Sobolev Spaces and Partial Differential Equations
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Imbedding Theorems: Case p Less Than N - Part 1

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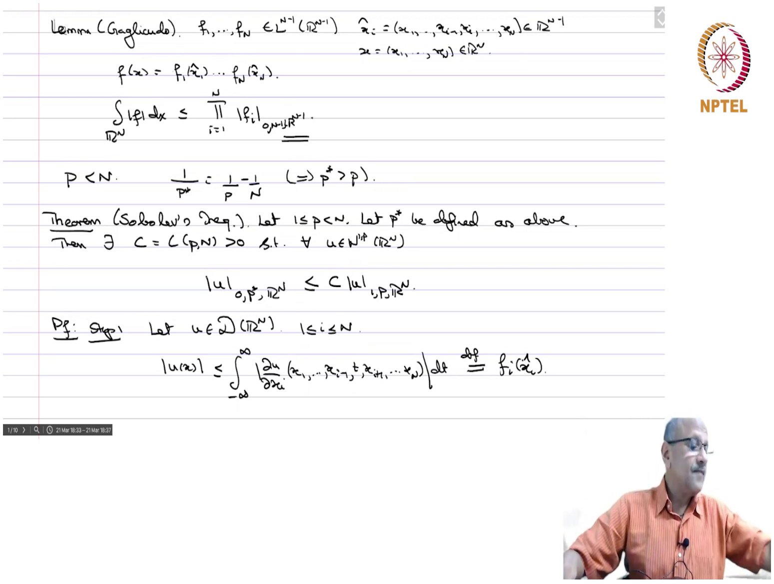

We are discussing Embedding theorems and yesterday we, let me recall the lemma of Gagliardo.

Lemma: Let $N \geq 2$, f_1, ..., $f_N \in L^{N-1}(\mathbb{R}^{N-1})$, $x \in \mathbb{R}^N$, $define$

$$1 \leq i \leq N,\ \hat{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \in \mathbb{R}^{N-1}.$$

$$f(x) = f_1(\hat{x}_1) .... f_N(\hat{x}_N).$$

Then $\int_{\mathbb{R}^N} |f| dx \leq \Pi_{i=1}^{N} |f_i|_{0, N-1, \mathbb{R}^{N-1}}$.

So, this was Gagliardo's lemma, we checked it in case 2 equal to 2 and 3, 2 was just separation of variables, 3 was Cauchy Schwarz inequality and the general case follows by induction on n and using Holder inequality, instead of Cauchy Schwarz inequality.

Now, we are going to discuss the case $p < N$. So, we want to define

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$$

So, this implies that of course, that $p^* > p$. So, now we have the important theorem , which is called Sobolev's inequality.

Theorem: Let $1 \le p < N$ *and* p^* *be as defined above*. Then there exists a constant

$C = C(p, N) > 0$ *s.t.* $\forall u \in W^{1,p}(\mathbb{R}^N)$,

$$|u|_{0,p^*,\mathbb{R}^N} \le C|u|_{1,p,\mathbb{R}^N} \quad \text{------(*)}$$

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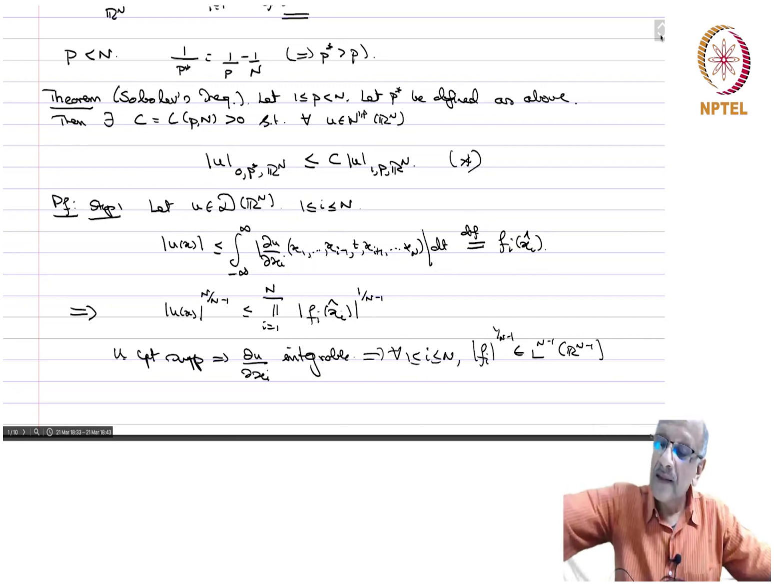

proof. *Step 1*. So, let $u \in D(\mathbb{R}^N)$, $1 \le i \le N$. so, you can write

$$|u(x)| \le \int_{-\infty}^{\infty} |\frac{\partial u}{\partial x_i}(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_N)|dt := f_i(\hat{x}_i).$$

$$\Rightarrow |u(x)|^{\frac{N}{N-1}} \le \Pi_{i=1}^{N} |f_i(\hat{x}_i)|^{\frac{1}{N-1}}.$$

u cpt support $\Rightarrow \frac{\partial u}{\partial x_i}$ is integrable $\Rightarrow \forall 1 \le i \le N$, $|f_i|^{\frac{1}{N-1}} \in L^{N-1}(\mathbb{R}^{N-1})$.

So, therefore, you have this. So, we can apply Gagliardo's lemma immediately.

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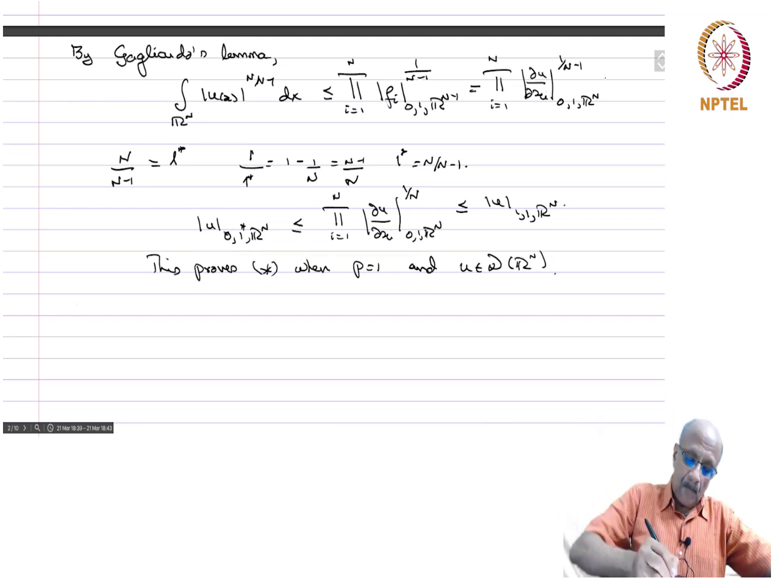

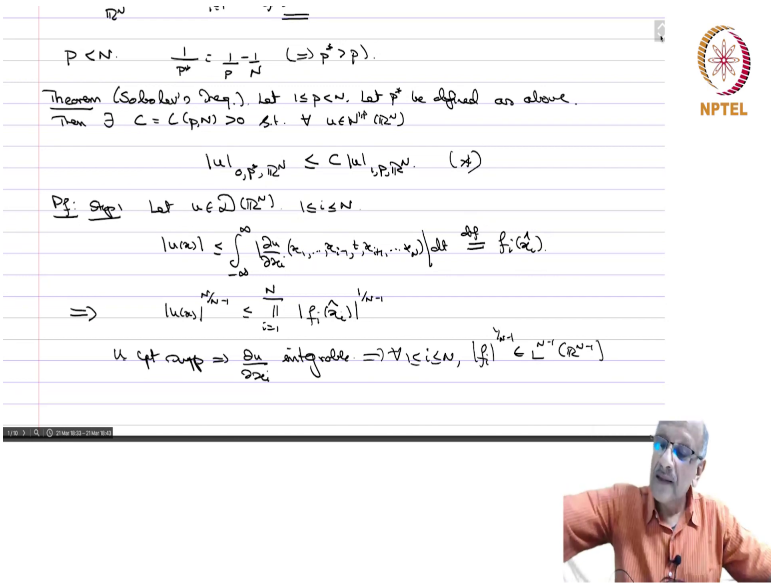

So, by Gagliardo, we get the

$$\int_{\mathbb{R}^N} |u(x)|^{\frac{N}{N-1}} dx \leq \Pi_{i=1}^{N} |f_i|^{\frac{1}{N-1}}_{0,1,\mathbb{R}^{N-1}} = \Pi_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{1}{N-1}}_{0,1,\mathbb{R}^N} .$$

Now, if you look at $\frac{N}{N-1}$, this is nothing but 1 star because what is 1, 1 by 1 star equals 1 1 by p minus 1 by n. So, this is equal to n minus 1 by n and therefore, 1 star equals n by n minus 1. So, whatever you have here you are nothing so, this is nothing but mod u of 0, 1 star, Rn. And it is so, I must take it to the power of n minus 1 by n on both sides. So, I will get here

$$|u|_{0,1^*,\mathbb{R}^N} \leq \Pi_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{1}{N}}_{0,1,\mathbb{R}^N} \leq |u|_{1,1,\mathbb{R}^N} .$$

So, this proves (*) when p =1.

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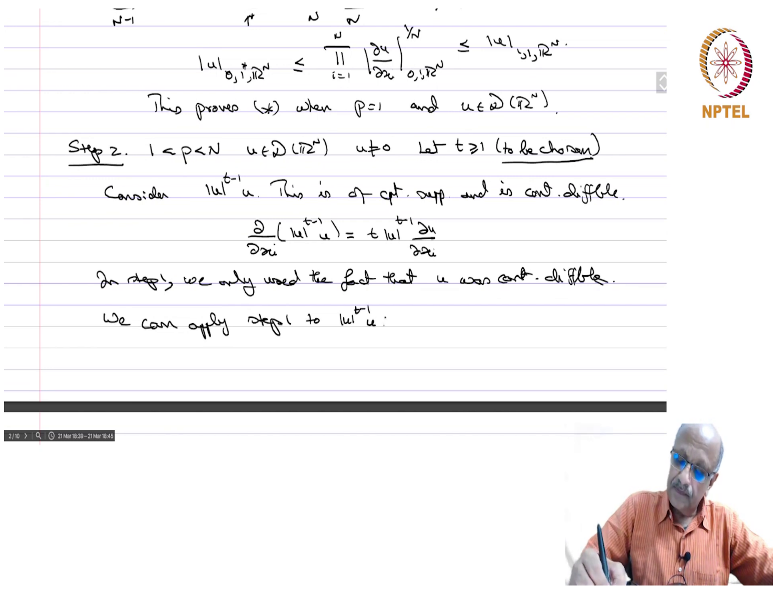

So, *step 2.* So, now we assume that $1 < p < N$ and let $u \in D(\mathbb{R}^N)$, $u \neq 0$, because if u is 0 the inequality is trivial. So, let $t \geq 1$ to be chosen, we are not choosing it yet, we will choose it in, in the meantime. So, consider mod u to the t minus 1 u.

So, this is of compact support, no problem because u itself has compact support and is continuously differentiable because you are taking t power t is bigger than 1, greater than equal to 1 and you are multiplying mod u to the depends on you, so this such a function is always continuously differentiable, we know that.

So, and in fact, you can write du by dxi of mod u to the t minus 1 u is nothing but t times mod u to the t minus 1 du by dxi. So, this is simple elementary calculus which you can check for yourself. If you take fx going to mod x power t minus 1x, that certainly is the differentiable function and this will give you the derivative.

So, in step 1, we only used the fact that u was continuously differentiable. We did not use any higher derivative, though it is differentiable. We did not use any of the higher derivatives and so on. So, we can still apply, so we can apply step 1 to mod u to the t minus 1 u. So, we apply that to the function so, what, what is mod u? 1 star is n by n minus 1, so if you are going to apply to this function, so when you take modulus of this function, you get mod u power t and then you are going to raise it to the power of n by n minus 1.

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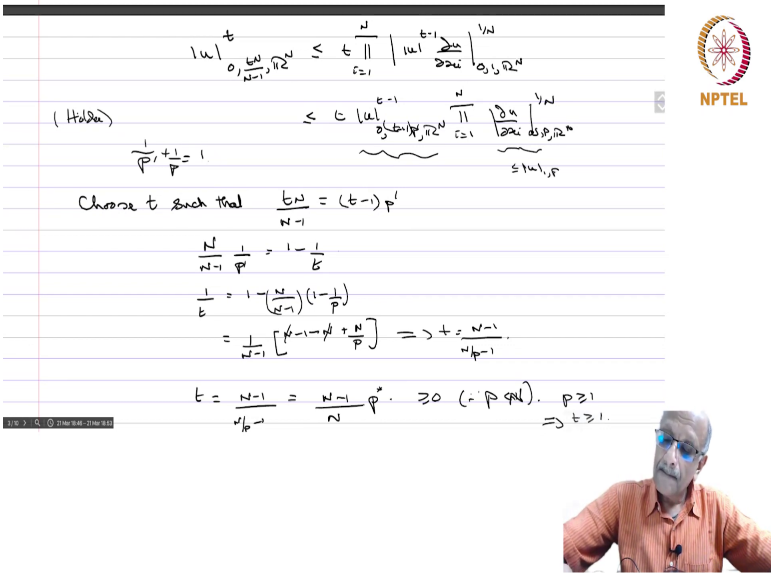

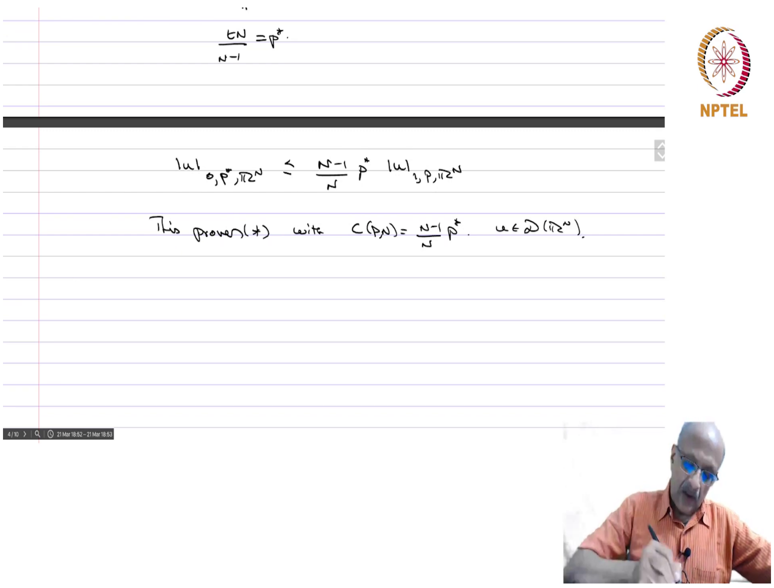

So, $|u|^{t}_{0,\frac{tN}{N-1},\mathbb{R}^N} \leq t\Pi_{i=1}^{N}||u|^{t-1}\frac{\partial u}{\partial x_i}|^{\frac{1}{N}}_{0,1,\mathbb{R}^N} \leq t|u|^{t-1}_{0,(t+1)p*,\mathbb{R}^N}\Pi_{i=1}^{N}|\frac{\partial u}{\partial x_i}|^{\frac{1}{N}}_{0,1,\mathbb{R}^N}$.

So, this is the Holder inequality which we have used.

So, now, we choose t, so choose t such that $\frac{tN}{N-1} = (t-1)p'$.

$$\frac{N}{N-1}\frac{1}{p'} = 1 - \frac{1}{t} \Rightarrow t = \frac{N-1}{\frac{N}{p}-1} \geq 0 \quad \Rightarrow t \geq 1 \, as \, p \geq 1.$$

$$p* = \frac{tN}{N-1}.$$

So, therefore, $|u|_{0,p^*,\mathbb{R}^N} \leq \frac{N}{N-1} p^* |u|_{1,p,\mathbb{R}^N}$. So, this proves (*) this with $C(N,p) = \frac{N-1}{N} p^*$, $u \in D(\mathbb{R}^N)$.

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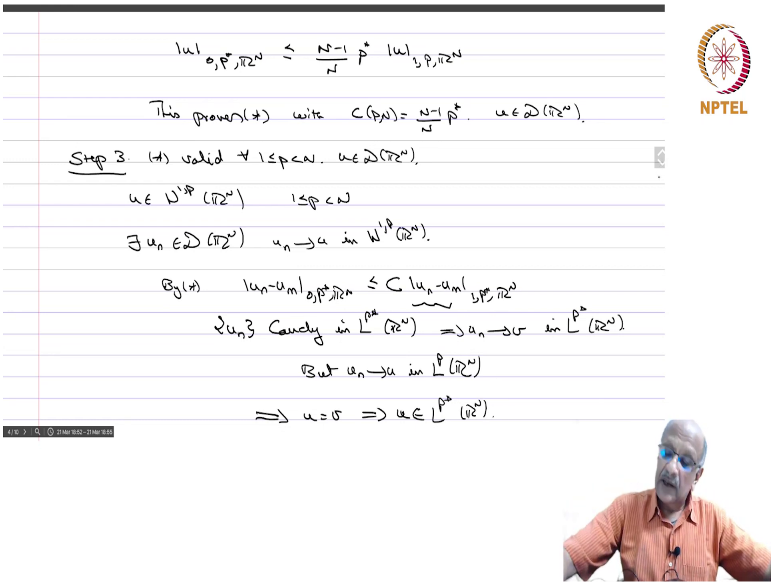

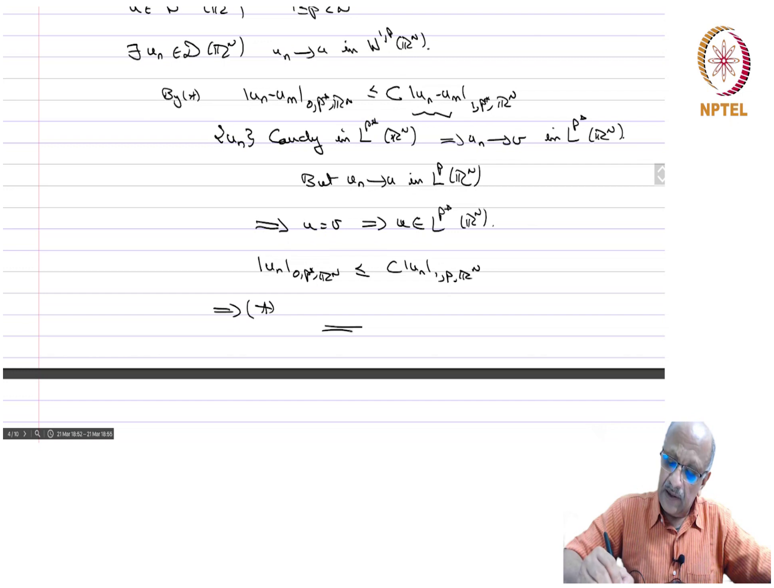

Step 3. (*) is valid for all $1 \leq p < N$, $u \in D(\mathbb{R}^N)$. So, if $u \in W^{1,p}(\mathbb{R}^N)$, there exists $u_n \in D(\mathbb{R}^N)$ converging to u in $W^{1,p}(\mathbb{R}^N)$ norm. So, by (*), what do you get,

$$|u_n - u_m|_{0,p^*,\mathbb{R}^N} \leq C|u_n - u_m|_{1,p^*,\mathbb{R}^N}.$$

So, this is Cauchy and therefore, this is also Cauchy. So, this implies that un Cauchy in Lp star Rn implies some $u_n \to v \; in \; L^{p^*}(\mathbb{R}^N)$, but $u_n \to u \; in \; L^{p}(\mathbb{R}^N)$.

So, in both of these you will have a sub sequence which goes pointwise almost everywhere and so, that should imply that

$$u = v \Rightarrow u \in L^{p^*}(\mathbb{R}^N) \Rightarrow |u_n|_{0,p^*,\mathbb{R}^N} \leq C|u_n|_{1,p,\mathbb{R}^N}$$

and therefore, you get star implies star by passing to the limit as n tends to infinity. So, this proves the Sobolev imbedding theorem completely.

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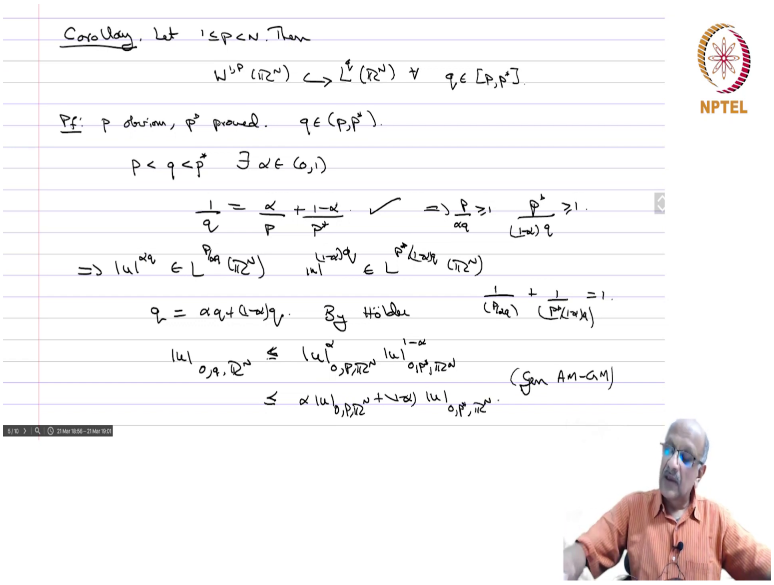

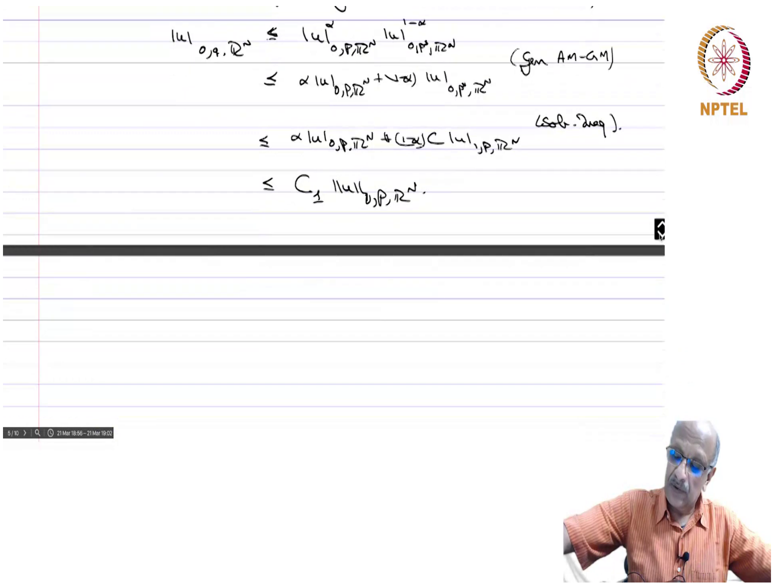

So, now, we will see some corollaries of this thing, corollary.

corollary. So, let $1 \leq p < N$. Then $W^{1,p}(\mathbb{R}^N) \to L^q(\mathbb{R}^N),\ for\, q \in [p, p^*]$

proof: p obvious, p* is proved in the Sobolev imbedding theorem. So, we only want to look for $q \in (p, p^*)$. Then there exists $\alpha \in (0, 1)$ (because then 1 by p star is less than 1 by q less than 1 by p. So, it can be written as a convex combination) such that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}$$

So, then what does this imply? This implies that $|u|^{\alpha q} \in L^{\frac{p}{\alpha q}}(\mathbb{R}^N)$. So, this implies that p by alpha q is greater than equal to 1, p star by 1 minus alpha q is also greater than equal to 1.

Similarly, mod u to the 1 minus alpha p star, 1 minus alpha q belongs to Lp star by 1 minus alpha q of Rn for the same reason, because u is an Lp star we know that and therefore, if I take p star by 1 minus alpha q, it belongs to Lp star of Rn. So, and you also have q is equal to alpha q plus 1 minus alpha q.

And so, by Holder, we get that

$$|u|_{0,q,\mathbb{R}^N} \leq |u|^{\alpha}_{0,p,\mathbb{R}^N}|u|^{1-\alpha}_{0,p^*,\mathbb{R}^N} \leq \alpha|u|_{0,p,\mathbb{R}^N} + (1 - \alpha)|u|_{0,p^*,\mathbb{R}^N} \text{ (Gen. AM-GM inequality)}$$

$$\leq \alpha|u|_{0,p,\mathbb{R}^N} + C(1 - \alpha)|u|_{1,p,\mathbb{R}^N} \text{ (Sobolev Ineq.)}$$

$$\leq C_1||u||_{1,p,\mathbb{R}^N}$$

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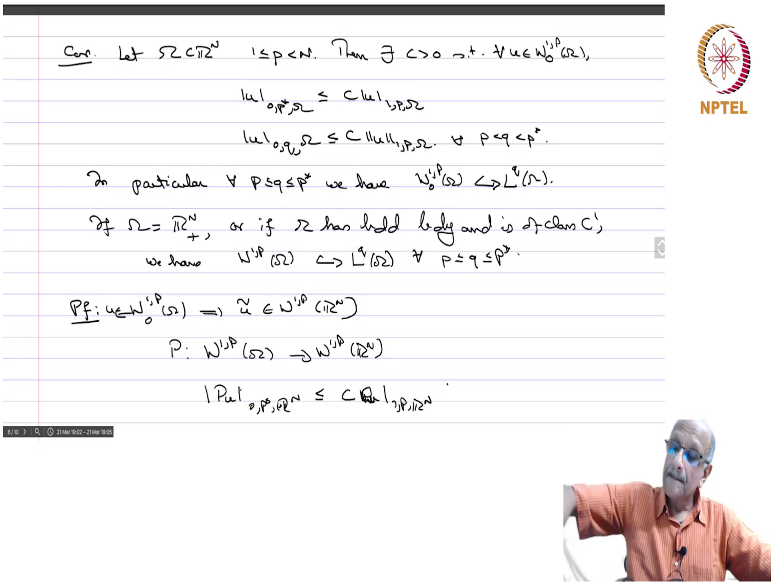

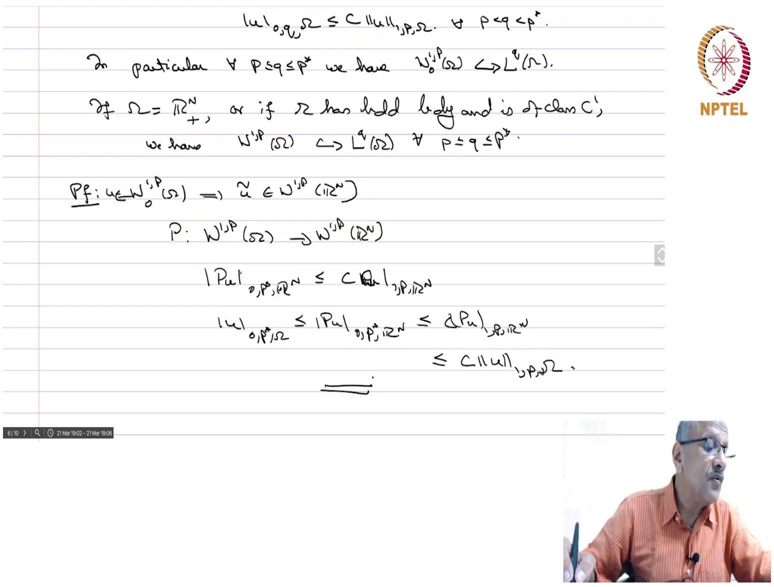

corollary. Let $\Omega \subset \mathbb{R}^N$ and $1 \le p < N$. Then there exists a C>0, such that, for all $u \in W^{1,p}(\Omega)$, you have

$$|u|_{0,p^*,\Omega} \le C|u|_{1,p,\Omega}$$

$$|u|_{0,q,\Omega} \le C||u||_{1,p,\Omega}\ ,\ \forall q \in (p, p\ *)\ .$$

In particular, for $q \in [p, p\ *]$, *we have* $W_0^{1,p}(\Omega) \to L^q(\Omega)$. If $\Omega = \mathbb{R}^N_+$ *or* if Ω has bounded boundary and of class C^1, then $W^{1,p}(\Omega) \to L^q(\Omega)$, $\forall\ q \in [p, p\ *]$.

proof. If $u \in W_0^{1,p}(\Omega)$, then you have $\tilde{u} \in W_0^{1,p}(\mathbb{R}^N)$. So, for u tilde you write down we have these two inequalities and then u tilde nothing happens outside omega and therefore, you have the inequalities for omega itself, so, that gives you the proof.

In case of omega equals Rn plus or if it has bounded boundary and so on then there exists a prolongation operator $P: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$. So, you apply the theorem

$$|Pu|_{0,p^*,\mathbb{R}^N} \leq C|u|_{1,p,\mathbb{R}^N}$$

$$|u|_{0,p^*,\Omega} \leq |Pu|_{0,p^*,\mathbb{R}^N} \leq C|Pu|_{1,p,\mathbb{R}^N} \leq C||u||_{1,p,\Omega} \, .$$