Sobolev Spaces and Partial Differential Equations
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Imbedding Theorems

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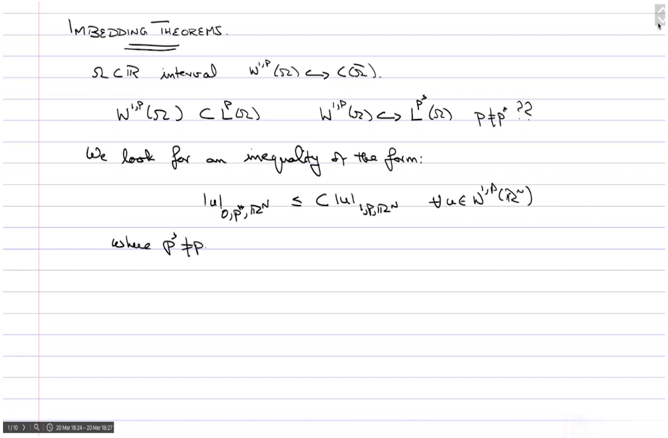

We will now discuss **Embedding theorems.**

We have already seen that if $\Omega \subset \mathbb{R}$ say an interval, then $W^{1,p}(\Omega) \to C(\overline{\Omega})$. In fact, it is an absolutely continuous function that is what we saw. So, we would like to generalize this, see if you have, if you have information on $W^{1,p}(\Omega)$ in some subspace.

Of course, we already have $W^{1,p}(\Omega) \subset L^p(\Omega)$, that is the, from the definition. Now, we want to know if you have because of the extra information we have on the derivatives namely that the distribution derivatives are also $L^p(\Omega)$, this is an important piece of information, does that tell you more about the function.

So, in particular, we would like to know if $W^{1,p}(\Omega)$ is contained in some space of continuous functions, differentiable functions or so on or at least in some other $L^p(\Omega)$ spaces. So, $L^p(\Omega)$ for instance, do you have W 1p omega continuously embedded in some other Lp space, some $L^{p^*}(\Omega)$, where p is not equal to p*, so, we want to know $W^{1,p}(\Omega)$, is it embedded in $L^{p^*}(\Omega)$ of omega, p not equal to p*, if such a thing is possible.

So, we would like to answer such questions and so, in particular, we looked for an inequality of the form

$$|u|_{0,p^*,\,\mathbb{R}^N} \leq |u|_{1,p,\,\mathbb{R}^N}\ , \qquad \forall u \in W^{1,p}(\mathbb{R}^N)\,.$$

So, this will mean that $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$ where p* star is not equal to p.

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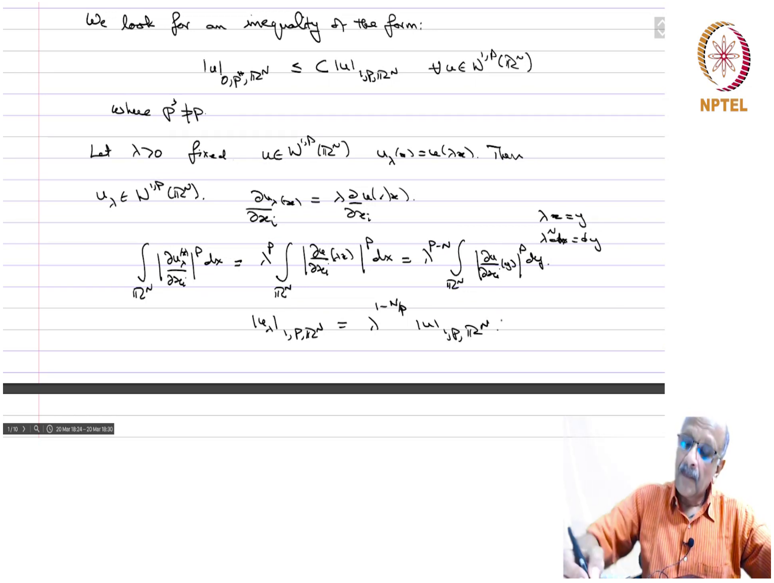

So, first of all we want to know when you can prove such a result and the proof may just look like magic or just manipulating p and L various p and then all these quantities, but as a simple analysis we will tell you where to look and why we should look there. So, let us assume such an inequality is true for all u in $W^{1,p}(\mathbb{R}^N)$.

And let lambda so, let $\lambda > 0$, $u \in W^{1,p}(\mathbb{R}^N)$. Define $u_\lambda(x) = u(\lambda x)$. Thus $u_\lambda \in W^{1,p}(\mathbb{R}^N)$, $\frac{\partial u_\lambda}{\partial x_i}(x) = \lambda \frac{\partial u}{\partial x_i}(\lambda x)$.

$$\int_{\mathbb{R}^N} |\frac{\partial u_\lambda}{\partial x_i}(x)|^p dx = \lambda^p \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_i}(\lambda x)|^p dx = \lambda^{p-N} \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_i}(y)|^p dy$$

$$\Rightarrow\ |u_\lambda|_{1,p,\mathbb{R}^N} = \lambda^{1-\frac{N}{p}} |u|_{1,p,\mathbb{R}^N}\ .$$

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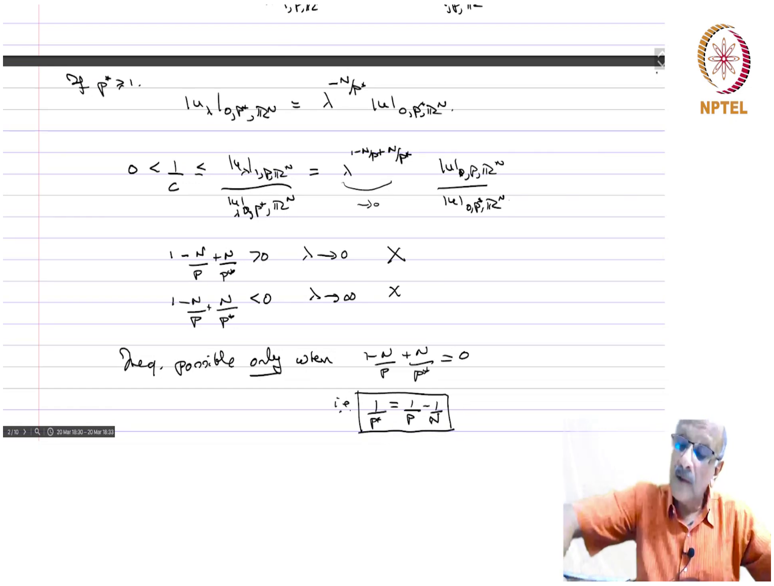

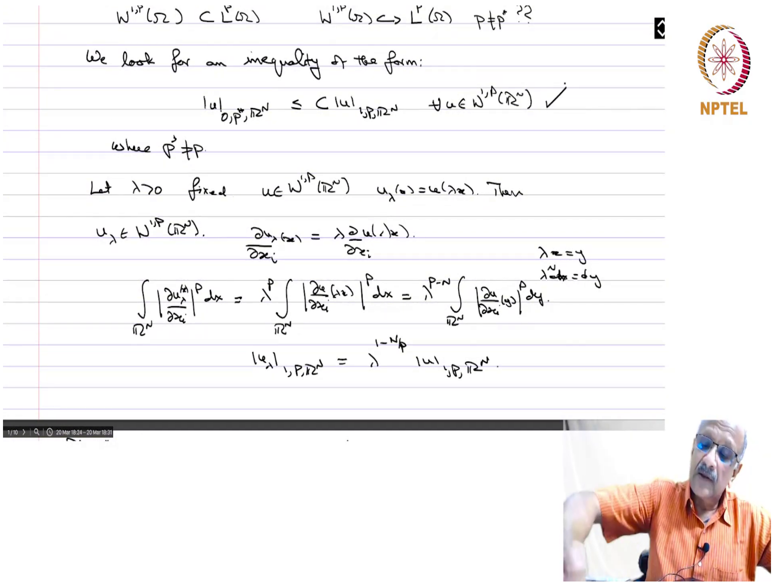

Now, in the same way we can look for some other p* so, if $p^* \geq 1$,

$$|u_\lambda|_{0,p^*,\mathbb{R}^N} = \lambda^{\frac{N}{p^*}} |u|_{0,p^*,\mathbb{R}^N} .$$

So, now, for all functions we are looking at the inequality of this form, thus we are looking at this inequality for all functions. So, this has to be satisfied by all the functions u, u lambda etcetera. So,

$$0 < \frac{1}{c} \leq \frac{|u_\lambda|_{1,p,\mathbb{R}^N}}{|u_\lambda|_{0,p^*,\mathbb{R}^N}} = \lambda^{1-\frac{N}{p}+\frac{N}{p^*}} .$$

Now, let us look at this number $1 - \frac{N}{p} + \frac{N}{p^*}$. Suppose this number is positive.

Now, if this number is positive, then, then you let $\lambda \to 0$, so, then this will go to 0 and therefore, you will not be able to get this inequality, it will violate this inequality because it is bigger than equal to 1 by C which is strictly positive.

Now, if so, you let lambda go to 0, so you to get a contradiction, if 1 minus n by p plus n by p star is less than 0, then you let lambda tend to infinity then again, this quantity will go to 0 and you will once more get a contradiction. Therefore, inequality is possible only when 1 minus n by p plus n by p star equal to 0 that is 1 by p star equal to 1 by p minus 1 by n. So, this is the defining relationship for p*.

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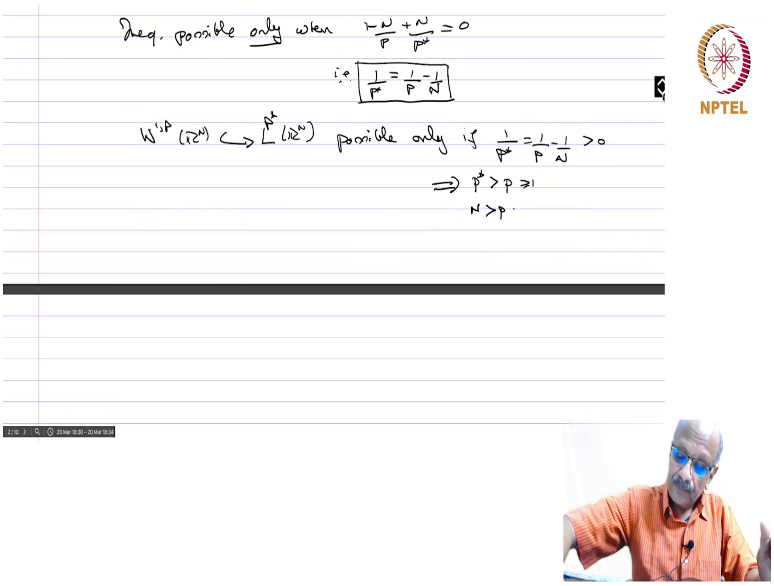

So, if at all you get $W^{1,p}(\mathbb{R}^N)$ if you want to show that it is in $L^{p^*}(\mathbb{R}^N)$ possible only if

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} > 0 \ \Rightarrow p^* > p,\ N > p .$$

So, we will therefore, restrict our attention to three cases.

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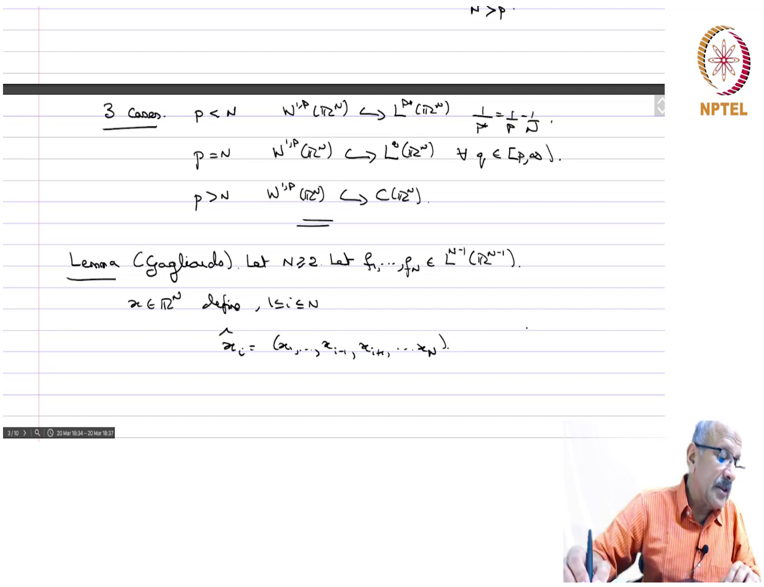

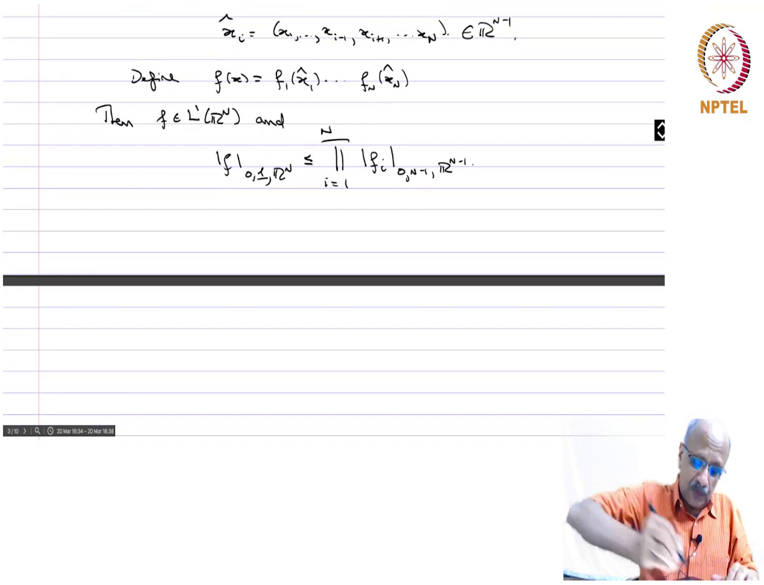

So, **in three cases:**

$$p < N, \quad W^{1,p}(\mathbb{R}^N) \to L^{p^*}(\mathbb{R}^N), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$$

$$p = N, \quad W^{1,p}(\mathbb{R}^N) \to L^{q}(\mathbb{R}^N), \qquad q \in [p, \infty).$$

$$p > N, \quad W^{1,p}(\mathbb{R}^N) \to C(\mathbb{R}^N).$$

So, these are the three kinds of theorems which we will use. So, we will first start with p less than n then we will do, from there deduce p equal to n and p bigger than n will require different arguments similar to. So, in the one-dimensional case you have p greater than or equal to 1 and therefore, you have p greater than equal to n and therefore, you had these two results, the second and third were valid.

So, now, before you go on, we need a technical lemma due to Gagliardo.

Lemma : Let $N \geq 2$, f_1, ..., $f_N \in L^{N-1}(\mathbb{R}^{N-1})$, $x \in \mathbb{R}^N$, $define$

$$1 \leq i \leq N,\ \hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}.$$

$$f(x) = f_1(\hat{x}_1) \ldots f_N(\hat{x}_N).$$

Then $f \in L^1(\mathbb{R}^N)$ and $|f|_{0,1,\mathbb{R}^N} \leq \Pi_{i=1}^{N} |f_i|_{0,N-1,\mathbb{R}^{N-1}}$.

So, this is the lemma of Gagliardo and which we will be using in the proof of the theorem.

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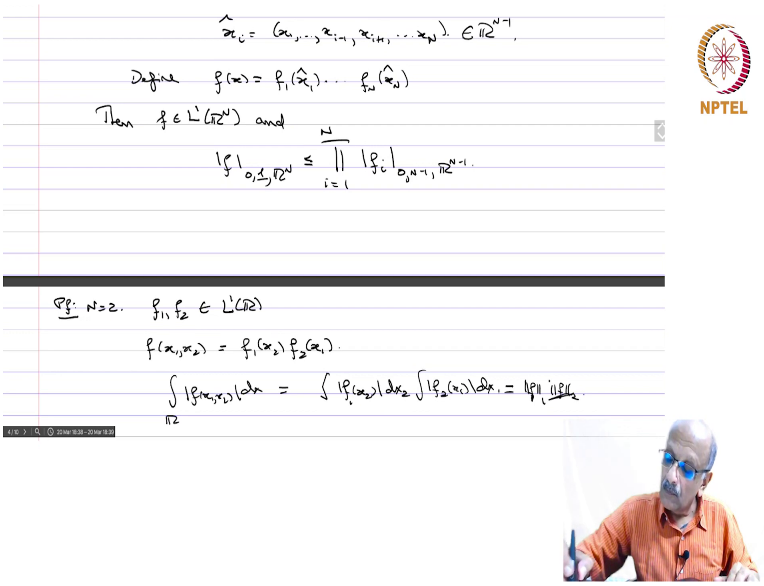

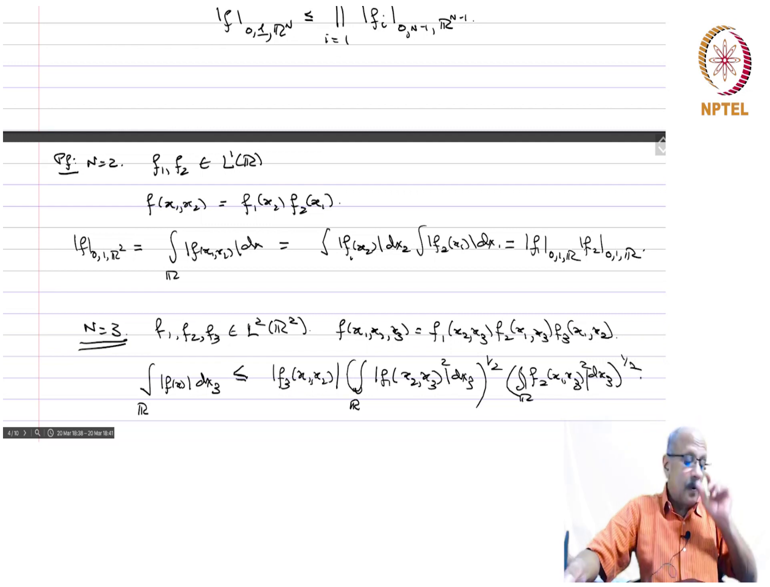

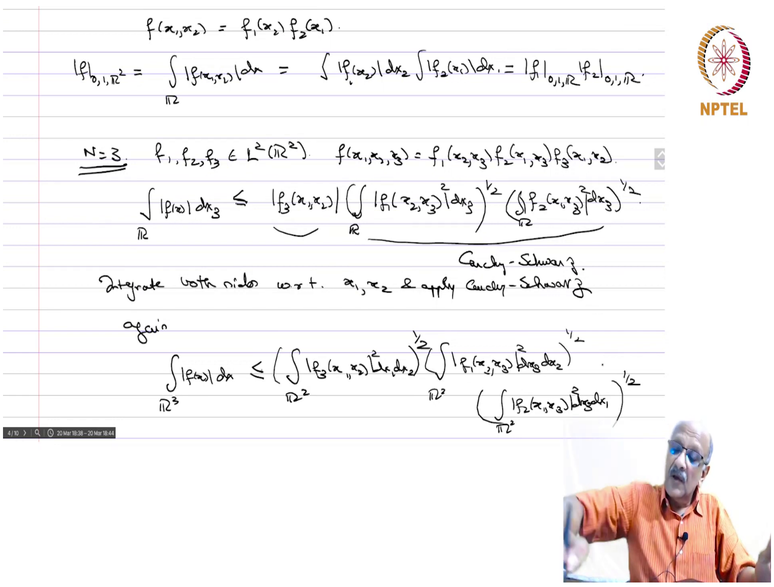

proof. Let us take $N = 2$, $f_1, f_2 \in L^1(\mathbb{R})$, $f(x_1, x_2) = f_1(x_2) f_2(x_1)$

$$|f|_{0,1,\mathbb{R}^2} = \int_{\mathbb{R}} |f(x_1, x_2)| dx = \int |f_1(x_2)| dx_2 \int |f_2(x_1)| dx_1 = |f_1|_{0,1,\mathbb{R}} \ |f_2|_{0,1,\mathbb{R}} .$$

Let us look at N equals 3. So, you have

$$f_1, f_2, f_3 \in L^2(\mathbb{R}^2), \ f(x_1, x_2, x_3) = f_1(x_2, x_3) f_2(x_1, x_3) f_3(x_1, x_2) .$$

$$\int_{\mathbb{R}} |f(x)| dx_3 \le |f_3(x_1, x_2)| (\int_{\mathbb{R}} |f_1(x_2, x_3)|^2 dx_3)^{\frac{1}{2}} (\int_{\mathbb{R}} |f_2(x_1, x_3)|^2 dx_3)^{\frac{1}{2}}$$

Now, we again integrate, integrate both sides with respect to x1 x2 and apply Cauchy Schwarz again. So, you will get

$$\int_{\mathbb{R}} |f(x)| dx_3 \le (\int_{\mathbb{R}} |f_3(x_1, x_2)|^2 dx_1 dx_2)^{\frac{1}{2}} (\int_{\mathbb{R}} |f_1(x_2, x_3)|^2 dx_3 dx_2)^{\frac{1}{2}} (\int_{\mathbb{R}} |f_2(x_1, x_3)|^2 dx_3 dx_1)^{\frac{1}{2}}$$

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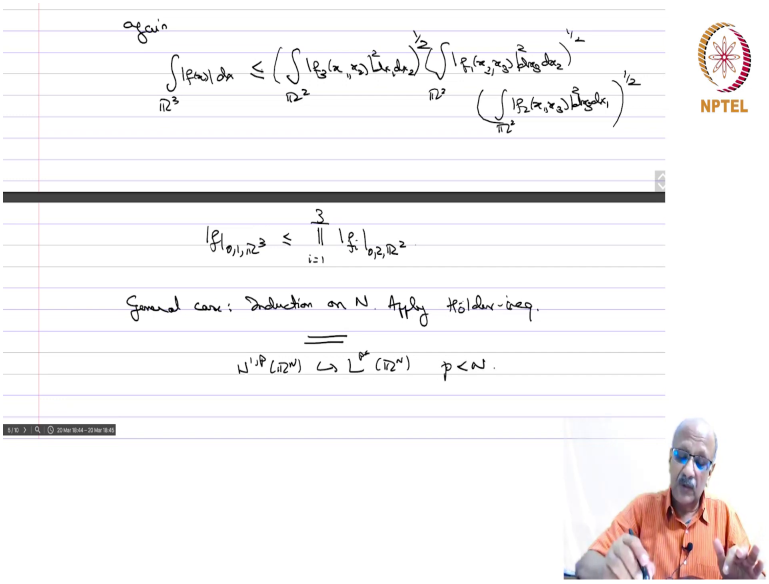

And that tells you that $|f|_{0,1,\mathbb{R}^3} \le \Pi_{i=1}^{3} |f_i|_{0,2,\mathbb{R}^2}$

So, in the general case I will skip the proof you can find it in the book. So, in the general case, induction on n we have proved it for n equals 2, n equals 3, you assume for n and apply Holder inequality in place of, in place of the Cauchy Schwarz which have been applied. So, this is just a technicality.

So, we will do so, this is the Gagliardo lemma which we will crucially use in the proof of the first theorem namely, next theorem which we are going to prove is $W^{1,p}(\mathbb{R}^N)$is contained in $L^{p^*}(\mathbb{R}^N)$, when $p < N$. So, this is a theorem which we next have to prove and we are going to use Gagliardo's lemma for that. We will do that next time.