorems: Case p = N - Part 2

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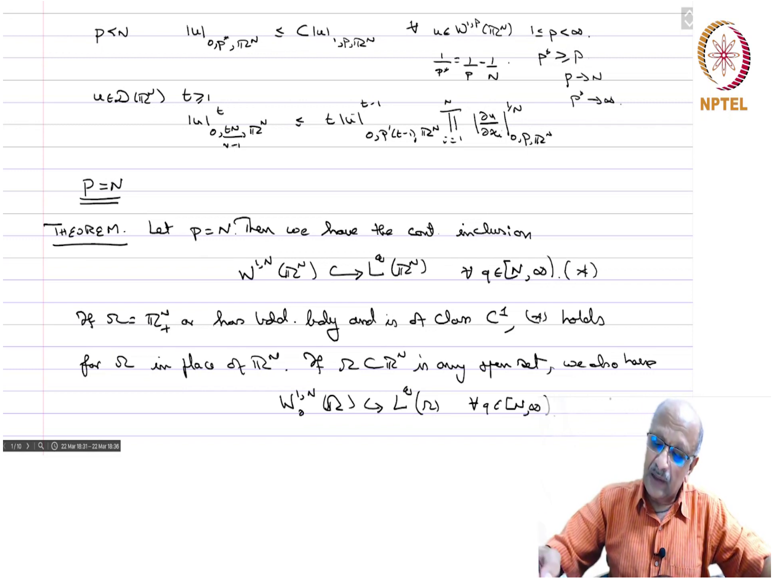

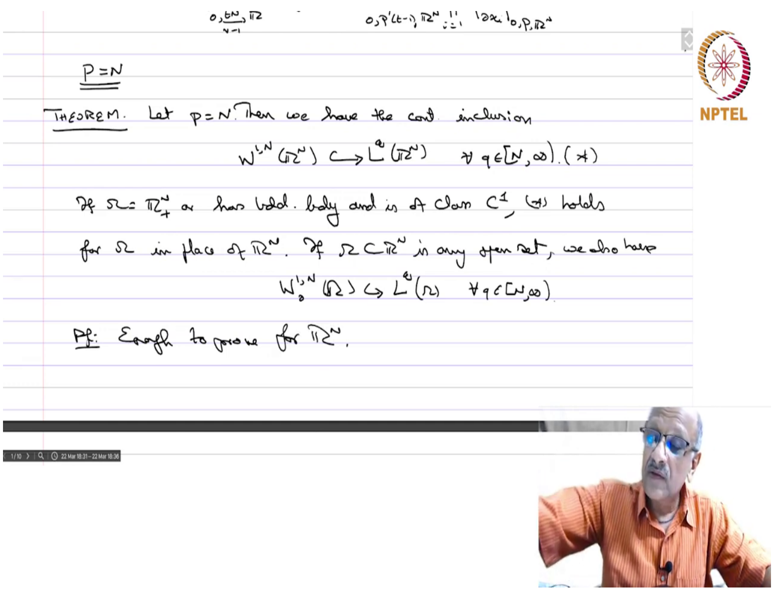

We were looking at embedding theorems. So, we want to show that, when $p < N$, we showed that $|u|_{0,p^*,\mathbb{R}^N} \leq C|u|_{1,p,\mathbb{R}^N}$, $\forall\, u \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p < \infty$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

So, how did we prove this? We first prove this for $u \in D(\mathbb{R}^N)$ and then for some t greater than equal to 1, we showed that

$$|u|^{t}_{0,\frac{tN}{N-1},\mathbb{R}^N} \leq t|u|^{t-1}_{0,p'(t-1),\mathbb{R}^N}\Pi^{N}_{i=1}\left|\frac{\partial u}{\partial x_i}\right|^{\frac{1}{N}}_{0,p,\mathbb{R}^N}$$

So, we will see what the theorem is now.

Theorem: Let $p = N$. Then we have the continuous inclusion

$$W^{1,N}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)\ ,\ \ \forall q \in [N,\infty) \quad ---------- \ (*)$$

If $\Omega = \mathbb{R}^N_+$, or has bounded boundary then (*) holds for Ω in place of $\mathbb{R}^N$. If $\Omega \subset \mathbb{R}^N$ is any open set we also have

$$W_0^{1,N}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)\ ,\ \ \forall q \in [N,\infty).$$

proof: So, it is enough to prove for $\mathbb{R}^N$. So, all the rest follows just because of the existence of extension operators.

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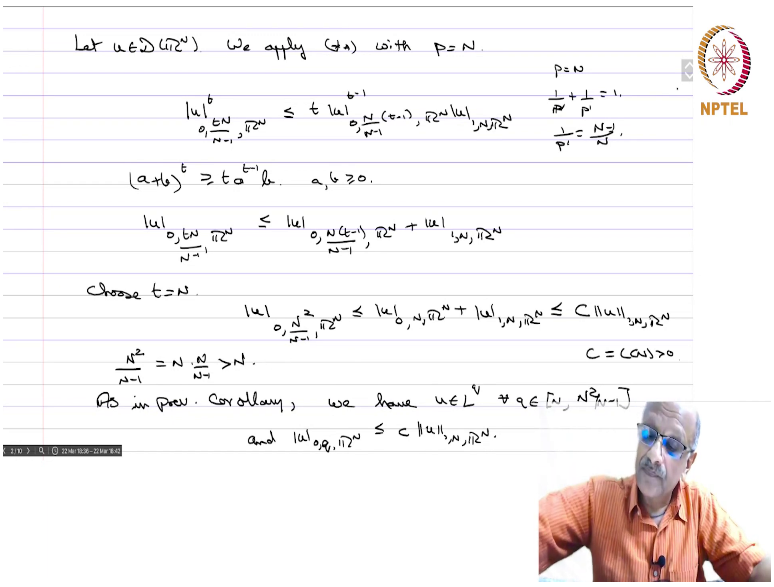

So, let, again $u \in D(\mathbb{R}^N)$, so, we apply (**). So, in applying that relationship, we have nothing actually, we have, we did not use anything about P less than N and so on. So, we applied a double star with P equals N. So, you have

$$|u|^t_{0,\frac{tN}{N-1},\mathbb{R}^N} \leq t|u|^{t-1}_{0,\frac{N}{N-1}(t-1),\mathbb{R}^N}|u|_{1,N,\mathbb{R}^N}$$

and then $\frac{1}{p'} = \frac{N-1}{N}$.

So, now, you have $(a + b)^t \geq ta^{t-1}b,\ \ a, b \geq 0$. So

$$|u|_{0,\frac{tN}{N-1},\mathbb{R}^N} \leq |u|_{0,\frac{N}{N-1}(t-1),\mathbb{R}^N} + |u|_{1,N,\mathbb{R}^N}\ .$$

Now you choose, now we have the liberty to choose t, so I am going to choose t =N. So that I get

$$|u|_{0,\frac{N^2}{N-1},\mathbb{R}^N} \leq |u|_{0,N,\mathbb{R}^N} + |u|_{1,N,\mathbb{R}^N} \leq C||u||_{1,N,\mathbb{R}^N}\ , \quad C = C(N, p) > 0.$$

And now you recall that $\frac{N^2}{N-1} > N$ and therefore, by previous corollary, we have

$u \in L^q\ ,\ \forall\, q \in [N, \frac{N^2}{N-1}]$.

And now $|u|_{0,q,\mathbb{R}^N} \leq C||u||_{1,N,\mathbb{R}^N}$.

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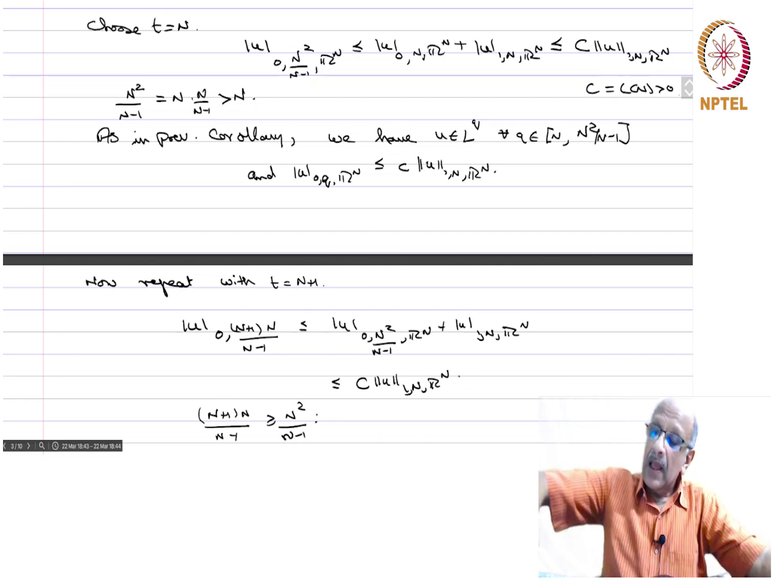

So, the see bootstrapping trick, which now repeats with t =N+1. So, what do you get

$$|u|_{0,\frac{(N+1)N}{N-1},\mathbb{R}^N} \leq |u|_{0,\frac{N^2}{N-1},\mathbb{R}^N} + |u|_{1,N,\mathbb{R}^N} \leq C||u||_{1,N,\mathbb{R}^N} .$$

and then recall that $\frac{(N+1)N}{N-1} \geq \frac{N^2}{N-1}$.

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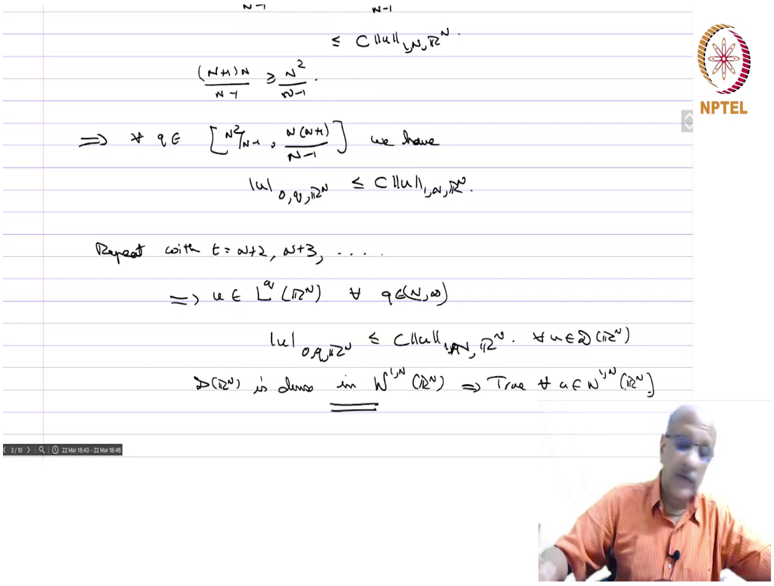

And so, again we use the previous corollary, therefore, this implies that

$\forall q \in [\frac{N^2}{N-1}, \frac{(N+1)N}{N-1}]$, we have

$$|u|_{0,q,\mathbb{R}^N} \leq C||u||_{1,N,\mathbb{R}^N}$$

So, now repeat with $t = N + 2, N + 3, \ldots$. and so on. And so, each time you will increase a certain interval and it will go on forward so that you get all so, this implies that

$u \in L^q(\mathbb{R}^N),\ \forall q \in (N, \infty)$ and we have

$$|u|_{0,q,\mathbb{R}^N} \leq C||u||_{1,N,\mathbb{R}^N}\ ,\ \ \forall u \in D(\mathbb{R}^N)$$

and then $D(\mathbb{R}^N)$ is dense in $W^{1,N}(\mathbb{R}^N)$ and this implies true for all $u \in W^{1,N}(\mathbb{R}^N)$. So, that proves the theorem, when $p = N$.

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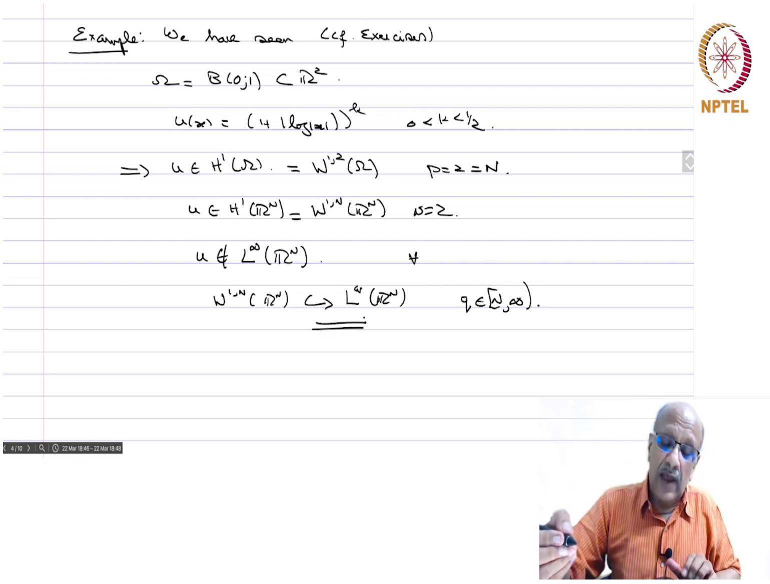

So, when p and then as I said for all subdomains you just have to use the prolongation operator. So, example we have seen, Cf exercises;

Example: we have seen in the last set of exercises that we did, $\Omega = B(0; 1) \subset \mathbb{R}^2$ and

$$u(x) = (1 + |\log|x||)^k,\ \ 0 < k < \frac{1}{2}.$$

Then we saw that $u \in H^1(\Omega) = W^{1,2}(\Omega),\ \ p = 2 = N$. So, you have

$u \in H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N),\ N = 2$ and $u \in L^\infty(\mathbb{R}^N)$, because it blows up near the origin face unbounded function and therefore, you do not have that it is in L^∞.

So, the result is sharp, you can only get from for all, so you only have

$$W^{1,N}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \ , \ \forall q \in [N, \infty)$$

So that proves that, so our next thing is to look at $p > N$, which we will do next time.