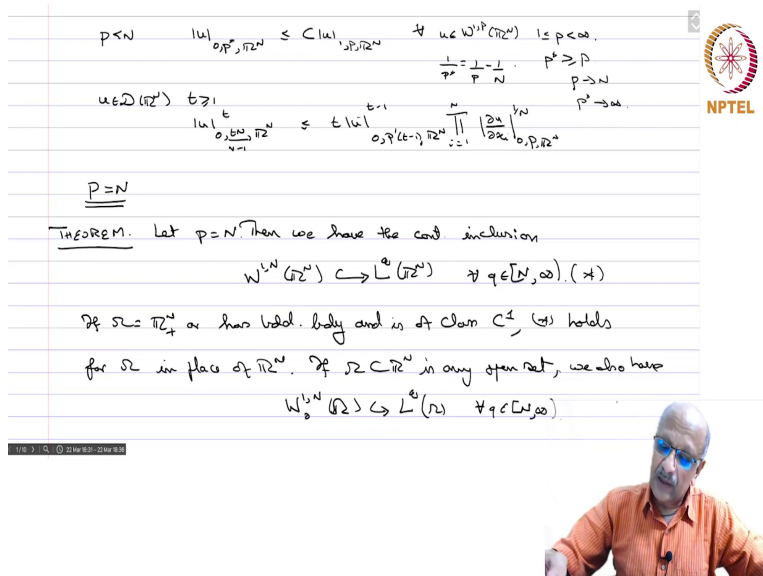


Sobolev Spaces and Partial Differential Equations
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Imbedding theorems: Case $p = N$ - Part 2

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$p < N$ $|u|_{0,p^*,\mathbb{R}^N} \leq C |u|_{1,p,\mathbb{R}^N}$ $\forall u \in W^{1,p}(\mathbb{R}^N)$ $1 \leq p < \infty$.

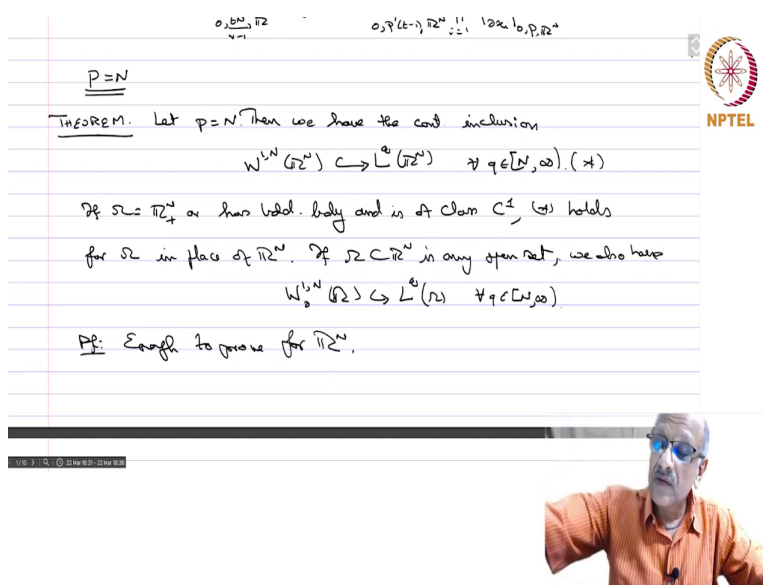
$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ $p^* > p$
 $p \rightarrow N$
 $p^* \rightarrow \infty$

$u \in \mathcal{D}(\mathbb{R}^N)$ $t \geq 1$
 $|u|_{0,\frac{2N}{N-2},\mathbb{R}^N} \leq C |u|_{1,p,\mathbb{R}^N}^{t-1} \prod_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|_{0,p,\mathbb{R}^N}^{1/N}$ $p^* \rightarrow \infty$

$p = N$

THEOREM. Let $p = N$. Then we have the cond inclusion
 $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \quad \forall q \in [N, \infty) \quad (*)$

If $\Omega \subset \mathbb{R}^N$ has vol. bdy and is of class C^1 , $(*)$ holds
for Ω in place of \mathbb{R}^N . If $\Omega \subset \mathbb{R}^N$ in any open set, we also have
 $W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [N, \infty)$



$0, \frac{2N}{N-2}, \mathbb{R}^N$ $0, \frac{2N}{N-2}, \mathbb{R}^N$ $1, \frac{2N}{N-2}, \mathbb{R}^N$

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Pr: Enough to prove for \mathbb{R}^N .

We were looking at embedding theorems. So, we want to show that, when $p < N$, we showed that $|u|_{0,p^*,\mathbb{R}^N} \leq C |u|_{1,p,\mathbb{R}^N}$, $\forall u \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p < \infty$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

So, how did we prove this? We first prove this for $u \in D(\mathbb{R}^N)$ and then for some t greater than equal to 1, we showed that

$$|u|^t_{0, \frac{tN}{N-1}, \mathbb{R}^N} \leq t |u|^{t-1}_{0, p'(t-1), \mathbb{R}^N} \prod_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{\frac{1}{N}}_{0, p, \mathbb{R}^N}$$

So, we will see what the theorem is now.

Theorem: Let $p = N$. Then we have the continuous inclusion

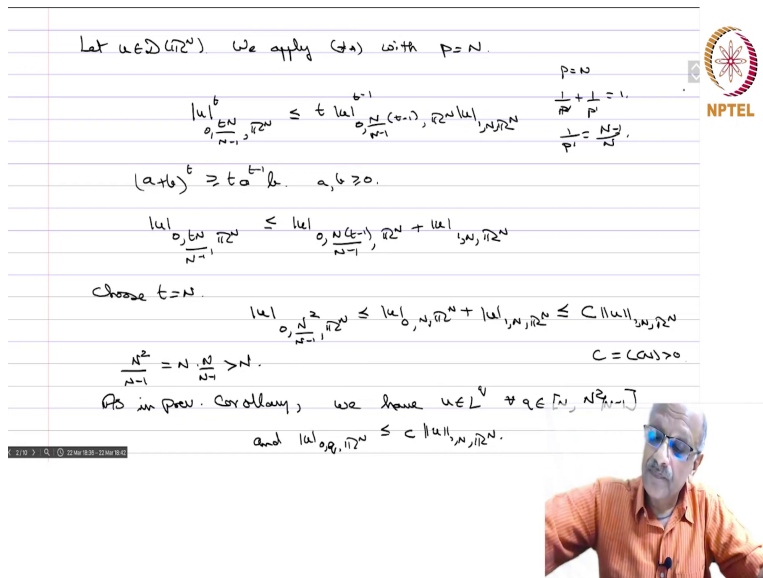
$$W^{1,N}(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N), \quad \forall q \in [N, \infty) \quad \text{-----} (*)$$

If $\Omega = \mathbb{R}^N_+$, or has bounded boundary then (*) holds for Ω in place of \mathbb{R}^N . If $\Omega \subset \mathbb{R}^N$ is any open set we also have

$$W^{1,N}_0(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N), \quad \forall q \in [N, \infty).$$

proof: So, it is enough to prove for \mathbb{R}^N . So, all the rest follows just because of the existence of extension operators.

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Let $u \in D(\mathbb{R}^N)$. We apply (*) with $p = N$.

$$|u|^t_{0, \frac{tN}{N-1}, \mathbb{R}^N} \leq t |u|^{t-1}_{0, \frac{N}{N-1}, \mathbb{R}^N} \prod_{i=1}^N |u|_{i, N, \mathbb{R}^N}$$

$p = N$
 $\frac{1}{p} + \frac{1}{p'} = 1$
 $\frac{1}{p'} = \frac{N-1}{N}$

$$(a+b)^t \geq t a^{t-1} b, \quad a, b \geq 0.$$

$$|u|_{0, \frac{tN}{N-1}, \mathbb{R}^N} \leq |u|_{0, \frac{N}{N-1}, \mathbb{R}^N} + |u|_{i, N, \mathbb{R}^N}$$

Choose $t = N$.

$$|u|_{0, \frac{N^2}{N-1}, \mathbb{R}^N} \leq |u|_{0, \frac{N}{N-1}, \mathbb{R}^N} + |u|_{i, N, \mathbb{R}^N} \leq C \|u\|_{i, N, \mathbb{R}^N}$$

$\frac{N^2}{N-1} = N \cdot \frac{N}{N-1} > N$.
 $C = C(N) > 0$

As in prev. Corollary, we have $u \in L^q \quad \forall q \in [N, \frac{N^2}{N-1}]$

and $|u|_{0, \frac{N^2}{N-1}, \mathbb{R}^N} \leq C \|u\|_{i, N, \mathbb{R}^N}$.

So, let, again $u \in D(\mathbb{R}^N)$, so, we apply (**). So, in applying that relationship, we have nothing actually, we have, we did not use anything about P less than N and so on. So, we applied a double star with P equals N . So, you have

$$|u|_{0, \frac{tN}{N-1}, \mathbb{R}^N}^t \leq t |u|_{0, \frac{N}{N-1}(t-1), \mathbb{R}^N}^{t-1} |u|_{1, N, \mathbb{R}^N}$$

and then $\frac{1}{p'} = \frac{N-1}{N}$.

So, now, you have $(a + b)^t \geq t a^{t-1} b$, $a, b \geq 0$. So

$$|u|_{0, \frac{tN}{N-1}, \mathbb{R}^N} \leq |u|_{0, \frac{N}{N-1}(t-1), \mathbb{R}^N} + |u|_{1, N, \mathbb{R}^N}.$$

Now you choose, now we have the liberty to choose t , so I am going to choose $t = N$. So that I get

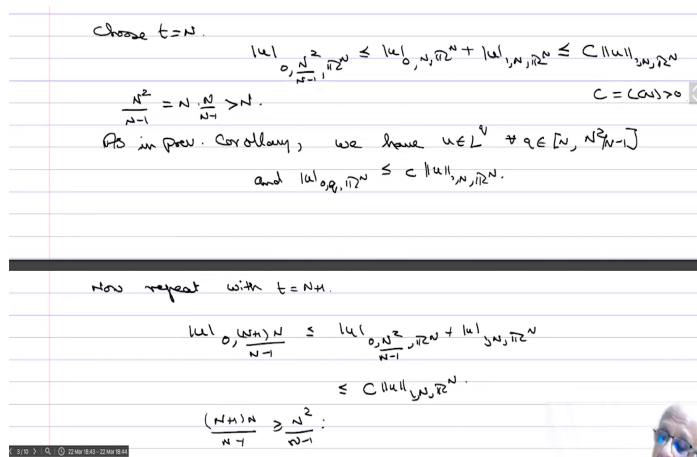
$$|u|_{0, \frac{N^2}{N-1}, \mathbb{R}^N} \leq |u|_{0, N, \mathbb{R}^N} + |u|_{1, N, \mathbb{R}^N} \leq C \|u\|_{1, N, \mathbb{R}^N}, \quad C = C(N, p) > 0.$$

And now you recall that $\frac{N^2}{N-1} > N$ and therefore, by previous corollary, we have

$$u \in L^q, \quad \forall q \in [N, \frac{N^2}{N-1}].$$

And now $|u|_{0, q, \mathbb{R}^N} \leq C \|u\|_{1, N, \mathbb{R}^N}$.

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Choose $t=N$.

$$|u|_{0, \frac{N^2}{N-1}, \mathbb{R}^N} \leq |u|_{0, \mathbb{R}^N} + |u|_{1, \mathbb{R}^N} \leq C \|u\|_{1, \mathbb{R}^N}$$

$$\frac{N^2}{N-1} = N \cdot \frac{N}{N-1} > N.$$


As in prev. Corollary, we have $u \in L^q$ $\forall q \in [N, \frac{N^2}{N-1}]$

and $|u|_{0, \mathbb{R}^N} \leq C \|u\|_{1, \mathbb{R}^N}$.

Now repeat with $t=N+1$.

$$|u|_{0, \frac{(N+1)N}{N-1}, \mathbb{R}^N} \leq |u|_{0, \frac{N^2}{N-1}, \mathbb{R}^N} + |u|_{1, \mathbb{R}^N}$$

$$\leq C \|u\|_{1, \mathbb{R}^N}.$$

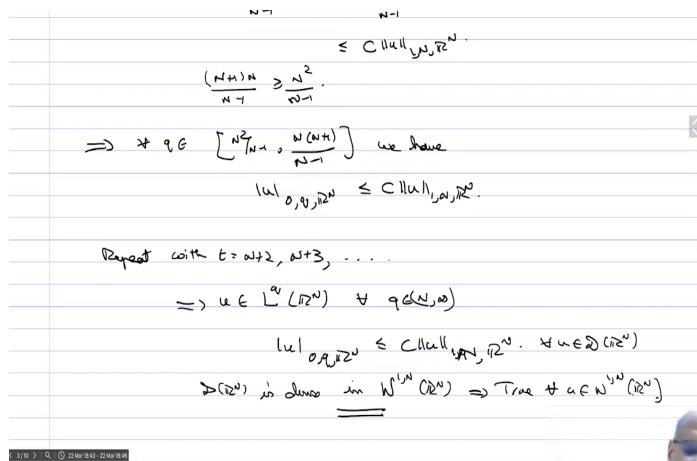
$$\frac{(N+1)N}{N-1} \geq \frac{N^2}{N-1}.$$


So, the see bootstrapping trick, which now repeats with $t=N+1$. So, what do you get

$$|u|_{0, \frac{(N+1)N}{N-1}, \mathbb{R}^N} \leq |u|_{0, \frac{N^2}{N-1}, \mathbb{R}^N} + |u|_{1, \mathbb{R}^N} \leq C \|u\|_{1, \mathbb{R}^N}.$$

and then recall that $\frac{(N+1)N}{N-1} \geq \frac{N^2}{N-1}$.

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$$\frac{(N+1)N}{N-1} \geq \frac{N^2}{N-1}.$$

$$\Rightarrow \forall q \in \left[\frac{N^2}{N-1}, \frac{(N+1)N}{N-1} \right] \text{ we have}$$


$$|u|_{0, q, \mathbb{R}^N} \leq C \|u\|_{1, \mathbb{R}^N}.$$

Repeat with $t=N+2, N+3, \dots$

$$\Rightarrow u \in L^q(\mathbb{R}^N) \quad \forall q \in [N, \infty)$$

$$|u|_{0, q, \mathbb{R}^N} \leq C \|u\|_{1, \mathbb{R}^N}, \quad \forall q \in [N, \infty)$$

$\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N) \Rightarrow \text{True } \forall u \in W^{1,p}(\mathbb{R}^N)$



And so, again we use the previous corollary, therefore, this implies that

$\forall q \in \left[\frac{N^2}{N-1}, \frac{(N+1)N}{N-1} \right]$, we have

$$|u|_{0,q,\mathbb{R}^N} \leq C \|u\|_{1,N,\mathbb{R}^N}$$

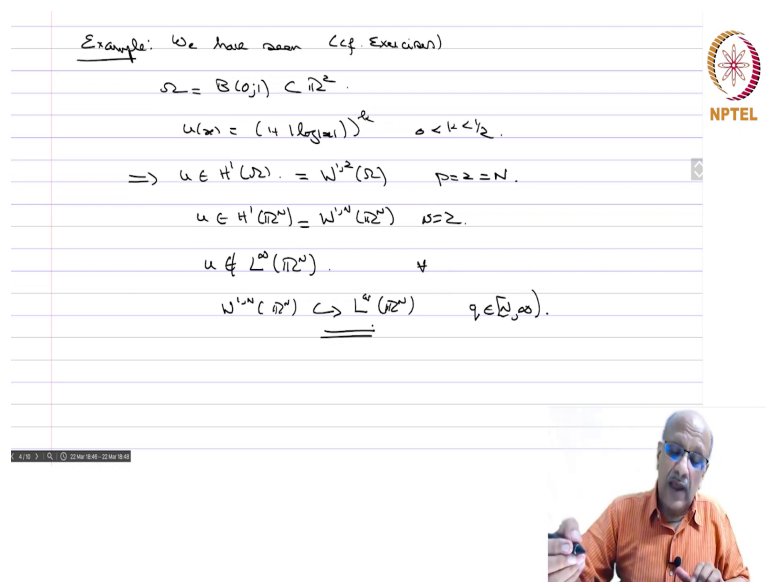
So, now repeat with $t = N + 2, N + 3, \dots$ and so on. And so, each time you will increase a certain interval and it will go on forward so that you get all so, this implies that

$u \in L^q(\mathbb{R}^N)$, $\forall q \in (N, \infty)$ and we have

$$|u|_{0,q,\mathbb{R}^N} \leq C \|u\|_{1,N,\mathbb{R}^N}, \quad \forall u \in D(\mathbb{R}^N)$$

and then $D(\mathbb{R}^N)$ is dense in $W^{1,N}(\mathbb{R}^N)$ and this implies true for all $u \in W^{1,N}(\mathbb{R}^N)$. So, that proves the theorem, when $p = N$.

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Example: We have seen (cf. Exercises)

$$\Omega = B(0,1) \subset \mathbb{R}^2.$$

$$u(x) = (1 + |\log |x||)^k, \quad 0 < k < \frac{1}{2}.$$

$$\Rightarrow u \in H^1(\Omega) = W^{1,2}(\Omega), \quad p=2=N.$$

$$u \in H^1(\mathbb{R}^2) = W^{1,2}(\mathbb{R}^2), \quad p=2.$$

$$u \notin L^\infty(\mathbb{R}^2).$$

$$W^{1,q}(\mathbb{R}^2) \subset L^q(\mathbb{R}^2), \quad q \in [2, \infty).$$

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So, when p and then as I said for all subdomains you just have to use the prolongation operator. So, example we have seen, Cf exercises;

Example: we have seen in the last set of exercises that we did, $\Omega = B(0; 1) \subset \mathbb{R}^2$ and

$$u(x) = (1 + |\log |x||)^k, \quad 0 < k < \frac{1}{2}.$$

Then we saw that $u \in H^1(\Omega) = W^{1,2}(\Omega)$, $p = 2 = N$. So, you have

$u \in H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N)$, $N = 2$ and $u \in L^\infty(\mathbb{R}^N)$, because it blows up near the origin face unbounded function and therefore, you do not have that it is in L^∞ .

So, the result is sharp, you can only get from for all, so you only have

$$W^{1,N}(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N) , \quad \forall q \in [N, \infty)$$

So that proves that, so our next thing is to look at $p > N$, which we will do next time.