

Sobolev Spaces and Partial Differential Equations
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Exercises - Part 4

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EXERCISES.

① Let $\Omega = B(0;1) \subset \mathbb{R}^2$. Define $u(x) = (1 + |\log|x||)^k$, $0 < k < \frac{1}{2}$.
 Show that $u \in H^1(\Omega)$.

Sol. $\int_{\Omega} |u|^2 dx = 2\pi \int_0^1 (1 + |\log r|)^{2k} r dr$

Repeated use of L'Hopital's rule $\Rightarrow \lim_{r \rightarrow 0} |\log r|^m r = 0$.

$\Rightarrow \int_{\Omega} |u|^2 dx < +\infty$.



Now it is time to do some exercises.

Exercises:

(1) Let $\Omega = B(0;1) \subset \mathbb{R}^2$. Define $u(x) = (1 + |\log|x||)^k$, $0 < k < \frac{1}{2}$. Show that

$$u \in H^1(\Omega).$$

$$\text{solution: } \int_{\Omega} |\nabla u|^2 dx = 2\pi \int_0^1 (1 + |\log r|)^{2k} r dr$$

Now, the repeated use of L'hospital's rule $\Rightarrow \lim_{r \rightarrow 0} |\log r|^m r = 0$.

$$\int_{\Omega} |\nabla u|^2 dx < \infty.$$

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Repeated use of L'Hopital's rule $\Rightarrow \lim_{r \rightarrow \infty} \log r = \infty$.

$\Rightarrow \int_{\Omega} |u|^2 dx < +\infty$.

Let $\phi \in D(\Omega)$. $\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = \lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_{\epsilon}} u \frac{\partial \phi}{\partial x_i} dx$.

$\int_{\Omega \setminus B_{\epsilon}} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega \setminus B_{\epsilon}} \frac{\partial u}{\partial x_i} \phi dx + \int_{|x|=\epsilon} u \phi \nu_i$.

$\int_{|x|=\epsilon} u \phi \nu_i = -\epsilon \int_{\Omega} \cos \theta \phi (1 + \log \epsilon) dx \rightarrow 0$ as $\epsilon \rightarrow 0$.

$\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega} \frac{\partial u}{\partial x_i} \phi dx$.



So, now, we have to check what the distribution derivatives are. So, here, like in the case of finding the fundamental solution of the Laplacian, we have to check if the distribution derivative is really obtained by the ordinary differentiation rules.

So, let $\phi \in D(\Omega)$. $\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = \lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_{\epsilon}} u \frac{\partial \phi}{\partial x_i} dx$.

And now, so, you have an Ω is a unit ball and the only singularity of the function is at the origin. So, I remove a ball of radius ϵ and therefore, I have a remaining, remaining place, $\Omega \setminus B_{\epsilon}$. Now, u is a L^2 function on a set of finite measure, it is integrable $d\phi$ by dx is a C^∞ function with compact support and it is bounded whatever you want.

And therefore, by the dominated convergence theorem, you will have that this is nothing but the limit ϵ tending to 0 integral on the $\Omega \setminus B_{\epsilon}$ of $u d\phi$ by dx . So, let us look at the integral on $\Omega \setminus B_{\epsilon}$ of $u d\phi$ by dx . Now, we are all in a smooth function case and therefore, we have, we can do all calculus formulae.

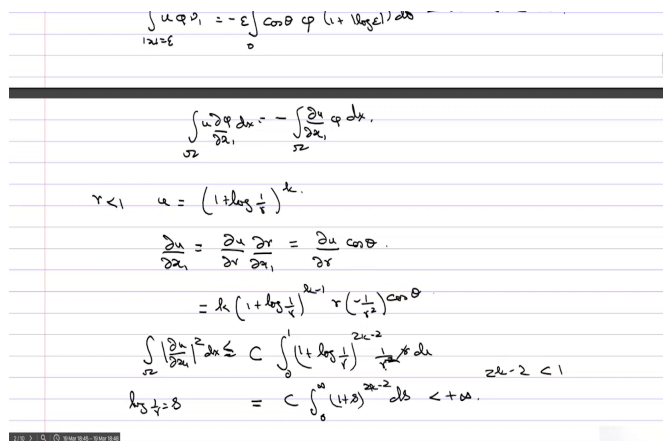
So, we use integration by parts:

$$\int_{\Omega \setminus B_{\epsilon}} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega \setminus B_{\epsilon}} \phi \frac{\partial u}{\partial x_i} dx + \int_{|x|=\epsilon} u \phi \nu_i.$$

$$\int_{|x|=\epsilon} u \phi v_1 = - \epsilon \int_0^\theta \cos \theta \phi (1 + |\log \epsilon|^k) d\theta \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

$$\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega} \phi \frac{\partial u}{\partial x_i} dx.$$

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$$\int_{|x|=\epsilon} u \phi v_1 = - \epsilon \int_0^\theta \cos \theta \phi (1 + |\log \epsilon|^k) d\theta$$

$$\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega} \phi \frac{\partial u}{\partial x_i} dx$$

$$r < 1, \quad u = \left(1 + \log \frac{1}{r}\right)^k$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x_1} = \frac{\partial u}{\partial r} \cos \theta$$

$$= k \left(1 + \log \frac{1}{r}\right)^{k-1} r \left(-\frac{1}{r^2}\right) \cos \theta$$

$$\int_{\Omega} \left|\frac{\partial u}{\partial x_1}\right|^2 dx \leq C \int_0^1 \left(1 + \log \frac{1}{r}\right)^{2k-2} \frac{1}{r} dr, \quad 2k-2 < 1$$

$$\int_{\Omega} \left|\frac{\partial u}{\partial x_1}\right|^2 dx \leq C \int_0^\infty (1+s)^{2k-2} ds < +\infty$$



So, now, $r < 1$, because $|x| < 1$. So,

$$u = \left(1 + \log \frac{1}{r}\right)^k.$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x_1} = \frac{\partial u}{\partial r} \cos \theta$$

$$= k \left(1 + \log \frac{1}{r}\right)^{k-1} r \left(-\frac{1}{r^2}\right) \cos \theta.$$

$$\int_{\Omega} \left|\frac{\partial u}{\partial x_1}\right|^2 dx \leq C \int_0^1 \left(1 + \log \frac{1}{r}\right)^{2k-2} \frac{1}{r} dr, \quad 2k-2 < 1.$$

$$= C \int_0^\infty (1+s)^{2k-2} ds < \infty, \quad \left[\log \frac{1}{r} = s\right].$$

$$\Rightarrow \frac{\partial u}{\partial x_1} \in L^2(\Omega), \text{ similarly, } \frac{\partial u}{\partial x_2} \in L^2(\Omega).$$

$$\Rightarrow u \in H^1(\Omega).$$

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$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x_1} = \frac{\partial u}{\partial r} \cos \theta \\ &= h_1 \left(1 + h_2 \frac{1}{r}\right)^{2k-1} r \left(-\frac{1}{r^2}\right) \cos \theta \\ \int_{\mathbb{R}^2} \left| \frac{\partial u}{\partial x_1} \right|^2 dx &\leq C \int_0^1 \left(1 + h_2 \frac{1}{r}\right)^{2k-2} \frac{1}{r^2} dr \\ h_2 \frac{1}{r} &= \frac{1}{r} \quad \Rightarrow \quad \int_0^1 \left(1 + \frac{1}{r}\right)^{2k-2} \frac{1}{r^2} dr < +\infty \quad 2k-2 < 1 \\ \Rightarrow \quad \frac{\partial u}{\partial x_1} &\in L^2(\Omega) \quad \text{by} \quad \frac{\partial u}{\partial x_2} \in L^2(\Omega) \\ \Rightarrow \quad u &\in H^1(\Omega).\end{aligned}$$



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So, this is just an extra straightforward exercise where we have to compute the distribution derivative and see that it is integral.

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$$\begin{aligned}(2) \quad \text{Let } \Omega &= (a, b), \quad a = x_0 < x_1 < \dots < x_n = b. \quad I_j = (x_{j-1}, x_j), \quad 1 \leq j \leq n. \\ \text{Let } u_k &\in H^1(I_k) \quad \forall 1 \leq k \leq n, \quad \text{define } u: (a, b) \rightarrow \mathbb{R} \\ u|_{I_k} &= u_k. \\ \text{Show that } u &\in H^1(\Omega) \Leftrightarrow u \in C(\bar{\Omega}). \\ \text{Soln } u &\in H^1(\Omega) \Rightarrow u \in C(\bar{\Omega}) \quad (\text{proved}). \\ \text{Let } u &\text{ defined as above belong to } C(\bar{\Omega}). \quad \text{Let } \varphi \in \mathcal{D}(\Omega). \\ \int_a^b u \varphi' dx &= \sum_{k=1}^n \int_{I_k} u \varphi' dx = - \sum_{k=1}^n \int_{I_k} u' \varphi dx \\ &\quad + \sum_{k=1}^n u(x_k) \varphi(x_k) - u(x_{k-1}) \varphi(x_{k-1}).\end{aligned}$$



(2) Let $\Omega = (a, b)$, $a = x_0 < x_1 < \dots < x_n = b$, $I_j = (x_{j-1}, x_j)$, $1 \leq j \leq n$. Let $u_k \in H^1(I_k) \forall 1 \leq k \leq n$. Define $u: (a, b) \rightarrow \mathbb{R}$, $u|_{I_k} = u_k$. Show that

$$u \in H^1(\Omega) \Leftrightarrow u \in C(\bar{\Omega}).$$

solution: if $u \in H^1(\Omega) \Rightarrow u \in C(\bar{\Omega})$ (proved).

So, now conversely let $u \in C(\overline{\Omega})$ and $\phi \in D(\Omega)$. Then

$$\begin{aligned} \int_a^b u \phi' dx &= \sum_{k=1}^n \int_{I_k} u \phi' dx = - \sum_{k=1}^n \int_{I_k} u' \phi dx \\ &\quad + \sum_{k=1}^n u(x_k) \phi(x_k) - u(x_{k-1}) \phi(x_{k-1}) \\ &= u(a) \phi(b) - u(b) \phi(b). \\ &= 0. \end{aligned}$$

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Let u defined as above belong to $C(\overline{\Omega})$. Let $\phi \in D(\Omega)$.

$$\int_a^b u \phi' dx = \sum_{k=1}^n \int_{I_k} u \phi' dx = - \sum_{k=1}^n \int_{I_k} u' \phi dx + \sum_{k=1}^n u(x_k) \phi(x_k) - u(x_{k-1}) \phi(x_{k-1}).$$

The term vanishes. $= u(a) \phi(b)$



Let u defined as above belong to $C(\bar{\Omega})$. Let $\varphi \in \mathcal{D}(\Omega)$.

$$\int_{\Omega} u \varphi' dx = \sum_{k=1}^n \int_{I_k} u \varphi' dx = - \sum_{k=1}^n \int_{I_k} u' \varphi dx + \sum_{k=1}^n u(x_k) \varphi(x_k) - u(x_{k-1}) \varphi(x_{k-1}).$$

The term vanishes. $= u(x_1) \varphi(x_1) - u(x_0) \varphi(x_0) = 0$ (since φ is compactly supported).

$$\Rightarrow u'|_{I_k} = u'_k \Rightarrow u' \in L^2(a,b) \Rightarrow u \in H^1(a,b);$$



And therefore, you have that $u'|_{I_k} = u'_k \Rightarrow u' \in L^2(a,b) \Rightarrow u \in H^1(a,b)$.

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$$\Rightarrow u'|_{I_k} = u'_k \Rightarrow u' \in L^2(a,b) \Rightarrow u \in H^1(a,b).$$

③ $\Omega \subset \mathbb{R}^N$ bounded open set. $u \in H_0^1(\Omega)$. Show that

$$\|u\|_{1,\Omega} = \|u\|_{H^1(\Omega)}.$$

Step 1: $u \in H_0^1(\Omega)$. $\frac{\partial u}{\partial x_i} = \text{sgn}(u) \frac{\partial u}{\partial x_i}$

$$\int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \int_{u \neq 0} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx$$

Since on $\{u=0\}$ $\frac{\partial u}{\partial x_i} = 0$ a.e.



(3) Let $\Omega \subset \mathbb{R}^N$ bounded open set and $u \in H_0^1(\Omega)$. Show that $\|u\|_{1,\Omega} = \|u\|_{H^1(\Omega)}$.

solution: we know that $|u| \in H_0^1(\Omega)$. $\frac{\partial |u|}{\partial x_i} = \text{sgn}(u) \frac{\partial u}{\partial x_i}$.

$$\int_{\Omega} \left| \frac{\partial |u|}{\partial x_i} \right|^2 dx = \int_{u \neq 0} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx.$$

Since on $\{u = 0\}$, $\frac{\partial u}{\partial x_i} = 0$ a.e. this also we have shown and therefore, there is the new, no new contribution from that side because this function is integrand is 0 almost everywhere. So, you can just add that and so, you will get this and that completes the proof of this.

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(4) Let $\Omega \subset \mathbb{R}^n$ bounded open set. Let $\rho \in C^1(\bar{\Omega})$.
 Show that $u \mapsto \rho u$ defines a bounded lin. op. of $H^1(\Omega)$
 into itself. If $\rho > 0$ on $\bar{\Omega}$, show that this is an isomorphism
 of $H^1(\Omega)$ onto itself.

Sol $\rho \in C^1(\bar{\Omega}) \Rightarrow$ bounded $\Rightarrow \rho u \in L^2(\Omega)$.
 Let $\phi \in D(\Omega)$. Let $\text{supp } \phi \subset \Omega' \subset \subset \Omega$.
 Let $u_n \rightarrow u$, $u_n \in D(\mathbb{R}^n)$ as in Friedrich's theorem.

$$\begin{aligned} \int_{\Omega} \rho u_m \frac{\partial \phi}{\partial x_i} dx &= \int_{\Omega'} \rho \frac{\partial}{\partial x_i} (u_m \phi) dx - \int_{\Omega'} \rho \frac{\partial u_m}{\partial x_i} \phi dx \\ &= - \int_{\Omega'} \frac{\partial \rho}{\partial x_i} u_m \phi dx - \int_{\Omega'} \rho \frac{\partial u_m}{\partial x_i} \phi dx \end{aligned}$$



(4) Let $\Omega \subset \mathbb{R}^N$ bounded open set and $\rho \in C^1(\Omega)$. Show that $u \rightarrow \rho u$ defines a bounded linear operator on $H^1(\Omega)$ into itself. If $\rho > 0$ on $\bar{\Omega}$, show that this is an isomorphism of $H^1(\Omega)$ onto itself.

solution: so, $\rho \in C^1(\Omega) \Rightarrow$ bounded $\Rightarrow \rho u \in L^1(\Omega)$. Let $\phi \in D(\Omega)$. Let

$\text{supp}(\phi) \subset \Omega' \subset \subset \Omega$. Let $u_n \rightarrow u$, $u_n \in D(\mathbb{R}^N)$ as in Friedrich's theorem.

$$\begin{aligned} \text{So, } \int_{\Omega} \rho u_m \frac{\partial \phi}{\partial x_i} dx &= \int_{\Omega'} \rho \frac{\partial}{\partial x_i} (u_m \phi) dx - \int_{\Omega'} \rho \frac{\partial u_m}{\partial x_i} \phi dx \\ &= - \int_{\Omega'} \frac{\partial \rho}{\partial x_i} u_m \phi dx - \int_{\Omega'} \rho \frac{\partial u_m}{\partial x_i} \phi dx \end{aligned}$$

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Let $u_n \rightarrow u$, $u_n \in \mathcal{D}(\Omega')$ as in Friedrich's theorem.

$$\begin{aligned} \int_{\Omega} g u_n \frac{\partial \varphi}{\partial x_i} dx &= \int_{\Omega} g \frac{\partial}{\partial x_i} (u_n \varphi) dx - \int_{\Omega} g \frac{\partial u_n}{\partial x_i} \varphi dx \\ &= - \int_{\Omega} \frac{\partial g}{\partial x_i} u_n \varphi dx - \int_{\Omega} g \frac{\partial u_n}{\partial x_i} \varphi dx. \end{aligned}$$

$\Rightarrow \frac{\partial}{\partial x_i} (g u_n)$



$$\begin{aligned} \int_{\Omega} g u \frac{\partial \varphi}{\partial x_i} dx &= - \int_{\Omega} \frac{\partial g}{\partial x_i} u \varphi dx - \int_{\Omega} g \frac{\partial u}{\partial x_i} \varphi dx \\ &= - \int_{\Omega} \left(\frac{\partial g}{\partial x_i} u + g \frac{\partial u}{\partial x_i} \right) \varphi dx \\ \Rightarrow \frac{\partial}{\partial x_i} (g u) &= \frac{\partial g}{\partial x_i} u + g \frac{\partial u}{\partial x_i} \in L^2(\Omega). \end{aligned}$$

$$\begin{aligned} \int_{\Omega} g^2 u^2 dx &\leq C \int_{\Omega} |u|^2 dx \\ \left| \frac{\partial g u}{\partial x_i} \right|_{L^2(\Omega)} &\leq C \|u\|_{L^2(\Omega)} \\ \Rightarrow \|g u\|_{H^1(\Omega)} &\leq C \|u\|_{L^2(\Omega)} \end{aligned}$$



So, when you pass to the limit everything is fine so you can use whatever result you want. So, you get

$$\begin{aligned} \int_{\Omega} \rho u \frac{\partial \varphi}{\partial x_i} dx &= - \int_{\Omega'} \frac{\partial \rho}{\partial x_i} u \varphi dx - \int_{\Omega'} \frac{\partial u}{\partial x_i} \rho \varphi dx \\ &= - \int_{\Omega} \left(\frac{\partial \rho}{\partial x_i} u + \frac{\partial u}{\partial x_i} \rho \right) \varphi dx \\ \Rightarrow \frac{\partial}{\partial x_i} (\rho u) &= \frac{\partial \rho}{\partial x_i} u + \frac{\partial u}{\partial x_i} \rho \in L^2(\Omega). \end{aligned}$$

$$\int_{\Omega} \rho^2 u^2 dx \leq c \int_{\Omega} |u|^2 dx \quad \text{and} \quad \left| \frac{\partial}{\partial x_i} (\rho u) \right|_{0,\Omega} \leq c \|u\|_{1,\Omega}.$$

$$\Rightarrow \|\rho u\|_{1,\Omega} \leq c \|u\|_{1,\Omega}.$$

and therefore, it defines a continuous linear operator of $H^1(\Omega)$ into itself.

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Now, if $\rho > 0$ on $\overline{\Omega}$, $\Rightarrow \frac{1}{\rho} \in C(\overline{\Omega})$ and if $v \in H^1(\Omega)$, $v = \rho \left(\frac{1}{\rho} v\right)$

which is also $H^1(\Omega)$.

So, the mapping is onto, it is 1-1 onto continuous and being in Banach spaces open mapping theorem says or anyway even otherwise with $\frac{1}{\rho}$ the inverse map is continuous and therefore, you have that it is an isomorphism.