Sobolev Spaces and Partial Differential Equations
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Exercises - Part 4

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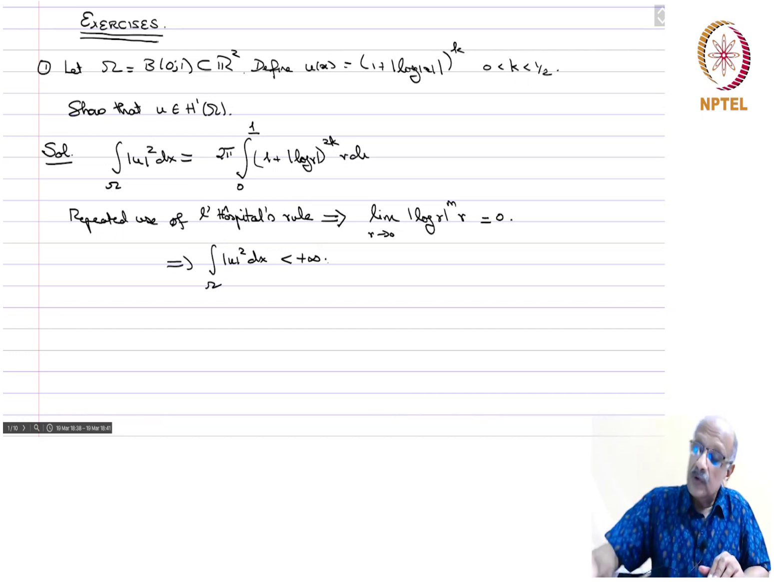

Now it is time to do some exercises.

Exercises:

(1) Let $\Omega = B(0; 1) \subset \mathbb{R}^2$. Define $u(x) = (1 + |log|x||)^k$, $0 < k < \frac{1}{2}$. Show that

$$u \in H^1(\Omega).$$

solution: $\int_\Omega |\nabla u|^2 dx = 2\pi \int_0^1 (1 + |log\, r|)^{2k} r\, dr$

Now, the repeated use of L'hopital's rule $\Rightarrow \lim_{r \to 0} |log\, r|^m r = 0.$

$$\int_\Omega |\nabla u|^2 dx < \infty.$$

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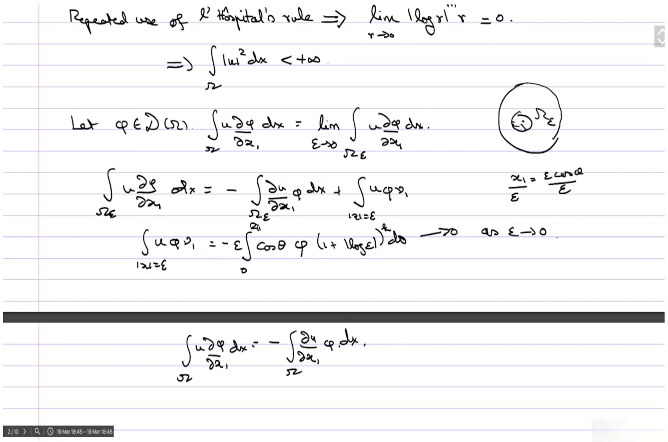

So, now, we have to check what the distribution derivatives are. So, here, like in the case of finding the fundamental solution of the Laplacian, we have to check if the distribution derivative is really obtained by the ordinary differentiation rules.

So, let $\phi \in D(\Omega)$. $\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} u \frac{\partial \phi}{\partial x_i} dx.$

And now, so, you have an omega is a unit ball and the only singularity of the function is at the origin. So, I remove a ball of radius epsilon and therefore, I have a remaining, remaining place, omega epsilon. Now, u is a l2 function on a set of finite measure, it is integrable d\ phi by dx is a C infinity function with compact support and it is bounded whatever you want.

And therefore, by the dominated convergence theorem, you will have that this is nothing but the limit epsilon tending to 0 integral on the ω\ omega epsilon of u d phi by dx1. So, let us look at the integral on omega epsilon u d phi by dx1. Now, we are all in a smooth function case and therefore, we have, we can do all calculus formulae.

So, we use integration by parts:

$$\int_{\Omega_\epsilon} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega_\epsilon} \phi \frac{\partial u}{\partial x_i} dx + \int_{|x| = \epsilon} u \phi \nu_1.$$

$$\int_{|x|=\epsilon} u\phi\nu_1 = - \epsilon\int_0^{\theta} \cos\,\theta\, \phi\, (1 + |\log \epsilon|^k)\, d\theta \to 0 \; as \; \epsilon \to 0.$$

$$\int_{\Omega} u\frac{\partial\phi}{\partial x_i}dx = - \int_{\Omega} \phi\frac{\partial u}{\partial x_i}dx.$$

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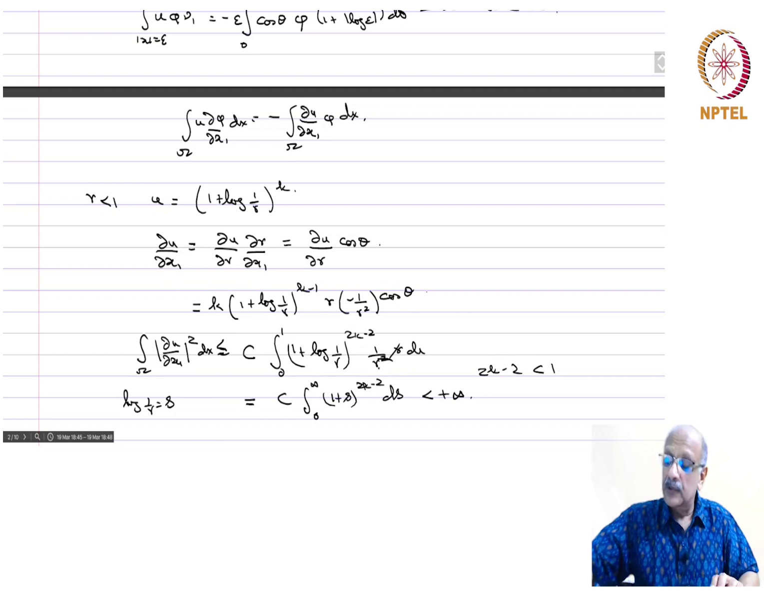

So, now, $r < 1$, because $|x| < 1$. So,

$$u = (1 + \log\,\frac{1}{r})^k.$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_1} = \frac{\partial u}{\partial r}\cos\theta$$

$$= k(1 + \log\,\frac{1}{r})^{k-1} r(\frac{-1}{r^2})\cos\theta.$$

$$\int_{\Omega} |\frac{\partial u}{\partial x_1}|^2 dx \le C\int_0^{r}(1 + \log\,\frac{1}{r})^{2k-2}\frac{1}{r}dr, \quad 2k - 2 < 1.$$

$$= C\int_0^{\infty}(1 + s)^{2k-2}ds < \infty, \quad [\log\,\frac{1}{r} = s].$$

$$\Rightarrow \frac{\partial u}{\partial x_1} \in L^2(\Omega),\; similarly,\; \frac{\partial u}{\partial x_2} \in L^2(\Omega).$$

$$\Rightarrow u \in H^1(\Omega).$$

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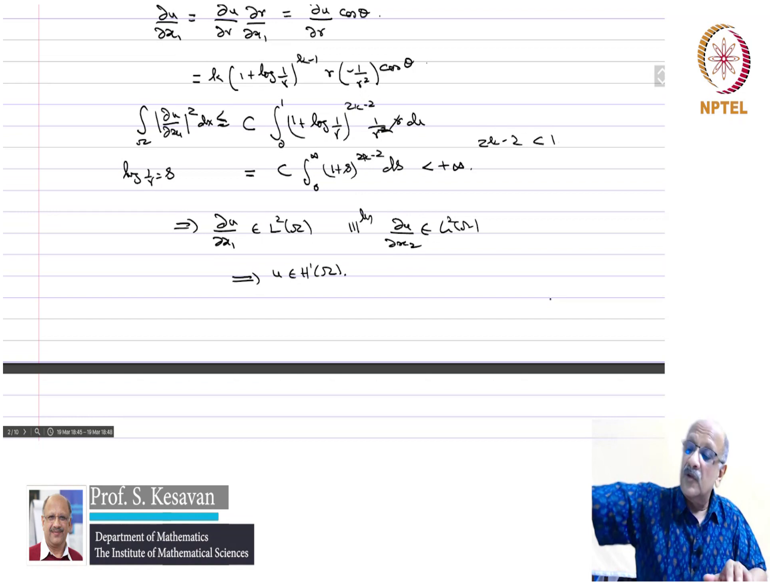

So, this is just an extra straightforward exercise where we have to compute the distribution derivative and see that it is integral.

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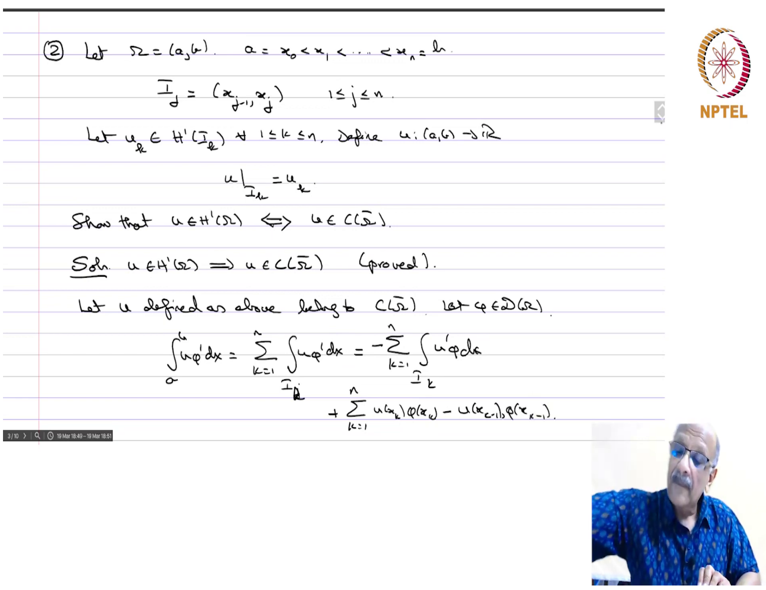

(2) Let $\Omega = (a, b)$, $a = x_0 < x_1 <.....< x_n = b$, $I_j = (x_{j-1}, x_j)$, $1 \le j \le n$. Let $u_k \in H^1(I_k)\ \forall\, 1 \le k \le n.$ *Define* $u: (a, b) \to \mathbb{R}$, $u|_{I_k} = u_k$. Show that

$$u \in H^1(\Omega) \Leftrightarrow u \in C(\overline{\Omega}).$$

solution: if $u \in H^1(\Omega) \Rightarrow u \in C(\overline{\Omega})$ (proved).

So, now conversely let $u \in C(\overline{\Omega})$ and $\phi \in D(\Omega)$. Then

$$\int_a^b u\phi' dx = \sum_{k=1}^{n} \int_{I_k} u\phi' dx = -\sum_{k=1}^{n} \int_{I_k} u'\phi dx$$

$$+ \sum_{k=1}^{n} u(x_k)\phi(x_k) - u(x_{k-1})\phi(x_{k-1})$$

$$= u(a)\phi(b) - u(b)\phi(b).$$

$$= 0.$$

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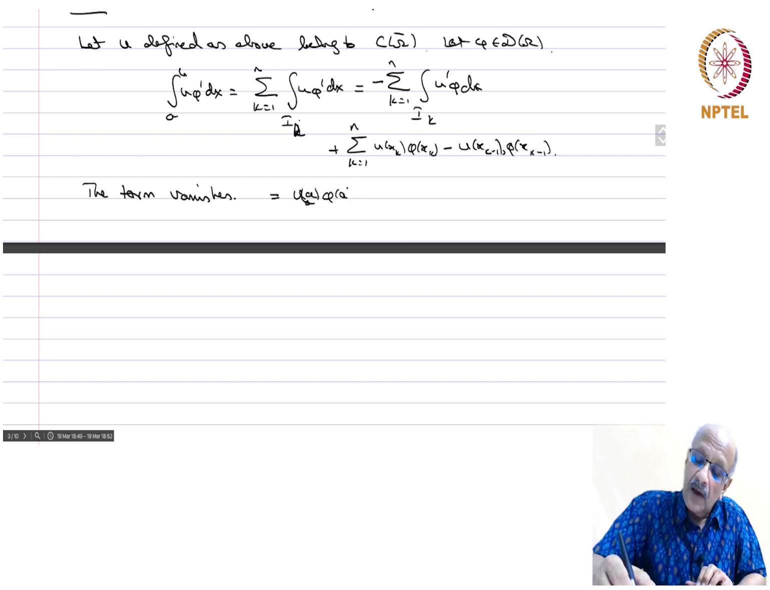

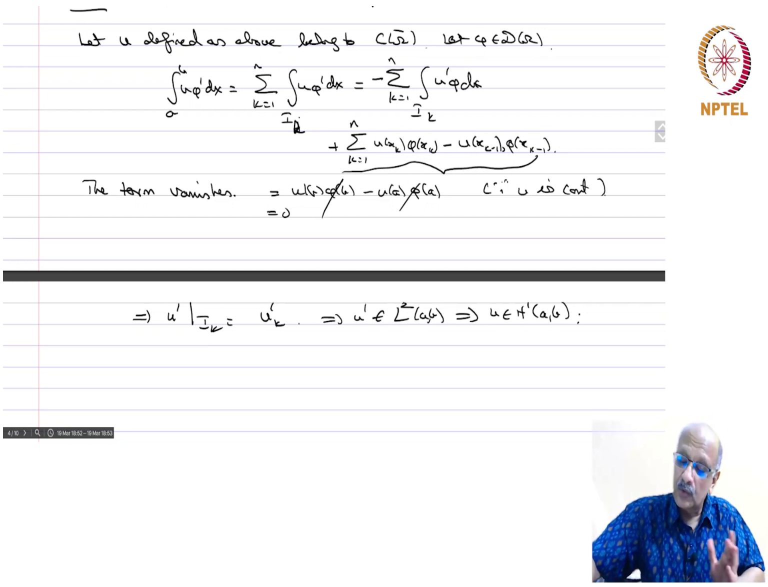

And therefore, you have that $u'|_{I_k} = u_k' \Rightarrow u' \in L^2(a, b) \Rightarrow u \in H^1(a, b)$.

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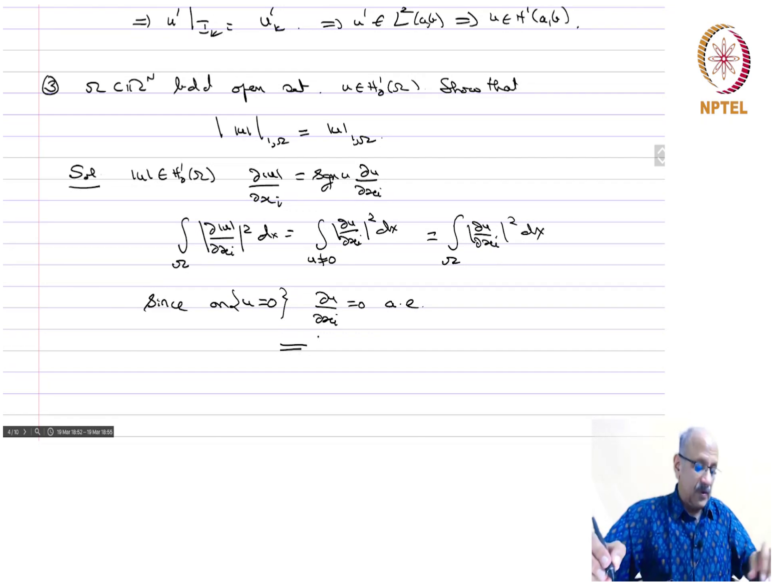

(3) Let $\Omega \subset \mathbb{R}^N$ bounded open set and $u \in H^1_0(\Omega)$. Show that $||u||_{1,\Omega} = |u|_{1,\Omega}$.

solution: we know that $|u| \in H^1_0(\Omega)$. $\frac{\partial |u|}{\partial x_i} = sgn(u) \frac{\partial u}{\partial x_i}$.

$$\int_{\Omega} |\frac{\partial |u|}{\partial x_i}|^2 dx = \int_{u \neq 0} |\frac{\partial u}{\partial x_i}|^2 dx = \int_{\Omega} |\frac{\partial u}{\partial x_i}|^2 dx.$$

Since on $\{u = 0\}$, $\frac{\partial u}{\partial x_i} = 0\ a.e.$ this also we have shown and therefore, there is the new, no new contribution from that side because this function is integrand is 0 almost everywhere. So, you can just add that and so, you will get this and that completes the proof of this.

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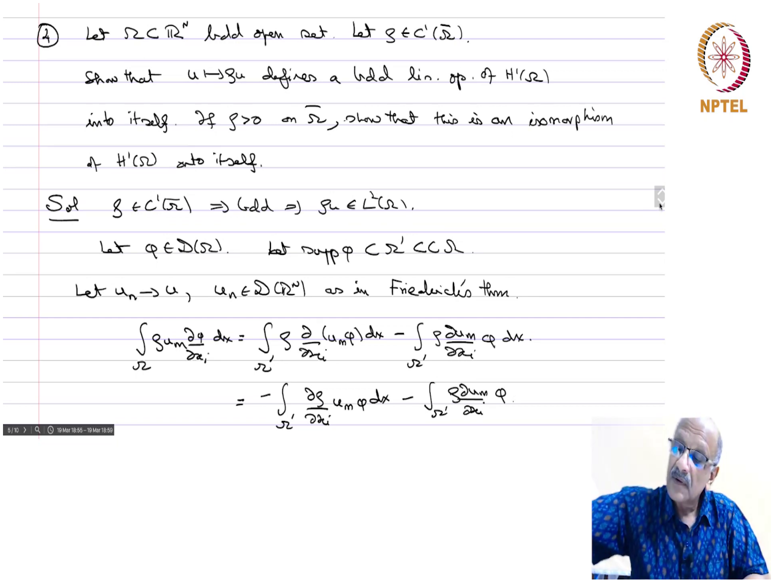

(4) Let $\Omega \subset \mathbb{R}^N$ bounded open set and $\rho \in C^1(\Omega)$. Show that $u \to \rho u$ defines a bounded linear operator on $H^1(\Omega)$ into itself. If $\rho > 0\ on\, \overline{\Omega}$, show that this is an isomorphism of $H^1(\Omega)$ onto itself.

solution: so, $\rho \in C^1(\Omega) \Rightarrow bounded \Rightarrow \rho u\ \in L^1(\Omega)$. Let $\phi \in D(\Omega)$. Let

$supp(\phi) \subset \Omega' \subset\subset \Omega$. Let $u_n \to u,\ u_n \in D(\mathbb{R}^N)$ as in Friedrich's theorem.

So, $\int\limits_{\Omega} \rho u_m \frac{\partial \phi}{\partial x_i} dx = \int\limits_{\Omega'} \rho \frac{\partial}{\partial x_i}(u_m \phi) dx - \int\limits_{\Omega'} \rho \frac{\partial u_m}{\partial x_i} \phi dx$

$$= - \int\limits_{\Omega'} \frac{\partial \rho}{\partial x_i} u_m \phi dx - \int\limits_{\Omega'} \frac{\partial u_m}{\partial x_i} \rho \phi dx\ .$$

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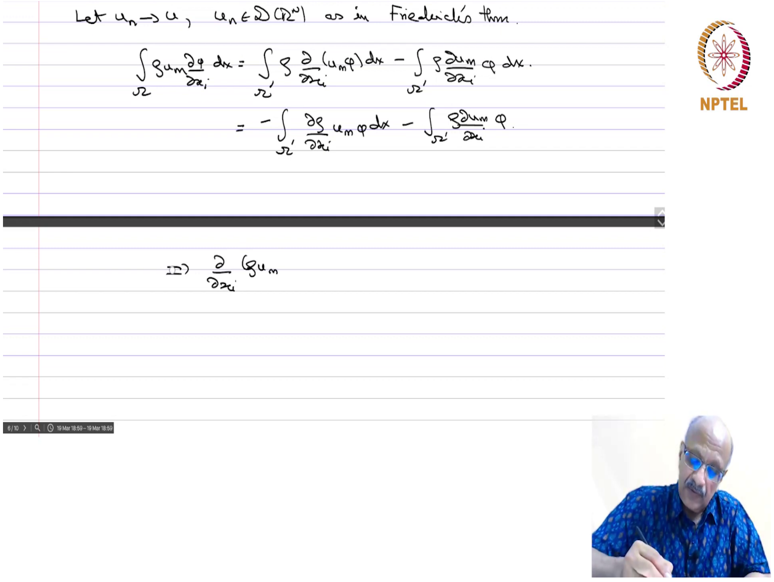

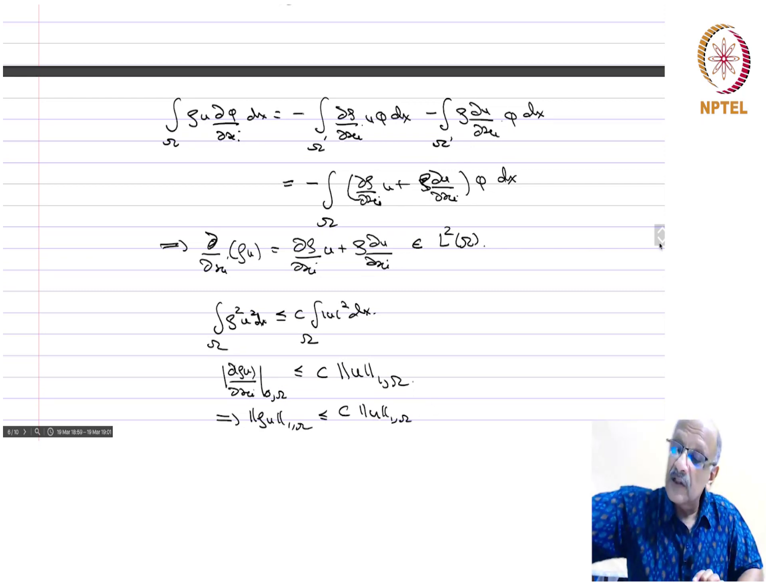

So, when you pass to the limit everything is fine so you can use whatever result you want. So, you get

$$\int_{\Omega} \rho u \ \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega'} \frac{\partial \rho}{\partial x_i} u \ \phi dx \ - \int_{\Omega'} \frac{\partial u}{\partial x_i} \rho \phi dx.$$

$$= - \int_{\Omega} \left(\frac{\partial \rho}{\partial x_i} u \ + \frac{\partial u}{\partial x_i} \rho \right) \phi dx \ .$$

$$\Rightarrow \frac{\partial}{\partial x_i} (\rho u) \ = \frac{\partial \rho}{\partial x_i} u \ + \frac{\partial u}{\partial x_i} \rho \in L^2(\Omega) \ .$$

$$\int_{\Omega} \rho^2 u^2 dx \le c \int_{\Omega} |u|^2 dx \quad and \quad |\frac{\partial}{\partial x_i}(\rho u)|_{0,\Omega} \le c||u||_{1,\Omega} \;.$$

$$\Rightarrow ||\rho u||_{1,\Omega} \le c||u||_{1,\Omega} \;.$$

and therefore, it defines a continuous linear operator of H1 omega into itself.

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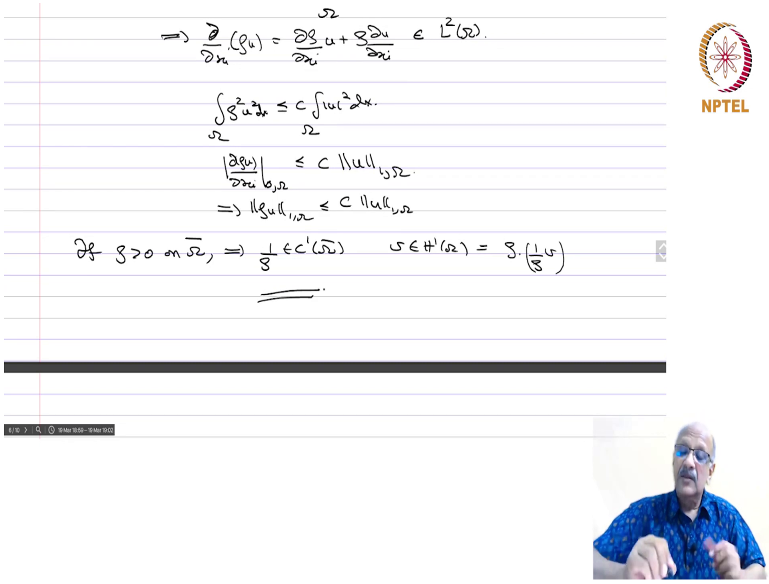

Now, if $\rho > 0 \; on \, \overline{\Omega}$, $\Rightarrow \frac{1}{\rho} \in C(\overline{\Omega}) \quad and \, if \quad v \in H^1(\Omega), \; v = \rho(\frac{1}{\rho} v)$

which is also $H^1(\Omega)$.

So, the mapping is onto, it is 1-1 onto continuous and being in Banach spaces open mapping theorem says or anyway even otherwise with $\frac{1}{\rho}$ the inverse map is continuous and therefore, you have that it is an isomorphism.